The Wiener Index of Unicyclic Graphs with Girth and Matching Number*

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Abstract

In this paper, we investigate how the Wiener index of unicyclic graphs varies with graph operations. These results are used to present a sharp lower bound for the Wiener index of unicyclic graphs of order n with girth g and matching number $\beta \geq \frac{3g}{2}$. Moreover, we characterize all extremal graphs which attain the lower bound.

Key words: Wiener index, unicyclic, the matching number, girth.

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1 Introduction

The Wiener index [19] of a simple connected graph is the sum of distances between all pairs of vertices, which has been much studied in both mathematical (see [2, 6, 7, 8, 9, 10]) and chemical (see [11, 12, 13, 14, 15, 16, 17, 18]) literatures. Gutman et al. in [12] gave some results for the Wiener indices of a unicyclic graph, which is a connected graph with a unique cycle. Recently, Yan and Yeh [20] investigated the relations between the

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matching number and the Wiener index. Du and Zhou in [3] determined the minimum Wiener index in the set of unicyclic graphs of order n with girth and the number of pendent vertices. Moreover, Du and Zhou in [5] gave the sharp lower bounds for the Wiener index of unicyclic graphs with the matching number. The Wiener index and related problems of trees and unicyclic graphs may be referred to [4, 21, 22].

Through this paper, all graphs are finite, simple and undirected. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The *girth* of a graph G with a cycle is the length of its shortest cycle. A *matching* in a graph G is a set of edges with no shared end vertices. The *matching number* of a graph G is the maximum size of all matching of graphs, and denoted by $\beta(G)$ or β. The *distance* between vertices v_i and v_j is the minimum number of edges between v_i and v_j and denoted by $d_G(v_i, v_j)$ (or for short $d(v_i, v_j)$). The *Wiener index* of a connected graph G is defined as

$$W(G) = \sum_{\{v_i, v_j\} \subseteq V(G)} d(v_i, v_j). \tag{1}$$

Moreover, the *distance of a vertex* v, denoted by $d_G(v)$, is the sum of of distances between v and all other vertices of G. Then

$$W(G) = \frac{1}{2} \sum_{u \in V(G)} d_G(u). \tag{2}$$

In this paper, motivated by the above results, we investigate, in Section 2, how the Wiener index of unicyclic graphs with girth and the matching number varies with some graph operations. In Section 3, we obtain a sharp lower bound for the Wiener index in the set consisting of all unicyclic graphs of order n with girth g and the matching number $\beta \geq \frac{3g}{2}$. Moreover, the all extremal graphs which attain the lower bound have been characterized.

2 Wiener index with switching operations

Let $G = (V(G), E(G)$ be a unicyclic graph of order n with girth g. Suppose that the only cycle is $C_g = u_1 u_2 \ldots u_g$. Then $G - E(C_g)$, which is obtained from G by deleted all edges of C_g, has g connected components, each of which is tree T_i of order n_i for $i = 1, \ldots, g$. The connected component T_i is called a *branch* of G at u_i. Clearly, $n_1 + n_2 + \ldots + n_g = n$. Moreover, any unicyclic graph of order n with girth g can be denoted by $U(T_1, \ldots, T_g)$. Let $\mathcal{U}_{(n,g,\beta)}$ be the set of all unicyclic graphs of order n with girth g and the matching number β. It is easy to get the following result.

Lemma 2.1 *Let $G = U(T_1, \ldots, T_g)$ be a unicyclic graph of order n with girth g. Then*

$$W(G) = (n - \frac{g}{2})\lfloor \frac{g^2}{4} \rfloor + (g-1)\sum_{i=1}^{g} d_{T_i}(u_i) + \sum_{i=1}^{g} W(T_i) + \tag{3}$$

$$\sum_{i=1}^{g-1}\sum_{j=i+1}^{g} [(n_i - 1)d_{T_j}(u_j) + (n_j - 1)d_{T_i}(u_i) + (n_i - 1)(n_j - 1)d_{C_g}(u_i, u_j)].$$

Proof. By the definition of the Wiener and $W(C_g) = \frac{g}{2}\lfloor \frac{g^2}{4} \rfloor$, we have

$$W(G) = \sum_{i=1}^{g} \sum_{\{u,v\}\subseteq V(T_i)} d_{T_i}(u,v) + \sum_{i=1}^{g-1}\sum_{j=i+1}^{g} \sum_{u\in V(T_i), v\in V(T_j)} d_G(u,v)$$

$$= \sum_{i=1}^{g} W(T_i) + \sum_{i=1}^{g-1}\sum_{j=i+1}^{g} \sum_{u\in V(T_i), v\in V(T_j)} [d_{T_i}(u, u_i) + d_{C_g}(u_i, u_j) + d_{T_j}(u_j, v)]$$

$$= \sum_{i=1}^{g} W(T_i) + \sum_{i=1}^{g-1}\sum_{j=i+1}^{g} [n_j d_{T_i}(u_i) + n_i n_j d_{C_g}(u_i, u_j) + n_i d_{T_j}(u_j)]$$

$$= (n - \frac{g}{2})\lfloor \frac{g^2}{4} \rfloor + (g-1)\sum_{i=1}^{g} d_{T_i}(u_i) + \sum_{i=1}^{g} W(T_i) +$$

$$\sum_{i=1}^{g-1}\sum_{j=i+1}^{g} [(n_i - 1)d_{T_j}(u_j) + (n_j - 1)d_{T_i}(u_i) + (n_i - 1)(n_j - 1)d_{C_g}(u_i, u_j)].$$

Hence (3) holds. ■

Corollary 2.2 *Let $G = U(T_1, \ldots, T_g)$ and $G_1 = U(\widetilde{T_1}, T_2, \ldots, T_g)$ be two unicyclic graphs of order n with girth g. If $|V(T_1) = |V(\widetilde{T_1})| = n_1, W(T_1) \geq W(\widetilde{T_1})$ and $d_{T_1}(u_1) \geq d_{\widetilde{T_1}}(u_1)$, then*

$$W(G) \geq W(G_1) \tag{4}$$

with equality if and only if $W(T_1) = W(\widetilde{T_1})$ and $d_{T_1}(u_1) = d_{\widetilde{T_1}}(u_1)$.

Proof. By (3) in Lemma 2.1, we have

$$W(G) - W(G_1) = W(T_1) - W(\widetilde{T_1}) + (g-1)(d_{T_1}(u_1) - d_{\widetilde{T_1}}(u_1)) + \sum_{j=2}^{g}(n_j - 1)(d_{T_1}(u_1) - d_{\widetilde{T_1}}(u_1)) \geq 0.$$

Hence the assertion holds. ■

For given two nonnegative integers a, b, let $T^*_{a,b}$ be a rooted tree of order $2a+b+1$ obtained from the root star $K_{1,a+b}$ at root u_1 by adding a pendent edges to a pendent vertices of $K_{1,a+b}$. In particular, $T^*_{0,0}$ is an isolated vertex. Then the matching number of $T^*_{a,b}$ is $a+1$ for $b \geq 1$ and a for $b=0$.

Lemma 2.3 *Let $G = U(T_1, \ldots, T_g)$ be a unicyclic graph of order n with girth g and the matching number β. Denote by β_1 the matching number of T_1 of order n_1. If $\beta_1 = 0$ or G has a maximum matching M containing an edge u_1x, $x \in V(T_1)$. let G_1 be the unicyclic graph from G by replacing T_1 with $T^*_{0,0}$ for $\beta_1 = 0$ and replacing T_1 with $T^*_{\beta_1-1,n_1-2\beta_1+1}$. Then the matching number of G_1 is β and*

$$W(G) \geq W(G_1) \tag{5}$$

with equality if and only if $T_1 = T^*_{0,0}$ or $T^*_{\beta_1-1,n_1-2\beta_1+1}$.

Proof. If $\beta_1 = 0$, the assertion obviously holds. Assume that $\beta_1 \geq 1$. Moreover, since G has a maximum matching M containing an edge u_1x, $x \in V(T_1)$, it is easy to see that the matching number of G_1 is β. Since the matching number of T_1 is β_1, there exist at most $n_1 - \beta_1$ vertices adjacent to u_1 in T_1 (otherwise the matching number of T_1 is less than β_1). Hence $d_{T_1}(u_1) \geq (n_1 - \beta_1) + 2(n_1 - (n_1 - \beta_1 + 1)) = n_1 + \beta_1 - 2$. Further by Corollary 5.7 in [21], we have $W(T_1) \geq W(T^*_{\beta_1-1,n_1-2\beta_1+1})$ with equality if and only if T_1 is $T^*_{\beta_1-1,n_1-2\beta_1+1}$. Hence by Corollary 2.2, the assertion holds. ■

Lemma 2.4 *Let T be a tree of order $n \geq 3$ and $u \in V(T)$. Suppose that the matching number of $T-u$ is β and $T-u$ has p connected components $T_1, \ldots, T_p$ of order $n_1, \ldots, n_p$, respectively. Then*

$$W(T) \geq W(T^*_{\beta,n-2\beta-1}) = n^2 + (\beta-2)n + (-3\beta+1) \tag{6}$$

with equality if and only if T is $T^*_{\beta,n-2\beta-1}$.

Proof. If $\beta = 0$, then T must be the star graph $K_{1,n-1}$, which is exact $T^*_{0,n-1}$. Hence without loss of generality, assume that $0 < \beta \leq \frac{n-1}{2}$ and the matching number of T_i is $\beta_i \geq 1$ for $i = 1, \ldots, t$ and 0 for $t+1 \leq i \leq p$. Assume the neighbor set of u is $\{w_1, \ldots, w_p\}$. Then by Theorem 4 in [2] and Corollary 5.7 in [21], we have

$$W(T) = n(n-1) + \sum_{i=1}^{p}[W(T_i) + (n-n_i)d_{T_i}(w_i) - n_i^2]$$

$$
\begin{aligned}
&\geq n(n-1)+\sum_{i=1}^{t}[(\beta_i-3)n_i-3\beta_i+4+(n-n_i)(n_i+\beta_i-2)]-p+t\\
&= (n-1)^2+(n-3)\beta+(-2n+4)t+n(n-p+t-1)-\sum_{i=1}^{t}n_i^2\\
&\geq (n-1)^2+(n-3)\beta+(-2n+4)t+n(n-p+t-1)\\
&\quad -4(t-1)-[(n-p+t-1)-2(t-1)]^2\\
&= (n-1)^2+(n-3)\beta-t^2-(n+2)t-3n+3-p^2+(n-2t+2)p\\
&\geq (n-1)^2+(n-3)\beta-t^2-(n+2)t-3n+3\\
&\quad -(n-t-1)^2+(n-2t+2)(n-t-1)\\
&= n^2+(\beta-2)n+(-3\beta+1)=W(T^*_{\beta,n-2\beta-1}),
\end{aligned}
$$

where $n=n_1+\ldots+n_t+p-t+1\geq p+t+1$, and $d_{T_i}(w_i)\geq n_i+\beta_i-2$, since w_i is at most adjacent to $n_i-\beta_i$ vertices in T_i. Moreover, if equality holds, then $n_1=\ldots=n_t=2$, which implies $t=\beta$ and $p=n-\beta-1$. Therefore T must be $T^*_{\beta,n-2\beta-1}$. ■

Lemma 2.5 *Let $G=U(T_1,\ldots,T_g)$ be a unicyclic graph of order n. Suppose that any maximum matching of G does not contain u_1x, $x\in V(T_1)$ and the matching number of T_1-u_1 of order n_1-1 is β_1. Let $G_1=U(T^*_{\beta_1,n_1-2\beta_1-1},T_2,\ldots,T_g)$ be the unicyclic graph obtained from G by replacing T_1 with $T^*_{\beta_1,n_1-2\beta_1-1}$. Then the matching numbers of G and G_1 are equal and*

$$W(G)\geq W(G_1) \tag{7}$$

with equality if and only if $T_1=T_{\beta_1,n_1-2\beta_1-1}$.

Proof. It is easy to see that the matching number of G is equal to the matching number of G_1 by the definition, since any maximum matching of G does not contain u_1x, $x\in V(T_1)$. On the other hand, by Lemma 2.4, we have $W(T_1)\geq W(T^*_{\beta_1,n_1-2\beta_1-1})$. Let T_1-u_1 have p components $T_{11},\ldots,T_{1p}$ with

$$|V(T_{11})|\geq\ldots\geq|V(T_{1t})|>|V(T_{1,t+1})|=\ldots=|V(T_{1p})|=1.$$

Since the matching number of T_1-u_1 is β_1, we have $t\leq\beta_1\leq\frac{n_1-1}{2}$ and $n_1\geq 2\beta_1+p-t+1\geq p+\beta_1+1$. Hence $d_{T_1}(u_1)\geq p+2(n_1-p-1)=2n_1-p-2\geq n_1+\beta_1-1=d_{T^*_{\beta_1,n_1-2\beta_1-1}}(u_1)$. Therefore by Corollary 2.2, $W(G)\geq W(G_1)$ with equality if and only if $T_1=T^*_{\beta_1,n_1-2\beta_1-1}$. ■

Lemma 2.6 *Let $G=U(T_1,\ldots,T_g)$ be a unicyclic graph of order n with girth g. Suppose that T_p of order $|V(T_p)|\geq 3$ and T_q of order $|V(T_q)|\geq 3$*

have pendant edges u_px and u_qy, respectively. Let $T_p^{(1)}$ be the tree from T_p and T_q by identifying u_p and u_q and deleting the edge u_qy, and let $T_q^{(1)}$ be the edge u_qy. Moreover, let $T_p^{(2)}$ be the edge u_px, and let $T_q^{(2)}$ be the tree from T_p and T_q by identifying u_p and u_q and deleting the edge u_px. Further, let $G_i = U(T_1, \ldots, T_p^{(i)}, \ldots, T_q^{(i)}, \ldots, T_g)$ for $i = 1, 2$. Then the matching numbers of G, G_1, G_2 are equal, and

$$W(G) > \min\{W(G_1), W(G_2)\}. \tag{8}$$

Proof. Clearly, by the definition, the matching numbers of G, G_1 and G_2 are equal. By Lemma 2.1, it is easy to see that

$$\begin{aligned} W(G) - W(G_1) &= (n_p - 2)(n_q - 2)d_G(u_p, u_q) \\ &+ (n_q - 2) \sum_{i=1, i\neq p,q}^{g} n_i[d_G(u_q, u_i) - d_G(u_p, u_i)] \end{aligned} \tag{9}$$

and

$$\begin{aligned} W(G) - W(G_2) &= (n_p - 2)(n_q - 2)d_G(u_p, u_q) \\ &- (n_p - 2) \sum_{i=1, i\neq p,q}^{g} n_i[d_G(u_q, u_i) - d_G(u_p, u_i)]. \end{aligned} \tag{10}$$

Hence by (9) and (10), the assertion holds. ■

Lemma 2.7 *Let $G = U(T_1, \ldots, T_g)$ be a unicyclic graph of order n with girth g. Suppose that T_p of order $|V(T_p)| \geq 3$ has no pendant vertices adjacent to u_p and T_q of order $|V(T_q)| \geq 3$ has an pendant adjacent vertex y adjacent to u_q. Let $T_p^{(1)}$ be the tree from T_p and T_q by identifying u_p and u_q with deleting the edge u_qy, and let $T_q^{(1)}$ be the edge u_qy. Moreover, let $T_p^{(2)}$ be isolated vertex u_p, and let $T_q^{(2)}$ be the tree from T_p and T_q by identifying u_p and u_q. Further, let $G_i = U(T_1, \ldots, T_p^{(i)}, \ldots, T_q^{(i)}, \ldots, T_g)$ for $i = 1, 2$. Then*

$$W(G) > \min\{W(G_1), W(G_2)\}. \tag{11}$$

Proof. By Lemma 2.1, it is easy to see that $W(G) - W(G_1) =$

$$(n_q - 2)[(n_p - 2)d_G(u_p, u_q) + \sum_{i=1, i\neq p,q}^{g} n_i(d_G(u_q, u_i) - d_G(u_p, u_i))]$$

and $W(G) - W(G_2) =$

$$(n_p - 1)[(n_q - 1)d_G(u_p, u_q) - \sum_{i=1, i\neq p,q}^{g} n_i(d_G(u_q, u_i) - d_G(u_p, u_i))].$$

Hence the assertion holds. ■

Corollary 2.8 *Let $G = U(T^*_{a_1,b_1}, \ldots, T^*_{a_g,b_g})$ be a unicyclic graph of order n with girth g. If $a_p \geq 1$, $b_p = 0$, and $b_q > 0$, $2a_q + b_q \geq 2$ for $1 \leq p \neq q \leq g$, then there exists a unicyclic graph G' of order n and girth g such that the matching numbers of G and G' are equal and $W(G) > W(G')$.*

Proof. Clearly, $|V(T_p)| \geq 3$ and $|V(T_q)| \geq 3$. Let $G_1 = U(T^*_{a_1,b_1}, \ldots, T^*_{a_p+a_q,b_q-1}, \ldots, T^*_{0,1}, \ldots, T^*_{a_g,b_g})$, $G_2 = U(T^*_{a_1,b_1}, T^*_{0,0}, \ldots, T^*_{a_p+a_q,b_q}, \ldots, T^*_{a_g,b_g})$. By Lemma 2.7, we have $W(G) > \min\{W(G_1), W(G_2)\}$. Moreover, let β, β_1, β_2 be the matching numbers of G, G_1 and G_2, respectively. Then $\beta = \beta_2 \leq \beta_1 \leq \beta + 1$. If $\beta_1 = \beta + 1$, let $G_3 = U(T^*_{a_1,b_1}, \ldots, T^*_{a_p+a_q-1,b_q+1}, \ldots, T^*_{0,1}, \ldots, T^*_{a_g,b_g})$. It is easy to see that $W(G_1) > W(G_3)$ and the matching number of G_3 is β. Hence the assertion holds. ■

Lemma 2.9 *Let $G = U(T_1, \ldots, T_g)$ be a unicyclic graph of order n with girth g and $|V(T_i)| = n_i$ for $i = 1, \ldots, g$. Let $T_p^{(1)}$ be the tree from T_p and T_q by identifying u_p and u_q, and let $T_q^{(1)}$ be the isolated vertex. Moreover let $T_p^{(2)}$ be the isolated vertex u_p, and let $T_q^{(2)}$ be the tree from T_p and T_q by identifying u_p and u_q. Further, let $G_i = U(T_1, \ldots, T_p^{(i)}, \ldots, T_q^{(i)}, \ldots, T_g)$ for $i = 1, 2$. Then*

$$W(G) > \min\{W(G_1), W(G_2)\}. \tag{12}$$

Proof. Assume that $p < q$. By Lemma 2.1, we have $W(G) - W(G_1) =$

$$(n_q - 1)[(n_p - 1)d_G(u_p, u_q) + \sum_{i=1, i\neq p,q}^{g} n_i(d_G(u_q, u_i) - d_G(u_p, u_i))]$$

and $W(G) - W(G_2) =$

$$(n_p - 1)[(n_q - 1)d_G(u_p, u_q) - \sum_{i=1, i\neq p,q}^{g} n_i(d_G(u_q, u_i) - d_G(u_p, u_i))].$$

Hence it is easy to see that the assertion holds. ■

Corollary 2.10 *Let $G = U(T^*_{a_1,b_1}, \dots, T^*_{a_p,0}, \dots, T^*_{a_q,0}, \dots, T^*_{a_g,b_g})$ be a unicyclic graph of order n with girth g. If $a_p, a_q \geq 1$, let $G_1 = U(T^*_{a_1,b_1}, \dots, T^*_{a_p+a_q,0}, \dots, T^*_{0,0}, \dots, T^*_{a_g,b_g})$ and $G_2 = U(T^*_{a_1,b_1}, \dots, T^*_{0,0}, \dots, T^*_{a_p+a_q,0}, \dots, T^*_{a_g,b_g})$, then the matching numbers of G, G_1 and G_2 are equal and $W(G) > \min\{W(G_1), W(G_2)\}$.*

Proof. It follows from Lemma 2.9 that the assertion holds. ■

Now we can present the main result in this section.

Theorem 2.11 *Let $G = U(T_1, \dots, T_g)$ be a unicyclic graph of order n with girth g. Then there exist nonnegative integers $a_1, b_1, \dots, b_g$ with $b_j \leq 1$ for $j = 2, \dots, g$ such that G and $\widetilde{G} = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \dots, T^*_{0,b_g})$ have the same the matching number and*

$$W(G) = W(U(T_1, \dots, T_g)) \geq W(\widetilde{G}) = W(U(T^*_{a_1,b_1}, T^*_{0,b_2}, \dots, T^*_{0,b_g})) \quad (13)$$

with equality if and only if $G = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \dots, T^*_{0,b_g})$.

Proof. We consider the following two cases.

Case 1: $|V(T_i)| \geq 3$ and G has a maximum matching M containing an edge u_1x, $x \in V(T_i)$. Then by Lemma 2.3, there exists a $G_1 = U(T_1, \dots, T^*_{c_i,d_i}, \dots, T_g)$ such that $W(G) \geq W(G_1)$ with equality if and only if $T_i = T^*_{c_i,d_i}$, where $c_i + 1$ is the matching number of T_i and $2c_i + d_i + 1 = |V(T_i)|$. Moreover, the matching numbers of G and G_1 are equal.

Case 2: $|V(T_i)| \geq 3$ and any maximum matching of G does not contain u_ix, $x \in V(T_i)$. Let the matching number of $T_i - u_i$ of order $n_i - 1$ be a_i. Then by Lemma 2.5, there exists a $G_2 = U(T_1, \dots, T^*_{c_i,d_i}, \dots, T_g)$ such that $W(G) \geq W(G_2)$ with equality if and only if $G = G_2$, where $2c_i + d_i + 1 = n_i$. Moreover, the matching numbers of G and G_2 are equal.

Hence there exists a $G_3 = U(T^*_{c_1,d_1}, \dots, T^*_{c_g,d_g})$ such that $W(G) \geq W(G_3)$, and the matching numbers of G and G_3 are equal. By the repeated use of Lemma 2.6 and Corollaries 2.8 and 2.10, it is easy to see that the assertion holds. ■

3 Wiener index of unicyclic graphs with girth and the matching number

In this section, we give a sharp lower bound for the Wiener index of unicyclic graphs of order n with girth g and the matching number $\beta \geq \frac{3g}{2}$ and characterize all extremal graphs which attain the lower bound. But we need some lemmas and notations

Lemma 3.1 *Let G_1 and G_2 be two simple connected graphs. Let G be the graph obtained from G_1 and G_2 by identifying a vertex x of G_1 and a vertex y of G_2. Then*

$$W(G) = W(G_1)+W(G_2)+d_{G_1}(x)(|V(G_2)|-1)+d_{G_2}(y)(|V(G_1)|-1). \quad (14)$$

Proof. By the definition, we have $W(G) =$

$$\sum_{u,v\subseteq V(G_1)} d_G(u,v) + \sum_{u,v\subseteq V(G_2)} d_G(u,v) + \sum_{u\in V(G_1)\setminus\{x\}} \sum_{v\in V(G_2)\setminus\{y\}} d_G(u,v)$$

$$\begin{aligned} &= W(G_1) + W(G_2) + \sum_{u\in V(G_1)\setminus\{x\}} \sum_{v\in V(G_2)\setminus\{y\}} (d_{G_1}(u,x) + d_{G_2}(y,v)) \\ &= W(G_1) + W(G_2) + d_{G_1}(x)(|V(G_2)| - 1) + d_{G_2}(y)(|V(G_1)| - 1). \end{aligned}$$

Hence we finish the proof. ■

Assume that $n \geq 2\beta \geq 3g \geq 9$. If g is odd, let $G^*_{(n,g,\beta)}$ be the unicyclic graph of order n obtained by identifying a vertex of a cycle C_g of odd order g and the rooted vertex with degree $n-\beta-\frac{g-1}{2}$ of $T^*_{\beta-\frac{g+1}{2},n-2\beta+1}$ of order $n-g+1$. If g is even, let $G^*_{(n,g,\beta)}$ be a unicyclic graph of order n obtained by identifying vertex u_1 of a cycle $C_g = u_1u_2\dots u_g$ of even order g and the rooted vertex with degree $n-\beta-\frac{g}{2}$ of $T^*_{\beta-\frac{g}{2}-1,n-2\beta+1}$ of order $n-g$, and adding a pendent edge u_2v at vertex u_2. In other words,

$$\begin{aligned} G^*_{(n,g,\beta)} &= U(T^*_{\beta-\frac{g+1}{2},n-2\beta+1}, T^*_{0,0}, \dots, T^*_{0,0}) \text{ for } g \text{ is odd,} \\ G^*_{(n,g,\beta)} &= U(T^*_{\beta-\frac{g}{2}-1,n-2\beta+1}, T^*_{0,1}, \dots, T^*_{0,0}) \text{ for } g \text{ is even.} \end{aligned}$$

Then $G^*_{(n,g,\beta)}$ is a unicyclic graph of order n with girth g and the matching number β. Moreover

Corollary 3.2 *(1). If g is odd, then* $W(G^*_{(n,g,\beta)}) = n^2+$

$$\left(\beta - \frac{3g+1}{2} + \lfloor\frac{g^2}{4}\rfloor\right) n + \left(1-\frac{g}{2}\right)\lfloor\frac{g^2}{4}\rfloor + g^2 + (-2\beta+1)g - 2\beta + 1. \quad (15)$$

(2). If g is even, then

$$W(G^*_{(n,g,\beta)}) = n^2 + \left(\beta - \frac{3g}{2} - 1 + \lfloor\frac{g^2}{4}\rfloor\right) n - \frac{g}{2}\lfloor\frac{g^2}{4}\rfloor + \frac{3g}{2} - 3\beta + 2. \quad (16)$$

Proof. It follows from Lemma 3.1 and some calculation. ■

Lemma 3.3 *Let $G = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \dots, T^*_{0,b_g})$ be a unicyclic graph of order n with odd girth g and the matching number β, where $b_i \le 1$ for $i = 2, \dots, g$. If $\beta \ge \frac{3g}{2}$, then*

$$W(G) \ge W(G^*_{(n,g,\beta)}) = W(U(T^*_{\beta-\frac{g+1}{2},n-2\beta+1}, T^*_{0,0}, \dots, T^*_{0,0}))$$

with equality if and only if $G = U(T^*_{\beta-\frac{g+1}{2},n-2\beta+1}, T^*_{0,0}, \dots, T^*_{0,0})$.

Proof. Let $t = \sum_{j=2}^{g} b_j$. We consider the following two cases.

Case 1: $t = 0$. If $b_1 > 0$, then $a_1 + \frac{g+1}{2} = \beta$, so $a_1 = \beta - \frac{g+1}{2}$ and $b_1 = n - 2\beta + 1$ Hence G must be $U(T^*_{\beta-\frac{g+1}{2},n-2\beta+1}, T^*_{0,0}, \dots, T^*_{0,0})$ and the assertion holds. If $b_1 = 0$, then $a_1 = \beta - \frac{g-1}{2}$ and $n = \beta + g$. Hence $a_1 - 2 = \beta - \frac{g+1}{2}$ and $n - 2\beta + 1 = 2$. Further we have

$$W(G) = W(U(T^*_{a_1,0}, T^*_{0,0}, \dots, T^*_{0,0})) > W(U(T^*_{a_1-1,2}, T^*_{0,0}, \dots, T^*_{0,0})).$$

Hence the assertion holds.

Case 2: $t \ge 1$. Suppose that the only cycle $C_g = u_1 \dots u_g$ and $T^*_{0,b_{i_1}}, \dots, T^*_{0,b_{i_t}}$ consist of an edge $u_{i_1}v_{i_1}, \dots, u_{i_t}v_{i_t}$, respectively, where $2 \le i_1 < \dots < i_t \le g$. Let $V_1 = \{v_{i1}, \dots, v_{it}\}$ and $V_2 = V \setminus V_1$. Then $b_j = 0$ for $j \ne 1, i_1, \dots, i_t$. Clearly, $\beta - g \le a_1 \le \beta - \frac{g+1}{2}$. Then $s \equiv \beta - \frac{g+1}{2} - a_1 \ge 0$ and

$$r \equiv t - 2s = (n - 2a_1 - b_1 - g) - 2(\beta - \frac{g+1}{2} - a_1) = n - 2\beta - b_1 + 1 \ge 0.$$

Then $U(T^*_{a_1-1,2}, T^*_{0,0}, \dots, T^*_{0,0})$ may be obtained from G by deleting $u_{i_j}v_{i_j}$ for $j = 1, \dots, t$ and adding s paths of length 2, i.e., $u_1v_{i_1}v_{i_2}, \dots, u_1v_{i_{2s-1}}v_{i_{2s}}$ and r edges $u_1v_{i_{2s+1}}, \dots, u_1v_{i_t}$. Therefore,

$$\begin{aligned} W(G) &= \sum_{\{u,v\}\subseteq V_1} d_G(u,v) + \sum_{u\in V_1, v\in V_2} d_G(u,v) + \sum_{\{u,v\}\subseteq V_2} d_G(u,v) \\ &\ge \frac{3t(t-1)}{2} + t(g + \lfloor \frac{g^2}{4} \rfloor + 7a_1 + 3b_1) + \sum_{\{u,v\}\subseteq V_2} d_G(u,v). \end{aligned}$$

On the other hand,

$$\begin{aligned} W(G_1) = &\sum_{\{u,v\}\subseteq V_1} d_{G_1}(u,v) + \sum_{u\in V_1, v\in V_2} d_{G_1}(u,v) + \sum_{\{u,v\}\subseteq V_2} d_{G_1}(u,v) \\ = &\ [6s^2 + (5r - 5)s] + [(3s + r)g + t\lfloor \frac{g^2}{4} \rfloor + (12s + 5r)a_1 \\ &+ (5s + 2r)b_1] + \sum_{\{u,v\}\subseteq V_2} d_{G_1}(u,v). \end{aligned}$$

Hence

$$\begin{aligned} W(G) - W(G_1) &= (r+2)s + \frac{3r(r-1)}{2} + (s+r)b_1 + (2s+2r)a_1 - gs \\ &> 0, \end{aligned}$$

since $a_1 \geq \beta - g \geq \frac{3}{2}g - g = \frac{g}{2}$. ■

Theorem 3.4 *Let G be a unicyclic graph of order n with odd girth g and the matching number β. If $\beta \geq \frac{3g}{2}$, then*

$$W(G) \geq n^2 + (\beta - \frac{3g+1}{2} + \lfloor \frac{g^2}{4} \rfloor)n + (1 - \frac{g}{2})\lfloor \frac{g^2}{4} \rfloor + g^2 + (-2\beta + 1)g - 2\beta + 1$$

with equality if and only if G is $G^*_{(n,g,\beta)}$.

Proof. It follow from Theorem 2.11, Lemmas 3.3 and Corollary 3.2 that the assertion holds. ■

Lemma 3.5 *Let $G = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \ldots, T^*_{0,b_g})$ be a unicyclic graph of order n with even girth g and the matching number $\beta \geq \frac{3g}{2}$. If $a_1 \leq \beta - \frac{g}{2} - 1$ and $b_j \leq 1$ for $j = 2, \ldots, g$, then*

$$W(G) \geq W(G^*_{(n,g,\beta)}) \equiv W(U(T^*_{\beta - \frac{g}{2} - 1, n-2\beta+1}, T^*_{0,1}, \ldots, T^*_{0,0})).$$

with equality if and only if $G = G^*_{(n,g,\beta)}$.

Proof. Since $a_1 \leq \beta - \frac{g}{2} - 1$, we have $t \equiv \sum_{j=2}^{t} b_j \geq 1$. Suppose that the only cycle $C_g = u_1 \ldots u_g$ and $T^*_{0,b_{i_j}}$ is an edge $u_{i_j}v_{i_j}$, $j = 1, \ldots, t$, where $1 \leq i_1 < \ldots < i_t \leq g$. Let $V_1 = \{v_{i1}, \ldots, v_{it}\}$ and $V_2 = V \setminus V_1$. Then $s \equiv \beta - \frac{g}{2} - 1 - a_1 \geq 0$ and

$$r \equiv t - 2s - 1 = (n - 2a_1 - b_1 - g) - 2(\beta - \frac{g}{2} - 1 - a_1) - 1 = n - 2\beta - b_1 + 1 \geq 0.$$

Hence $G^*_{(n,g,\beta)}$ may be obtained from G by deleting $\{v_{i_1}, \ldots, v_{i_t}\}$ and adding s paths $u_1 v_{i_{2l-1}} v_{i_{2l}}$ for $l = 1, \ldots, s$ and adding edges $u_1 v_{i_l}$ for $l = 2s+1, \ldots, t-1$ and $u_2 v_{i_t}$. Further,

$$\begin{aligned} W(G) &= \sum_{\{u,v\} \subseteq V_1} d_G(u,v) + \sum_{u \in V_1, v \in V_2} d_G(u,v) + \sum_{\{u,v\} \subseteq V_2} d_G(u,v) \\ &\geq 3\frac{t(t-1)}{2} + t\left(g + \lfloor \frac{g^2}{4} \rfloor + 7a_1 + 3b_1\right) + \sum_{\{u,v\} \subseteq V_2} d_G(u,v) \\ &= 6s^2 + (6r+3)s + \frac{3r(r+1)}{2} + t\left(g + \lfloor \frac{g^2}{4} \rfloor + 7a_1 + 3b_1\right) \\ &+ \sum_{\{u,v\} \subseteq V_2} d_G(u,v). \end{aligned}$$

On the other hand.

$$\sum_{\{u,v\}\subseteq V_1} d_{G^*_{(n,g,\beta)}}(u,v) = 6s^2 + (5r+2)s + r(r+1)$$

and $\sum_{u\in V_1, v\in V_2} d_{G^*_{(n,g,\beta)}}(u,v) =$

$$(3s+r+1)g + t\lfloor\frac{g^2}{4}\rfloor + (12s+5r+7)a_1 + (5s+2r+3)b_1.$$

Hence

$$W(G) - W(G^*_{(n,g,\beta)}) = (r+1)s + \frac{r(r+1)}{2} + (s+r)b_1 + 2(s+r)a_1 - sg \geq 0$$

with equality if and only if $r = s = 0$, since $a_1 \geq \beta - g \geq \frac{g}{2}$. Hence the assertion holds. ■

Lemma 3.6 *Let $G = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \ldots, T^*_{0,b_g})$ be a unicyclic graph of order n with even girth g and the matching number $\beta \geq \frac{3g}{2}$. If $a_1 = \beta - \frac{g}{2}$ and $b_j \leq 1$ for $j = 2, \ldots, g$, then*

$$W(G) \geq W(U(T^*_{a_1,b_1+t}, T^*_{0,0}, \ldots, T^*_{0,0}))$$

with equality if and only if G is $U(T^*_{a_1,b_1+t}, T^*_{0,0}, \ldots, T^*_{0,0})$, where $t = \sum_{i=2}^{g} b_i$.

Proof. If $t = \sum_{i=2}^{g} b_i = 0$, then the assertion holds. Suppose that $t \geq 1$. Then the matching number of $U(T^*_{a_1,b_1+t}, T^*_{0,0}, \ldots, T^*_{0,0})$ is β. Moreover, $G_1 \equiv U(T^*_{a_1,b_1+t}, T^*_{0,0}, \ldots, T^*_{0,0})$ may be obtained from G by deleting vertices $v_{i_1}, \ldots, v_{i_t}$ and adding edges $u_1v_{i_1}, \ldots, u_1v_{i_t}$. Then it is easy to see that $W(G) > W(G_1)$. Therefore the proof is finished. ■

Theorem 3.7 *Let G be a unicyclic graph of order n with even girth g and the matching number β. If $\beta \geq \frac{3g}{2}$, then*

$$W(G) \geq W(G^*_{(n,g,\beta)}) = n^2 + \left(\beta - \frac{3g}{2} - 1 + \lfloor\frac{g^2}{4}\rfloor\right)n - \frac{g}{2}\lfloor\frac{g^2}{4}\rfloor + \frac{3g}{2} - 3\beta + 2$$

with equality if and only if G is $G^*_{(n,g,\beta)}$.

Proof. By Theorem 2.11, there exists a unicyclic graph G_1 of order n with even girth g and the matching number β such that $G_1 = U(T^*_{a_1,b_1}, T^*_{0,b_2}, \ldots, T^*_{0,b_g})$ of order n with girth g and the matching number β such that $W(G) \geq W(G_1)$, where $b_i \leq 1$ for $i = 2, \ldots, g$. If $a_1 \leq \beta - \frac{g}{2} - 1$, then by Lemma 3.5, $W(G_1) \geq W(U^*_{(n,g,\beta)})$ with equality if and only if G_1 is $U^*_{(n,g,\beta)}$. If $a_1 = \beta -$

$\frac{g}{2}$, then by Lemma 3.6, we have $W(G_1) \geq W(U(T^*_{\beta-\frac{g}{2},n-2\beta}, T^*_{0,0}, \ldots, T^*_{0,0})$. Further, it is easy to see that

$$W(U(T^*_{\beta-\frac{g}{2},n-2\beta}, T^*_{0,0}, \ldots, T^*_{0,0}) > W(U(T^*_{\beta-\frac{g}{2}-1,n-2\beta+1}, T^*_{0,1}, \ldots, T^*_{0,0})).$$

Therefore by Corollary 3.2, the assertion holds. ■

Combining Theorems 3.4 and 3.7, we obtain the main result in this paper

Theorem 3.8 *Let G be a unicyclic graph of order n with girth g and the matching number β. If $\beta \geq \frac{3g}{2}$, then*

$$W(G) \geq W(G^*_{(n,g,\beta)})$$

with equality if and only if G is $G^*_{(n,g,\beta)}$.

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