Mutually Independent Hamiltonian Cycles in Some Graphs

Cheng-Kuan Lin [α], Yuan-Kang Shih [α],
Jimmy J. M. Tan [α,*], and Lih-Hsing Hsu [β,†]
[α] Department of Computer Science,
National Chiao Tung University
[β]Department of Computer Science
and Information Engineering,
Providence University

Abstract

Let $G = (V, E)$ be a hamiltonian graph. A hamiltonian cycle C of G is described as $\langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ to emphasize the order of vertices in C. Thus, v_1 is the beginning vertex and v_i is the i-th vertex in C. Two hamiltonian cycles of G beginning at u, $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1\rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1\rangle$ of G are *independent* if $u_1 = v_1 = u$, and $u_i \neq v_i$ for every $2 \leq i \leq n(G)$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G are *mutually independent* if they are pairwise independent. The *mutually independent hamiltonianicity* of graph G, $IHC(G)$, is the maximum integer k such that for any vertex u there are k-mutually independent hamiltonian cycles of G beginning at u. In this paper, we prove that $IHC(G) \leq \delta(G)$ for any hamiltonian graph and $IHC(G) \geq 2\delta(G) - n(G) + 1$ if $\delta(G) \geq \frac{n(G)}{2}$. Moreover, we present some graphs that meet the bound mentioned above.

Keywords: hamiltonian cycle, Dirac theorem, mutually independent hamiltonian

*This work was supported in part by the National Science Council of the Republic of China under Contract NSC 95-2221-E-009-134-MY3.

†This work was supported in part by the National Science Council of the Republic of China under Contract NSC 95-2221-E-233-002.

1 Basic definitions

For graph definitions and notations we follow [1]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. Let S be a subset of V. The subgraph of G *induced* by S, $G[S]$, is the graph with $V(G[S]) = S$ and $E(G[S]) = \{(x, y) \mid (x, y) \in E(G)$ and $x, y \in S\}$. We use $G - S$ to denote the subgraph of G induced by $V - S$. Two vertices u and v are *adjacent* if (u, v) is an edge of G. We use K_n to denote the complete graph with n vertices. The degree $deg_G(u)$ of a vertex u of G is the number of edges incident with u. The *minimum degree* of G, denoted by $\delta(G)$, is $\min\{\deg_G(x) \mid x \in V\}$. A *path* P is a sequence of vertices represented by $\langle v_0, \ldots, v_k \rangle$ with no repeated vertex and (v_i, v_{i+1}) is an edge of G for all $0 \leq i \leq k - 1$. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one.

A *hamiltonian cycle* of G is a cycle that traverses every vertex of G. A graph is *hamiltonian* if it has a hamiltonian cycle. Let G be a hamiltonian graph. A hamiltonian cycle C of G is described as $\langle v_1, v_2, \ldots, v_{n(G)}, v_1 \rangle$ to emphasize the order of vertices of C. Thus, v_1 is the beginning vertex and v_i is the i-th vertex in C. Two hamiltonian cycles of G beginning at u, $C_1 = \langle u_1, u_2, \ldots, u_{n(G)}, u_1 \rangle$ and $C_2 = \langle v_1, v_2, \ldots, v_{n(G)}, v_1 \rangle$ of G are *independent* if $u_1 = v_1 = u$, and $u_i \neq v_i$ for every $2 \leq i \leq n(G)$. A set of hamiltonian cycles $\{C_1, C_2, \ldots, C_k\}$ of G are *mutually independent* if they are pairwise independent. The *mutually independent hamiltonianicity* of graph G, $IHC(G)$, is the maximum integer k such that for any vertex u there are k-mutually independent hamiltonian cycles of G beginning at u.

The concept of mutually independent hamiltonian cycles arises from the following application: Suppose that there is a warehouse, daily products are to be shipped to several outlets by trucks each day. We may use a graph to serve as the model, where one node represents the warehouse, and the rest of the nodes represent the outlets. Suppose there are several different categories of products, and each truck is designed to carry only one category, for example, frozen food etc. Each truck loaded with the product from the warehouse has to travel through every outlet to supply the demand, and return to the warehouse. Due to the limited amount of equipment or space, each outlet can handle only one coming truck at a time to download the product. To avoid long line of truck waiting, we wish to find, starting from the warehouse, as many mutually independent hamiltonian cycles as possible, so that we may distribute the trucks to different route.

It is proved that $IHC(Q_n) = n$ if $n \geq 4$ and $IHC(Q_n) = n-1$ if $n \in \{1,2,3\}$ where Q_n is the n-dimensional hypercube [4]. Moreover, $IHC(P_n) = n-1$ if $n \geq 4$ and $IHC(P_n) = n-2$ if $n \in \{2,3\}$; and $IHC(S_n) = n-1$ if $n \geq 5$ and $IHC(S_n) = n-2$ if $n \in \{2,3,4\}$ where P_n is the n-dimensional pancake graph and S_n is the n-dimensional star graph [3].

In this paper, we study $IHC(G)$ for those graph with $\delta(G) \geq \frac{n(G)}{2}$. The motivation that we are interesting in these families of graphs is inspired by the classical Dirac's Theorem which states that those graphs G with at least three vertices and $\delta(G) \geq \frac{n(G)}{2}$ are hamiltonian.

In the following section, we prove that $IHC(G) \leq \delta(G)$ for any hamiltonian graph and $IHC(G) \geq 2\delta(G) - n(G) + 1$ if $\delta(G) \geq \frac{n(G)}{2}$. In section 3, we present some graphs that meet the bound mentioned above.

2 Bounds

Lemma 1. *Assume that G is a hamiltonian graph. Then $IHC(G) \leq \delta(G)$.*

Lemma 2. *Let x be a vertex of a graph G such that $\deg_G(x) \geq n(G)/2$ and $G - \{x\}$ is hamiltonian. Then there are $(2\deg_G(x) - n(G) + 1)$-mutually independent hamiltonian cycles of G beginning at x.*

Proof. Assume that $C = \langle v_1, v_2, \ldots, v_{n(G)-1}, v_1\rangle$ is a hamiltonian cycle of $G - \{x\}$. Suppose that $\deg_G(x) = n(G) - 1$. We set $C_i = \langle x, v_i, v_{i+1}, \ldots, v_{n(G)-1}, v_1, v_2, \ldots, v_{i-1}, x\rangle$ for every $1 \leq i \leq n(G) - 1$. Then $\{C_1, C_2, \ldots, C_{n(G)-1}\}$ forms a set of $(n(G)-1)$-mutually independent hamiltonian cycles of G beginning at x. Note that $n(G) - 1 = 2\deg_G(x) - n(G) + 1$.

Suppose that $\deg_G(x) \leq n(G) - 2$. Without loss of generality, we may assume that $(x, v_1) \in E(G)$ and $(x, v_{n(G)-1}) \notin E(G)$. Let $S = \{v_i \mid (x, v_i) \in E(G)$ and $(x, v_{i+1}) \in E(G)$ for $1 \leq i \leq n(G) - 2\}$, and let $H = \{v_i \mid (x, v_i) \in E(G)$ and $(x, v_{i+1}) \notin E(G)$ for $1 \leq i \leq n(G) - 2\}$. We have $|H| = \deg_G(x) - |S|$.

Suppose that $|S| \leq 2\deg_G(x) - n(G)$. Then we have

$$\begin{aligned} n(G) &\geq |S| + 2|H| + 1 \\ &= |S| + 2(\deg_G(x) - |S|) + 1 \\ &= 2\deg_G(x) - |S| + 1 \\ &\geq 2\deg_G(x) - (2\deg_G(x) - n(G)) + 1 \\ &= n(G) + 1. \end{aligned}$$

We obtain a contradiction. Thus, $|S| \geq 2\deg_G(x) - n(G) + 1$.

We set $C_i = \langle x, v_i, v_{i+1}, \ldots, v_{n(G)-1}, v_1, v_2, \ldots, v_{i-1}, x\rangle$ for every $v_i \in S$. Then $\{C_i \mid \text{for every } v_i \in S\}$ forms a set of $|S|$-mutually independent hamiltonian cycles of G beginning at x. Therefore, we obtain at least $2\deg_G(x) - n(G) + 1$ mutually independent hamiltonian cycles beginning at x. □

With Lemma 2, we have the following theorem.

Theorem 1. *Assume that* G *is a graph with* $\delta(G) \geq \frac{n(G)}{2}$. *Then* $IHC(G) \geq 2\delta(G) - n(G) + 1$.

Proof. Since $\delta(G) \geq \frac{n(G)}{2}$, $n(G) \geq 3$.

Case 1: $n(G) = 3$. Then $G = K_3$. Obviously, $IHC(G) = 2 = 2\delta(G) - n(G) + 1$.

Case 2: $n(G) = 2k$ for some positive integer k with $k \geq 2$.

Suppose that $\delta(G) = \frac{n(G)}{2}$. By Dirac's Theorem, G is hamiltonian. Thus, $IHC(G) \geq 1$.

Suppose that $\delta(G) \geq \frac{n(G)}{2} + 1$. We have $n(G) \geq 4$. Let x be an arbitrary vertex of G. Obviously, $\delta(G - \{x\}) \geq \delta(G) - 1 \geq \frac{n(G)}{2} > \frac{n(G-\{x\})}{2}$ and $n(G - \{x\}) = n(G) - 1 \geq 3$. By Dirac's Theorem, $G - \{x\}$ is hamiltonian. Then by Lemma 2, there exist $(2\deg_G(x) - n(G) + 1)$-mutually independent hamiltonian cycles of G beginning at x. Since $\deg_G(x) \geq \delta(G)$, $IHC(G) \geq 2\delta(G) - n(G) + 1$.

Case 3: $n(G) = 2k + 1$ for some positive integer k with $k \geq 2$.

Obviously, $n(G) \geq 5$. Let x be an arbitrary vertex of G. Obviously, $\delta(G - \{x\}) \geq \delta(G) - 1 \geq k = \frac{n(G-\{x\})}{2}$ and $n(G - \{x\}) = n(G) - 1 \geq 4$. By Dirac's Theorem, $G - \{x\}$ is hamiltonian. Then by Lemma 2, there exist $(2\deg_G(x) - n(G) + 1)$-mutually independent hamiltonian cycles of G beginning at x. Since $\deg_G(x) \geq \delta(G)$, $IHC(G) \geq 2\delta(G) - n(G) + 1$.

The theorem is proved. □

3 $IHC(G)$ for some graph G

In this section, we present some graphs that meet the bound mentioned above.

Theorem 2. *$IHC(K_n) = n - 1$ if $n \geq 3$.*

Proof. By Lemma 1, $IHC(K_n) \leq n-1$. By Theorem 1, $IHC(K_n) \geq n-1$. Thus, $IHC(K_n) = n - 1$. □

Theorem 3. *$IHC(G) = n(G)-3$ if G is a graph with $\delta(G) = n(G)-2 \geq 4$.*

Proof. By Theorem 1, $IHC(G) \geq n(G) - 3$. Thus, we only need to show $IHC(G) \leq n(G) - 3$.

Let x be any vertex of G with $\deg_G(x) = n(G)-2$. Let $\{C_1, C_2, \ldots, C_r\}$ be a set of r-mutually hamiltonian cycles beginning at x. We may write $C_i = \langle x = v_1^i, v_2^i, \ldots, v_{n(G)}^i, x\rangle$ for every $1 \leq i \leq r$. Since $\deg_G(x) = n(G) - 2$, there is exactly one vertex y with $(x, y) \notin E(G)$. Let i be any index with $1 \leq i \leq r$. Obviously, $y \notin \{v_1^i, v_2^i, v_{n(G)}^i\}$. Thus, $y = v_i^{i(y)}$ for some $i(y)$ with $3 \leq i(y) < n(G)$. Since $i(y) \neq j(y)$ for any $1 \leq i < j \leq r$, $r \leq n(G) - 3$. Thus, $IHC(G) \leq n(G) - 3$.

The theorem is proved. □

Let G and H be two graphs. We use G^c to denote the complement of G, use $G + H$ to denote the disjoint union of G and H, and use $G \vee H$ to denote the graph obtained from $G + H$ by joining each vertex of G to each vertex of H.

Let m be a positive integer. We use $H(m)$ to denote the graph $K_2^c \vee (K_m + K_m)$. We illustrate $H(4)$ in Figure 1.

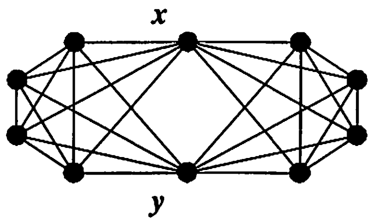

Figure 1: Illustration for H(4).

Theorem 4. *$IHC(H(m)) = 1$ for any positive integer m.*

Proof. Obviously, $n(H(m)) = 2m + 2$ and $\delta(H(m)) = m + 1 = \frac{n(H(m))}{2}$. By Theorem 1, $IHC(H(m)) \geq 1$.

Let x and y be the vertices in $H(m)$ corresponds to K_2^c. Let $C = \langle x = u_1, u_2, \ldots, u_{2m+2}, x \rangle$ be any hamiltonian cycle of $H(m)$ beginning at x. It is easy to see that $u_{m+2} = y$. Thus, beginning at x, there does not exist any other hamiltonian cycle of $H(m)$ independent with C. Thus, $IHC(H(m)) = 1$. □

References

[1] J. A. Bondy and U. S. R. Murty, Graph theory with applications, North Holland, New York, 1980.

[2] G. A. Dirac, Some Theorem on Abstract Graphs, Proceedings of the London Mathematical Society 2 (1952) 69–81.

[3] C.-K. Lin, H.-M. Huang, J. J. M. Tan, and L.-H. Hsu, Mutually Independent Hamiltonian Cycles of the Pancake Networks and the Star Networks, *Submitted.*

[4] C.-M. Sun, C.-K. Lin, H.-M. Huang, and L.-H. Hsu, Mutually Independent Hamiltonian Cycles in Hypercubes, Journal of Interconnection Networks 7 (2006) 235–255.