A Census of all 5-regular planar graphs with diameter 3

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Abstract

Using connectivity and planarity constraints we characterise all 5-regular planar graphs with diameter 3.

1 Introduction

During the writing of [6], all 5-regular simple planar graphs of order at most 24 were generated. Only those of orders 12 and 16 had diameter three. Hell and Seyffarth [5] proved that planar graphs of diameter 3 with maximum degree 5 have at most 52 vertices. Regular planar graphs with degree 5 are more heavily constrained. We use these constraints to prove results linking their connectivity and diameter. These suffice to describe the 5-regular planar graphs of diameter 3.

We follow [2] for terminology and notation. A *cut-set* in a graph G is a set S of vertices in G such that $G - S$ is disconnected. We use $d_H(v)$ for the degree of a vertex v in a graph (or subgraph) H. Our graphs have no loops or multiple edges.

2 Connectivity

We show first that any 5-regular planar graph G with diameter 3 is 5-connected and *weakly 6-connected*; that is, the only cut-sets with at most five vertices are single vertex neighbourhoods. Since edge contraction preserves planarity, and because $K_{3,3}$ and K_5 are not planar, every graph having $K_{3,3}$ or K_5 as a minor is non-planar.

Theorem 1. *Every 5-regular planar graph with diameter 3 is 5-connected and weakly 6-connected.*

Proof. Let G be a 5-regular graph with diameter 3, embedded in the plane. Whenever S is a cut-set in G, all paths connecting vertices from different components of $G - S$ pass through S; hence having diameter 3 requires that at most one component of $G - S$ has a vertex with no neighbour in S. Let S be a smallest cut-set in G. Let C be a smallest component of $G - S$ among those whose vertices all have a neighbour in S. However, if $|S| \leq 4$, then we also restrict the choice of S to make C as small as possible. Let C' be a component of $G - S$ other than C.

We prove that $|S| \geq 5$ and that if $|S| = 5$ then $|V(C)| = 1$. If $|V(C)| = 1$, then $|S| = 5$, so we may assume that $|V(C)| \geq 2$. To avoid crossings in the given embedding, C' must lie in a single face F of C. Since S is a smallest cut-set, each vertex of S has a neighbour in C', so each vertex of S also lies in F. Since each vertex of C has a neighbour in S, we thus conclude that C is outerplanar.

It is well known that every outerplanar graph with at least two vertices has at least two vertices of degree at most 2 that are not cut-vertices; let v and w be such vertices in C. Each such vertex has at least three neighbours in S, since G is 5-regular. Hence $|S| \geq 3$.

If v and w have more than two common neighbours in S, then contracting C' to a single vertex yields $K_{3,3}$ as a minor of G, since each vertex of S has a neighbour in C'. Suppose that v and w have exactly two common neighbours in S; call these x and y. Let $T = \{x, v, y, w\}$; note that these vertices in order form a 4-cycle.

Since v and w have two common neighbours in S and $|S| \leq 5$, at least one of them has two neighbours in C. Hence $|V(C)| \geq 3$. Since C is connected, it contains a v, w-path. Also, there is an x, y-path through C'. These paths cannot cross, so in the embedding they lie in opposite faces of the 4-cycle through T. Now T is another minimum cut-set with size at most 4, and $C - \{v, w\}$ contains a component of $G - T$ whose vertices all have a neighbour in T. Since $C - \{v, w\}$ is smaller than C, this contradicts the choice of S.

Thus v and w have at most one common neighbour in S, which requires $|S| = 5$. Let $S = \{u_1, u_2, u, v_1, v_2\}$, with $N(v) \cap S = \{u, u_1, u_2\}$ and $N(w) \cap S = \{u, v_1, v_2\}$. Let $T = \{u_1, v, u, v_1, v_2\}$. If u_2 has no neighbour other than v in C, then T together with a component of $C - v$ contradicts the choice of S. Hence we may assume that u_2 has a neighbour in C other than v, and the same is true by symmetry for u_1.

Since v was chosen not to be a cut vertex of C, we can now contract all of $C - v$ to a single vertex. In this minor, the resulting vertex is adjacent to $\{u, u_1, u_2\}$, as is v. When we also contract C', we again obtain $K_{3,3}$ as a minor of G. The contradiction implies that $|V(C)| \geq 2$ is impossible, and S is a vertex neighbourhood. □

3 Complete Census

The icosahedron is a 5-regular planar graph with 12 vertices that has diameter 3. It has a unique embedding in the plane, in which every face is a triangle. Every

5-regular plane graph other than the icosahedron has at least one nontriangular face, say f. Because of 5-regularity, each vertex v of f has a neighbour w that does not lie on a face neighbouring f. In this situation we call v the *base vertex* of w and w the *summit vertex* of v (relative to f). We use u, v*-path* to mean a path from a vertex u to a vertex v. A cut-set that is a vertex neighbourhood is a *trivial cut.*

Theorem 2. *There is exactly one 5-regular planar graph with diameter 3 having a face of length 4, and it has 16 vertices.*

Proof. Let G be such a graph, and let f be a 4-face in G. Let v_1, v_2, v_3, and v_4 be the vertices of f in clockwise order in the embedding, and let w_1, w_2, w_3, and w_4 be their summit vertices relative to f, as shown in figure 1. No two summit vertices are adjacent, since this would create a separating cycle of length 4 or a separating cycle of length 5 with at least two vertices both inside and out, contradicting Theorem 1.

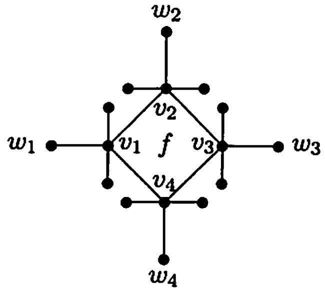

Figure 1: A face of size 4 and its base and summit vertices

Let $P_{1,3}$ be a w_1, w_3-path and $P_{2,4}$ be a w_2, w_4-path of length at most 3 in G. Such paths via f have length at least 4, so these paths use other vertices. Identifying a summit vertex with any neighbour of a vertex on f creates a cut-set of size at most 4, contradicting Theorem 1. Therefore, the internal vertices of $P_{1,3}$ and $P_{2,4}$, which are adjacent to summit vertices, are not in Figure 1. Since G is planar, $P_{1,3}$ and $P_{2,4}$ cross at some vertex x. Without loss of generality, x is a common neighbour of w_1 and w_2.

The cycle C_2 with vertices x, w_1, v_1, v_2, w_2 in order now encloses neighbours of v_1 and v_2 and separates them from the rest of G. By Theorem 1, there is exactly one vertex z_2 inside C_2, with neighbourhood $\{x, w_1, v_1, v_2, w_2\}$.

If w_3 or w_4 (say w_3 by symmetry) also is adjacent to x, then the same argument applies to the cycle through $\{x, w_2, v_2, v_3, w_3\}$, but then w_2 has only three possible neighbours. Hence $P_{1,3}$ and $P_{2,4}$ have length exactly 3, and the other internal vertices t and s of $P_{1,3}$ and $P_{2,4}$ are neighbours of w_3 and w_4, respectively, as shown in Figure 2

If $s = t$, then the cycle through $\{s, w_3, v_3, v_4, w_4\}$ is a separating cycle C_4 that must have exactly one vertex z_4 inside, by the same argument as given for C_2.

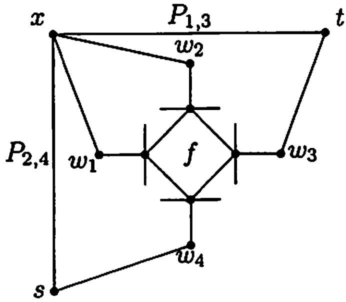

Figure 2: Crossing paths joining summit vertices

Now w_3 must have two neighbours inside the cycle $C_3 = \{x, s, w_3, v_3, v_2, w_2\}$, and w_4 must have two neighbours inside the cycle $C_1 = \{x, s, w_4, v_4, v_1, w_1\}$. Now $\{s, w_3, v_3, v_2, w_2\}$ or $\{s, w_4, v_4, v_1, w_1\}$ is a non-trivial cut-set of size 5 unless x has a neighbour inside C_3 and a neighbour inside C_1. This is impossible, since x already has four neighbours. We conclude that $s \neq t$, as shown in Figure 2.

Since x now has five neighbours, it has no neighbour inside C_3 or C_1. By the cut-set argument just given, this implies that w_1 has only one neighbour inside C_1, and w_2 has only one neighbour inside C_3. Since $d(w_1) = d(w_2) = 5$, we conclude that $w_1 s$ and $w_2 t$ are edges. We have now forced the induced subgraph H with 14 edges shown in Figure 3, and G has no additional vertices inside the bounded faces of H.

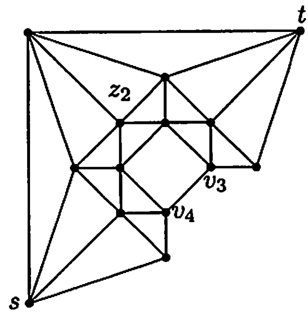

Figure 3: The induced subgraph H of G

In H, all vertices except those in $\{t, w_3, v_3, v_4, w_4, s\}$ already have five neighbours. Since multiple edges are forbidden, the degrees of these vertices cannot be increased to 5 without additional vertices (in fact Chvátal [4] proved that there is no 5-regular planar graph with 14 vertices). Since G is regular of odd degree, at least two more vertices must be added.

If any additional vertex does not have a neighbour in $\{t, v_3, v_4, s\}$, then its distance from z_2 exceeds 3. Those vertices already have degree 4, so we have at most four more vertices. Counting the two additional edges at each of w_3 and w_4, exactly eight edges join $V(H)$ to the new vertices. With four new vertices, this leaves degree-sum 12 for edges induced by them, forcing them to induce K_4. However, contracting H to a vertex then yields K_5 as a minor of G.

Hence there must be exactly two additional vertices, and in order to have the right number of edges, they must be adjacent. Because w_3 and w_4 must be adjacent to both new vertices, the only way to complete the picture is as shown in Figure 4. This graph has diameter 3, so it is the only graph satisfying the hypothesis. □

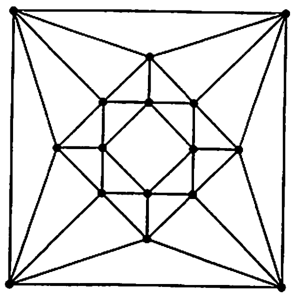

Figure 4: The unique graph with 16 vertices and diameter 3

Theorem 3. *Every 5-regular planar graph having a face of length greater than 4 has diameter greater than 3.*

Proof. Let G be a 5-regular plane graph with diameter 3. Suppose that G has a face f of length at least 5. Let $\{v_1, \ldots, v_k\}$ be the vertices of f in order, and let w_i be the summit vertex of v_i, for each i. No summit vertex has more than one neighbour on f (its base vertex), since G has no cut-set of size 3. Two consecutive summit vertices cannot be identical or adjacent, since G has no cut-set of size at most 4. Summit vertices with base vertices two apart on f cannot be identical or adjacent, since G has no nontrivial cut-set of size at most 5. If f has length at least 6 and summit vertices with base vertices more than 2 apart on f are identical or adjacent, then there is no path of length at most 3 from some base vertex to the summit vertex for a base vertex opposite to it along f.

If two consecutive summit vertices w_i and w_{i+1} have a common neighbour x that is not adjacent to a vertex on f, then $\{x, w_i, v_i, v_{i+1}, w_{i+1}\}$ forms a separating 5-cycle with at least one vertex inside it. Since G has no nontrivial cut-set of size 5, there is exactly one vertex z inside, and x is adjacent to it. If

$\{w_{i-1}, w_i, w_{i+1}\}$ are all adjacent to x, then let z' be the vertex inside the cycle formed by $\{x, w_{i-1}, v_{i-1}, v_i, w_i\}$. Now $\{x, z', v_i, v_{i+1}, w_{i+1}\}$ is a separating 5-cycle with $\{w_i, z\}$ inside, contradicting the prohibition of nontrivial cut-sets of size 5. Hence three consecutive summit vertices cannot have a common neighbour.

Suppose that summit vertices w_i and w_j with i and j nonconsecutive have a common neighbour x. Now the cycle C that traverses $\{w_j, x, w_i\}$ followed by a v_i, v_j-path along f separates summit vertices inside from summit vertices outside. Consider summit vertices w and w' separated by C. A w, w'-path of length at most 3 must cross C and hence include a neighbour of w or w' on C. The only vertex on C not already forbidden as a neighbour for these vertices is x, so x is a neighbour of at least one of them.

By applying this argument with $\{w, w'\} = \{w_{i-1}, w_{i+1}\}$ and with $\{w, w'\} = \{w_{j-1}, w_{j+1}\}$, we obtain two pairs of consecutive summit vertices that are all neighbours of x, because we have already excluded the possibility of three consecutive summit vertices being neighbours of x. However, now our claim about consecutive summit vertices with a common neighbour forces x to have at least six neighbours.

We conclude that no two summit vertices have a common neighbour that has no neighbour on f. However, paths connecting nonconsecutive summit vertices through vertices of f have length at least 4. Hence when i and j are not consecutive, some neighbours of w_i and w_j without neighbours on f are adjacent. Now contract the edges from w_i to its neighbours not on f; this makes w_i and w_j adjacent when i and j are not consecutive. In addition, for each i contract v_i into w_{i+1} along a path not using v_{i+1} or neighbours of summit vertices that are not neighbours of f. We now have a complete graph with at least five vertices as a minor of G. The contradiction implies that there is no such G. □

4 Conclusions

After including the results above, the best known bounds for the maximum order of regular planar graphs with given degree and diameter are as shown in Table 1.

Here the notation $[a, b]$ indicates that the largest known example has a vertices and that b has been proved to be an upper bound. An entry that is one number indicates that the answer has been determined completely. The current table of these bounds is kept updated and hosted at

`http://faculty.capebretonu.ca/jpreen/degdiam.html`

It may be possible to use methods like those in this paper to improve the bounds for 4-regular graphs with small diameter, although they lack the advantage of having summit vertices. Also, the connectivity restrictions are weaker; for example, the largest 4-regular graph with diameter 2 has connectivity 3. It should also be practical to reduce the upper bounds for 5-regular planar graphs

Table 1: The degree-diameter table for planar graphs [September 2008]

	Diameter			
Degree	2	3	4	5
2	5	7	9	11
3	6	12	[18, 30]	[28, 62]
4	9	[16, 33]	[27, 96]	[44, 291]
5		16	[28, 248]	[62, 984]

with larger diameters. Computer searches suggest that the lower bounds from examples found for diameter at most 4 are in fact optimal, but short proofs are still to be found.

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