

Transposition H_V -Groups

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ABSTRACT. This paper introduces and studies transposition H_V -groups and join H_V -groups. The algebra of these H_V -groups is developed, some of their fundamental properties are presented, examples are constructed and a study of sub- H_V -groups is conducted. Moreover, the identity elements and their properties are examined. Through the notion of strong identity, the fortified transposition and the fortified join H_V -groups are introduced. Canonical and quasicanonical H_V -groups are also introduced, through the notion of scalar identity.

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1. Introduction

Hypercompositional structures are algebraic structures equipped with multivalued compositions, which are called *hyperoperations* or *hypercompositions*. Thus, (H, \cdot) is a *hypercompositional structure*, if H is a non-void set and « \cdot » is a function from $H \times H$ to the powerset $P(H)$ of H . Hypercompositional structures were introduced in algebra by F. Marty in 1934, during the 8th congress of the Scandinavian Mathematicians, through the notion of the *hypergroup* [9]. A hypergroup is a hypercompositional structure that satisfies the following axioms:

- i. $(ab)c = a(bc)$ for all $a, b, c \in H$ (associativity),
- ii. $aH = Ha = H$ for all $a \in H$ (reproduction).

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Note that, if « \cdot » is a hypercomposition in a set H and A, B are subsets of H , then $A \cdot B$ signifies the union $\bigcup_{(a,b) \in A \times B} a \cdot b$. In both cases, aA and Aa have the

same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member a . In [9] F. Marty also defined the two induced hypercompositions (right and left division) that derive from the hypercomposition of the hypergroup, i.e.

$$\frac{a}{|b} = \{x \in H : a \in xb\} \quad \text{and} \quad \frac{a}{|b} = \{x \in H : a \in bx\}.$$

In a hypergroup, the results of the hypercomposition, as well as the results of the two induced hypercompositions are non-void sets [15]. It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, W. Prenowitz denotes division in commutative hypergroups by a/b and, later on, J. Jantosciak used the notation a/b for right division and $b \backslash a$ for left division [6]. Notations $a : b$ and $a .. b$ have also been used for the above two types of divisions [15].

In the following years, hypergroups were not only enriched with further axioms (thus leading to a number of special hypergroups - e.g. [6, 7, 14, 24, 28]), but more hypercompositional structures appeared as well, which created several new branches in this algebraic theory (e.g. [1, 2, 8, 11, 12, 21, 22, 26]). In this manner, W. Prenowitz enriched hypergroups with a new axiom, in order to use them in the study of geometry. Specifically, W. Prenowitz introduced into the commutative hypergroup the transposition axiom:

$$ab \cap cd \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H$$

and named this new hypergroup *join space* [31]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

$$b \backslash a \cap c/d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H.$$

He named this particular hypergroup *transposition hypergroup* [6].

On the other hand, certain axioms were removed from the hypergroup and weaker structures were studied. Thus the pair (H, \cdot) , where H is a non-empty set and « \cdot » a hypercomposition, was named *partial hypgroupoid*, while it was called *hypgroupoid* if $ab \neq \emptyset$ for all $a, b \in H$. A hypgroupoid in which associativity is valid was called *semi-hypergroup*, while it was called *quasi-hypergroup* if only reproductivity holds. The quasi-hypergroups in which weak associativity is valid, i.e. $(ab)c \cap a(bc) \neq \emptyset$ for all $a, b, c \in H$, were named H_V -groups by T. Vougiouklis [33]. An H_V -group is called *H_V -commutative* if weak commutativity holds (i.e. $ab \cap ba \neq \emptyset$ for all $a, b \in H$), while it is called *commutative* or *abelian* H_V -group if commutativity holds. In direct correspondence to what holds true in hypergroups [15], in Propositions 3.1 and 3.2 it is proven that the results of the hypercomposition and of the two induced hypercompositions in H_V -groups are non-void sets as well.

An interesting occurrence of H_V -groups is the one resulting from the *complement hyperoperation* of a hypergroup. A. Iranmanesh and A. Babareza defined the complement hyperoperation in [5] as follows: If « \circ » is a hyperoperation in a set H , then the hyperoperation $a * b = H - a \circ b$ is called complement hyperoperation of « \circ ». The authors of [5] have elaborated this notion in the following hypergroup, which resulted from the constructions of non-quotient hyperfields over a set H with $\text{card } H > 3$, by Ch. Massouros [10, 11, 17]:

$$a \circ b = \{a, b\}, \quad \text{for all } a, b \in H \text{ with } a \neq b,$$

$$a \circ a = H - \{a\}, \quad \text{for all } a \in H.$$

The complement hyperoperation « $*$ » of « \circ » in H is the following:

$$a * b = H - \{a, b\}, \quad \text{for all } a, b \in H \text{ with } a \neq b,$$

$$a * a = \{a\}, \quad \text{for all } a \in H.$$

This hyperoperation is essentially the one used by A. Nakassis, in order to prove the existence of non-quotient hyperrings [30]. As it is proven in [5], $(H, *)$ is not a hypergroup. However, a thorough verification of the axioms proves that $(H, *)$ is an H_V -group. Such an H_V -group will be named *complement H_V -group of a hypergroup*. Conversely, the hypergroup will be named *complement hypergroup of an H_V -group*. One can easily observe that the complement of a hypergroup might even be a partial hyperegroupoid. This occurs when equality $xy = H$ is valid in a hypergroup for some $x, y \in H$ (e.g. see [16]).

In this paper, the transposition axiom is introduced in H_V -groups and transposition H_V -groups are studied. Thus:

Definition 1.1. An H_V -group (H, \cdot) is called *transposition H_V -group*, if it satisfies the transposition axiom:

$$b \setminus a \cap c / d \neq \emptyset \text{ implies } ad \cap bc \neq \emptyset \text{ for all } a, b, c, d \in H.$$

Clearly, if A, B, C, D are subsets of H , then $B \setminus A \cap C / D \neq \emptyset$ implies $AD \cap BC \neq \emptyset$. Hereafter, the relational notation $A \approx B$ (read as A meets B) is used to assert that sets A and B have a non-void intersection [6]. From weak commutativity and the definition of left and right division, it follows that:

Proposition 1.1. If H is a commutative H_V -group, then $a / b = b \setminus a$ for all $a, b \in H$.

Definition 1.2. A transposition H_V -group (H, \cdot) is called *join H_V -group* if H is a commutative H_V -group, while it is called *weak join H_V -group* if H is an H_V -commutative group.

2. Construction Methods of Transposition H_V -groups

Four construction methods of transposition H_V -groups will be presented in this section. It is obvious that, through these construction methods a vast

number of specialized examples come into being. The first of these methods will be utilized in some of the following paragraphs, in which more specific types of H_V -groups are defined.

Construction I.

Let (H, \cdot) be a hypergroup. An arbitrary subset I_{ab} of H is associated to each pair of elements $(a, b) \in H^2$. Next, a new hypercomposition «*» is introduced into H , which is defined as follows: $a*b = ab \cup I_{ab}$. Then:

Proposition 2.1. $(H, *)$ is an H_V -group.

Proof. Since xH and Hx are subsets of $x*H$ and $H*x$ respectively, it follows that $H = H*x = x*H$. Also, since the set abc is a subset of both $a*(b*c)$ and $(a*b)*c$, it follows that $a*(b*c) \approx (a*b)*c$.

Remark 2.1. If H is a hypergroup and $I_{ab} = \{a, b\}$, then, as proven in [15], $(H, *)$ is a hypergroup as well (see also [26]).

Definition 2.1. A hypercomposition which always contains the two participating elements x, y in its result xy , is called a **containing hypercomposition**. If $x \in xy$ for all $x, y \in H$, this hypercomposition is called **left-containing**, while it is called **right-containing** if $y \in xy$ for all $x, y \in H$.

Proposition 2.2. If H is a commutative hypergroup and $I_{ab} = I_{ba}$ for all $a, b \in H$, then $(H, *)$ is a commutative H_V -group.

Next, it is obvious that:

Proposition 2.3. If $\bigcap_{a, b \in H} I_{ab} \neq \emptyset$, then $(H, *)$ is a transposition H_V -group,

and

Proposition 2.4. If H is a commutative hypergroup and $I_{ab} = I_{ba}$ for all $a, b \in H$ and $\bigcap_{a, b \in H} I_{ab} \neq \emptyset$, then $(H, *)$ is a join H_V -group.

Corollary 2.1. If H is a hypergroup and w is an arbitrary element of H , then H endowed with the hypercomposition

$$x * y = xy \cup \{x, y, w\}$$

is a transposition H_V -group, while it is a join H_V -group if H is a commutative hypergroup.

Remark 2.2. If H is a group, it follows that every group produces transposition H_V -groups. On the other hand, all the above constructions can be easily extended, assuming that H is an H_V -group. Thus, transposition H_V -groups

can be constructed from already known H_V -groups (in [34], it is shown that any extension of the hyperoperation of a given H_V -group also defines an H_V -group).

The following three construction methods originate from geometric applications of H_V -groups and comprise generalizations of those presented by A. Dramalidis in [2, 3, 4]. The following lemmata will be proven before proceeding to these constructions (see also [1]):

Lemma 2.1. *A right- or left-containing hypercomposition is weakly associative.*

P r o o f. If the hypercomposition is right-containing, then $x \in x(yz) \cap (xy)z$. If it is left-containing, then $z \in x(yz) \cap (xy)z$.

Lemma 2.2. *A non-empty set endowed with a containing hypercomposition is an H_V -commutative group.*

P r o o f. Since the hypercomposition is containing, $\{x, y\} \subseteq xy \cap yx$. Thus, weak commutativity is in effect. Next, $xH = \bigcup_{y \in H} xy$. But $\bigcup_{y \in H} \{x, y\} \subseteq \bigcup_{y \in H} xy$. Hence $H = xH$. Similarly, $H = Hx$.

Construction II.

Let V be a real vector space. In V , a hypercomposition is defined as follows:

$$xy = \{x + \lambda y \mid \lambda \in [0, 1]\}.$$

Obviously, this is a left-containing hypercomposition and, according to Lemma 2.1, is weakly associative. Next, let z be any element in V . Then:

$$z = x + 1(z - x) \in \{x + \lambda(z - x) \mid \lambda \in [0, 1]\} \subseteq xV$$

and so $V = xV$ for all $x \in V$. On the other hand,

$$Vx = \bigcup_{y \in V} \{y + \lambda x \mid \lambda \in [0, 1]\}.$$

Hence, for $\lambda = 0$, any element of V belongs to Vx and therefore $V = Vx$. Thus, reproduction is valid. Moreover, since $x + y \in xy \cap yx$ for all $x, y \in V$, it follows that V is an H_V -commutative group. Finally, suppose that $b \setminus a \approx c / d$ is valid for a, b, c, d in V . Then:

$$b \setminus a = \{x \in V \mid a \in bx\} = \{x \in V \mid a = b + \lambda x, \lambda \in [0, 1]\}$$

and $c / d = \{y \in V \mid c \in yd\} = \{y \in V \mid c = y + \kappa d, \kappa \in [0, 1]\}.$

Since the above two sets have a non-void intersection, there exists $z \in V$ and $\kappa, \lambda \in [0, 1]$, such that $a = b + \lambda z$ and $c = z + \kappa d$. Therefore, $z = c - \kappa d$ and

so $a = b + \lambda(c - \kappa d)$. Hence, $a + \lambda\kappa d = b + \lambda c$. But $a + \lambda\kappa d \in ad$ and $b + \lambda c \in bc$, thus $ad \approx bc$. Therefore:

Proposition 2.5. *If V is a real vector space, then V endowed with the hypercomposition $xy = \{x + \lambda y \mid \lambda \in [0, 1]\}$ for all $x, y \in V$, is a weak join H_V -group.*

Remark 2.3. In [29], hypergroups are associated to vector spaces. There, it is proven that a vector space V endowed with the hypercompositions

$$xy = \{x + \lambda y, y + \lambda x \mid \lambda \in (0, 1)\} \text{ or } xy = \{x + \lambda y, y + \lambda x \mid \lambda \in [0, 1]\}$$

becomes a join hypergroup.

Construction III.

Let V be a real vector space and p a fixed point in V . V is endowed with the hypercomposition $xy = \{x + \lambda y + \kappa p \mid \lambda, \kappa \in [0, 1]\}$, which belongs to the family of P-hypercompositions [34]. Obviously, this is a left-containing hypercomposition and, according to Lemma 2.1, is weakly associative. Next, let z be any element in V . Then:

$$z = x + 1(z - x) + 0p \in \{x + \lambda(z - x) + \kappa p \mid \lambda, \kappa \in [0, 1]\} \subseteq xV$$

and so $V = xV$, for all $x \in V$. On the other hand:

$$Vx = \bigcup_{y \in V} \{y + \lambda x + \kappa p \mid \lambda, \kappa \in [0, 1]\}.$$

Hence, for $\lambda = \kappa = 0$, any element of V belongs to Vx . Therefore, $V = Vx$. Thus, reproduction is valid. Moreover, since, for example, $x + y$ and $x + y + p$ belong to $xy \cap yx$ for all $x, y \in V$, it follows that V is an H_V -commutative group. Finally, suppose that $b \setminus a \approx c / d$ holds for a, b, c, d in V . Then,

$$b \setminus a = \{x \in V \mid a \in bx\} = \{x \in V \mid a = b + \lambda x + \kappa p, \lambda, \kappa \in [0, 1]\}$$

$$\text{and } c / d = \{y \in V \mid c \in yd\} = \{y \in V \mid c = y + \mu d + \nu p, \mu, \nu \in [0, 1]\}.$$

Since the above two sets have a non-void intersection, there exists $z \in V$ such that $a = b + \lambda z + \kappa p$ and $c = z + \mu d + \nu p$. Therefore, $z = c - \mu d - \nu p$ and so $a = b + \lambda c - \lambda \mu d - \lambda \nu p + \kappa p$. Hence, $a + \lambda \mu d + \lambda \nu p = b + \lambda c + \kappa p$. But $a + \lambda \mu d + \lambda \nu p \in ad$ and $b + \lambda c + \kappa p \in bc$, thus $ad \approx bc$. Therefore:

Proposition 2.6. *If V is a real vector space and p a fixed point of V , then V endowed with the hypercomposition $xy = \{x + \lambda y + \kappa p \mid \lambda, \kappa \in [0, 1]\}$ for all $x, y \in V$ is a weak join H_V -group.*

Construction IV.

Let V be a real vector space. Then, for any $\lambda \in [0,1]$, a hypercomposition can be defined in V as follows:

$$x \circ y = \{z \mid z \in [x - \lambda y, x + \lambda y]\},$$

which is equivalent to

$$x \circ y = \{x + \mu \lambda y, \mu \in [-1,1]\}.$$

It is proven in [4] that the hyperstructure (V, \circ) is an H_V -group, which becomes H_V -commutative when $\lambda = 1$. Now, suppose that $b \setminus a \approx c / d$ is valid for a, b, c, d in V . Then:

$$b \setminus a = \{x \in V \mid a \in bx\} = \{x \in V \mid a = b + \mu \lambda x, \mu \in [-1,1]\}$$

$$\text{and } c / d = \{y \in V \mid c \in yd\} = \{x \in V \mid c = y + \rho \lambda d, \rho \in [-1,1]\}.$$

Since the above two sets have a non-void intersection, there exists $z \in V$ and $\mu, \rho \in [-1,1]$, such that $a = b + \mu \lambda z$ and $c = z + \rho \lambda d$. Therefore, $z = c - \rho \lambda d$. Thus, $a = b + \mu \lambda (c - \rho \lambda d)$, which implies that $a + \mu \rho \lambda^2 d = b + \mu \lambda c$. But $a + (\mu \rho \lambda) \lambda d \in a \circ d$ and $b + \mu \lambda c \in b \circ c$, thus $a \circ d \approx b \circ c$. Therefore:

Proposition 2.7. *The vector space V endowed with the hyperoperation $x \circ y = \{x + \mu y, \mu \in [-1,1]\}$ for all $x, y \in V$ is a weak join H_V -group.*

Proposition 2.7 and the analysis of the hyperoperation given above yields:

Proposition 2.8. *The vector space V endowed with the hyperoperation $x \circ y = \{x + \mu \lambda y, \mu \in [-1,1]\}$ for all $x, y \in V$, where λ is an arbitrary element in the interval $[0,1)$, is a transposition H_V -group.*

3. Algebraic Properties

In [6] and then in [7] a principle of duality is established in the theory of hypergroups. In accordance to what is valid in the theory of hypergroups, a principle of duality is also valid for the theory of H_V -groups. More precisely, two statements of the theory of hypergroups are dual statements, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition ab with the hypercomposition ba . One can observe that the transposition axiom, as well as the weak associativity axiom

are self-dual. The left and the right division have dual definitions, thus they must be interchanged in the construction of a dual statement. Therefore, the following principle of duality holds for the theory of H_V -groups and for the theory of transposition H_V -groups:

Given a theorem, the dual statement resulting from interchanging the order of hypercomposition “.” (and, necessarily, interchanging of the left and the right divisions), is also a theorem.

Proposition 3.1. *The result of hypercomposition in an H_V -group H is always a non-empty set.*

Proof. Suppose that $ab = \emptyset$ for some $a, b \in H$. Then, $(ab)c = \emptyset$ for any $c \in H$. Therefore, $(ab)c \cap a(bc) = \emptyset$, which is absurd. Hence, ab is non-void.

Proposition 3.2. *The reproduction of hypercomposition in H_V -groups is equivalent to the non-void results of induced hypercompositions.*

Proof. Suppose that $x/a \neq \emptyset$ for all $x, a \in H$. Thus, there exists $y \in H$, such that $x \in ya$. Therefore, $x \in Ha$ for all $x \in H$; thus, $H \subseteq Ha$. Next, since $Ha \subseteq H$ for all $a \in H$, it follows that $H = Ha$. Conversely now, from equality $H = Ha$, it follows that, for every $x \in H$ there exists $y \in H$, such that $x \in ya$. Thus, $y \in x/a$ and, therefore, $x/a \neq \emptyset$. The proof that $H = aH$ for all $a \in H$ is equivalent to the non-void result of left division follows from the principle of duality.

Proposition 3.3. *In any H_V -group H , equalities (i) $H = H/a = a/H$ and (ii) $H = a \setminus H = H \setminus a$ are valid for all a in H .*

Proof. (i) Per Proposition 3.1, the result of hypercomposition in H is always a non-empty set. Thus, for every $x \in H$ there exists $y \in H$ such that $y \in xa$, which implies that $x \in y/a$. Hence, $H \subseteq H/a$. Moreover, $H/a \subseteq H$. Therefore, $H = H/a$. Next, let $x \in H$. Since $H = xH$, there exists $y \in H$ such that $a \in xy$, which implies that $x \in a/y$. Hence, $H \subseteq a/H$. Moreover, $a/H \subseteq H$. Therefore, $H = a/H$. (ii) follows by duality.

Proposition 3.4. *Let a, b be elements of an H_V -group H ; then:*

(i) $b \in (a/b) \setminus a$ and (ii) $b \in a/(b \setminus a)$.

Proof. (i) Let $x \in a/b$. Then $a \in xb$. Hence, $b \in x \setminus a$. Thus, $b \in (a/b) \setminus a$. Therefore, (i) is valid. (ii) is the dual of (i).

Corollary 3.1. *If A, B are non-empty subsets of a H_V -group H , then:*

(i) $B \subseteq (A/B) \setminus A$ and (ii) $B \subseteq A/(B \setminus A)$.

Proposition 3.5. Let a, b, c be elements of an H_V -group H ; then:

- i. $(b \setminus a) / c \approx b \setminus (a / c)$,
- ii. $(a / b) / c \approx a / cb$,
- iii. $c \setminus (b \setminus a) \approx bc \setminus a$.

P r o o f. For (i) it holds true that: $(b \setminus a) / c = \{x \in H \mid b \setminus a \approx xc\} = \{x \in H \mid a \in b(xc)\} \approx \{x \in H \mid a \in (bx)c\} = \{x \in H \mid a / c \approx bx\} = b \setminus (a / c)$.

Regarding (ii): $(a / b) / c = \{x \in H \mid a / b \approx xc\} = \{x \in H \mid a \in (xc)b\} \approx \{x \in H \mid a \in x(cb)\} = a / cb$.

Finally, (iii) is the dual of (ii).

Corollary 3.2. If A, B, C are non-empty subsets of a H_V -group H , then:

- i. $(B \setminus A) / C \approx B \setminus (A / C)$,
- ii. $(A / B) / C \approx A / CB$,
- iii. $C \setminus (B \setminus A) \approx BC \setminus A$.

Proposition 3.6. Let a be an arbitrary element of an H_V -group H .

Then:

- i. $H = H / a = a / H$ and
- ii. $H = a \setminus H = H \setminus a$.

P r o o f. (i) Obviously, $H / a \subseteq H$. Now, let $x \in H$; then $xa \subseteq H$ and so $x \in H / a$. Hence, $H \subseteq H / a$ and, therefore, $H = H / a$. Similarly, it can be shown that $H = a / H$. Thus, (i) is valid. (ii) is the dual of (i).

Proposition 3.7. Let a, b, c be arbitrary elements of a transposition H_V -group H . Then:

- i. $a(b / c) \cup a / (c / b) \subseteq (ab) / c$ and
- ii. $(c \setminus b) a \cup (b \setminus c) \setminus a \subseteq c \setminus (ba)$.

P r o o f. (i) Let $x \in a(b / c)$. Then, there exists $y \in b / c$, such that $x \in ay$ or $y \in a \setminus x$. Thus, $b / c \approx a \setminus x$ (1). Next, if $x \in a / (c / b)$, there exists $y \in c / b$, such that $x \in a / y$. Therefore, $a \in xy$ or $y \in x \setminus a$. Thus, $c / b \approx x \setminus a$ (2). From both (1) and (2) it follows that $xc \approx ab$. So, there exists $w \in ab$, such that $w \in xc$, which implies that $x \in w / c$. Therefore, $x \in (ab) / c$. Hence, (i) is valid. (ii) is the dual of (i).

Corollary 3.3. If A, B, C are non-void subsets of a transposition H_V -group H , then:

- i. $A(B / C) \cup A / (C / B) \subseteq (AB) / C$ and
- ii. $(C \setminus B) A \cup (B / C) A \subseteq C \setminus BA$.

Proposition 3.8. Let a, b, c, d be arbitrary elements of a transposition H_V -group H . Then:

- i. $(b \setminus a)(c / d) \subseteq (b \setminus ac) / d \cap b \setminus (ac / d)$,
- ii. $(b \setminus a) / (d / c) \subseteq (b \setminus ac) / d$,
- iii. $(c \setminus d) \setminus (a / b) \subseteq d \setminus (ca / b)$.

P r o o f. (i) Let $x \in (b \setminus a)(c / d)$. Then there exists $y \in b \setminus a$, such that $x \in y(c / d) \subseteq (yc) / d$ [Proposition 3.7.i]. Thus, $xd \approx yc$, or $xd \approx (b \setminus a)c \subseteq b \setminus ac$

[Proposition 3.7.ii]. Hence, $x \in (b \setminus ac)/d$. Next, since $x \in (b \setminus a)(c/d)$, there exists $z \in c/d$, such that $x \in (b \setminus a)z \subseteq b \setminus (az)$ [Proposition 3.7.ii]. Thus, $bx \approx az$ or $bx \approx a(c/d)c \subseteq ac/d$ [Proposition 3.7.i]. Therefore, $x \in b \setminus (ac/d)$.

(ii) Suppose that $x \in (b \setminus a)/(d/c)$. Then there exists $y \in b \setminus a$ such that $x \in y/(d/c) \subseteq (yc)/d$ [Proposition 3.7.i]. Thus, $xd \approx yc$ or $xd \approx (b \setminus a)c \subseteq b \setminus ac$ [Proposition 3.7.ii]. Hence, $x \in (b \setminus ac)/d$.

Working in a similar manner, we can prove (iii).

Corollary 3.4. *If A, B, C, D are non-void subsets of a transposition H_V -group H , then:*

i. $(B \setminus A)(C/D) \subseteq (B \setminus AC)/D \cap B \setminus (AC/D),$

ii. $(B \setminus A)/(D/C) \subseteq (B \setminus AC)/D,$

iii. $(C \setminus D) \setminus (A/B) \subseteq D \setminus (CA/B).$

4. Identities and Fortification

Let H be a H_V -group. An element e is called *right identity* if $x \in x \cdot e$ for all x in H . If $x \in e \cdot x$ for all x in H , then x is called *left identity*, while it is called *identity*, if it is both right and left identity. If $x = x \cdot e = e \cdot x$ for all x in H , then e is called *scalar identity*. When a scalar identity exists in H , then it is unique. An identity e is called *strong identity*, if $x \in x \cdot e = e \cdot x \subseteq \{e, x\}$ for all x in H [24]. The strong identity need not be unique [7]. An element x' is called *right inverse* of x , if a right identity $e \neq x'$ exists, such that $e \in x \cdot x'$. The definition of the *left inverse* is analogous, while x' is called *inverse* of x , if it is both right and left inverse.

Proposition 4.1. *If e is a strong or scalar identity in an H_V -group H and x is an element in H , distinct from e , then $x/e = e \setminus x = x$.*

Proof. Let e be a strong identity. Then, $y \in x/e$ implies that $x \in ye \subseteq \{y, e\}$. Since $x \neq e$, it follows that $y = x$. Thus, $x/e = x$ is true. $e \setminus x = x$ follows by duality. The proof is similar when e is scalar.

An H_V -group H with a strong identity e has a natural partition. Let $A = \{x \in H : ex = xe = \{e, x\}\}$ and $C = \{x \in H - \{e\} : ex = xe = x\}$.

Then $H = A \cup C$ and $A \cap C = \emptyset$. The elements of A are called *attractive elements* and the elements of C are called *canonical elements* (see [25] for the origin of the terminology).

Proposition 4.2. *If e is a strong identity, then $A = e/e = e \setminus e$.*

Proof. $x \in A$ is equivalent to $x \in xe$. Therefore, $x \in e/e$. By duality, $A = e \setminus e$.

Proposition 4.3. *In a transposition H_V -group with a scalar identity e , each element has a unique inverse.*

P r o o f. Let x be an arbitrary element of a transposition H_V -group H . Then, because of reproduction, there exists x' and x'' such that $e \approx x'x$ and $e \approx xx''$. Then, $x \in x' \setminus e$ and $x \in e / x''$. Hence, $x' \setminus e \approx e / x''$. Because of transposition $x'e \approx ex''$ and since e is scalar, it follows that $x' = x''$. Therefore, x has a unique inverse.

Definition 4.1. *A transposition H_V -group (H, \cdot) is called **fortified**, if H contains an element e , which satisfies the axioms:*

- i. $ee = e$,
- ii. $x \in ex = xe$ for all $x \in H$,
- iii. for every $x \in H - \{e\}$ there exists a unique $y \in H - \{e\}$, such that $e \in xy$ and, furthermore, y satisfies $e \in yx$.

If " \cdot " is commutative, then H is called a **fortified join H_V -group**.

In the above defined H_V -group, for $x \in H - \{e\}$, notation x^{-1} is used to signify the unique y that satisfies axiom (iii). Clearly, $(x^{-1})^{-1} = x$.

In [7, 25], a transposition hypergroup is constructed from a quasicanonical hypergroup G by introducing into G the hypercomposition $x \cdot y = xy \cup \{x, y\}$ for all $x, y \in G$. For simplicity, assume that G is a group with a neutral element e . We then apply Construction I to G , setting $I_{xy} = \{x, y\}$, if $y \neq x^{-1}$, $I_{xx^{-1}} = G - \{e, x, x^{-1}\}$ for all $x \neq e$ and $I_{ee} = \emptyset$. Thus:

Proposition 4.4. *Let G be a group with card $G > 3$ and with a neutral element e . Then, G can be endowed with a fortified transposition H_V -group structure, if one defines a hypercomposition as follows:*

$$\begin{aligned} x \cdot y &= xy \cup \{x, y\}, \text{ if } y \neq x^{-1}, \\ x \cdot x^{-1} &= x^{-1}x = G - \{x, x^{-1}\}, \text{ for all } x \neq e, \\ e \cdot e &= e. \end{aligned}$$

P r o o f. The hypercomposition is weakly associative. Indeed, for example, if $x, y \neq e$, then $y \cdot (x \cdot x^{-1}) = G$, while $(y \cdot x) \cdot x^{-1} = G - \{x\}$. Also, if $y = e$, then $e \cdot (x \cdot x^{-1}) = G - \{x, x^{-1}\}$, while $(e \cdot x) \cdot x^{-1} = G - \{x\}$. Next, for the induced hypercompositions, the following are valid:

$$\begin{aligned} x / y &= \{z \mid x \in z \cdot y\} = \{x, xy^{-1}, y^{-1}\}, \text{ if } y \neq x, x^{-1}, \\ y \setminus x &= \{z \mid x \in y \cdot z\} = \{x, y^{-1}x, y^{-1}\}, \text{ if } y \neq x, x^{-1}, \\ x / x &= x \setminus x = G - \{x^{-1}\}, \end{aligned}$$

$$\begin{aligned}
x / x^{-1} &= x^{-1} \setminus x = x^2, \\
x^{-1} / x &= x \setminus x^{-1} = x^{-2}, \\
e / e &= e \setminus e = G.
\end{aligned}$$

Based on the above, the verification of the transposition axiom is quite straightforward, albeit long.

Corollary 4.1. (G, \cdot) is a join H_V -group, if G is abelian.

Corollary 4.2. (G, \cdot) consists only of attractive elements.

In [22] and then in [24, example 2.1], a fortified join hypergroup is constructed via the use of another fortified join hypergroup and of an external element w . This construction can be also applied in the case of transposition H_V -groups. Thus:

Proposition 4.5. Let (E, \cdot) be a transposition H_V -group and w an element not in E . Then, the set $H = E \cup \{w\}$ can be endowed with a transposition H_V -structure, if one defines a hypercomposition as follows:

$$\begin{aligned}
x \circ y &= x \cdot y, \text{ for all } x, y \in E, \\
x \circ w &= w \circ x = w, \text{ for all } x \in E, \\
w \circ w &= H.
\end{aligned}$$

Remark 4.1. If (E, \cdot) is a transposition H_V -group such as those resulting from Proposition 4.4, then all the elements of E are attractive, while w is a canonical element.

Research Remark. The notion of the *fortified join hypergroup* first appeared in the study of formal languages and automata via hypercompositional structure tools by G. G. Massouros [22]. Fortified join hypergroups were subsequently studied in a series of papers [e.g. 18, 24, 25] and were later generalized via the removal of the commutativity axiom, thus resulting into the generation of the *fortified transposition hypergroups* [7]. In addition to *fortified transposition hypergroups*, other types of hypergroups emerged through the study of automata; examples are *transposition polysymmetrical hypergroups* [22, 27], and *transposition polysymmetrical hypergroups with strong identity* [22, 20]. In the meantime, hypergroup theory states that a transposition hypergroup is a *quasicanonical* one [14], if it has a scalar identity [6, 7, 19] and that a join space is a *canonical hypergroup* [19, 27], if it has a scalar identity [13, 19]. This paper opens a gate into a thus far unexplored area of hypercompositional algebra, one which involves the study of the above structures when associativity is weak, i.e the study of structures such as *transposition polysymmetrical H_V -groups*, *transposition polysymmetrical H_V -groups with strong identity*, *quasicanonical H_V -groups*, *canonical H_V -groups*, etc. This area can be expanded even further, if one also includes generalizations of canonical hypergroups in which the transposition axiom is not valid. A

characteristic example of the above is the case of *canonical polysymmetrical hypergroups* [27, Example 1.3], from which *canonical polysymmetrical H_V -groups* can potentially be derived.

Proposition 4.6. *Let H be a fortified transposition H_V -group and $x \in H - \{e\}$. Then, $e \in xy$ or $e \in yx$ implies that $y \in \{x^{-1}, e\}$.*

Also, per Proposition 4.3:

Proposition 4.7. *In a fortified transposition H_V -group H , the identity is strong.*

Proof. It must be proven that $ex \subseteq \{e, x\}$ for all x in H . This is true for $x = e$. Let $x \neq e$. Suppose that $y \in ex$. Then, $x \in e \setminus y$. However, $x \in e / x^{-1}$, since $e \in xx^{-1}$. Thus, $e \setminus y \approx e / x^{-1}$ and transposition yields $e = ee \approx yx^{-1}$. Hence, per Proposition 4.6, $y \in \{x, e\}$.

Proposition 4.8. *In a fortified transposition H_V -group H , the strong identity is unique.*

Proof. Suppose that u is an identity distinct from e . It then follows that there exists z distinct from u , such that $u \in ez$. But, $ez \subseteq \{e, z\}$, so $u \in \{e, z\}$, which is a contradiction.

The following now become clear:

Proposition 4.9. *In a fortified transposition H_V -group, if $x \neq e$, then:*

$$e / x = ex^{-1} = \{e, x^{-1}\} = x^{-1}e = x \setminus e.$$

Proposition 4.10. *In a fortified transposition H_V -group, if $x \neq y$:*

$$xy^{-1} = x / y \cup \{y^{-1}\} \text{ if } y^{-1} \notin xy^{-1} \text{ and } xy^{-1} = x / y \text{ if } y^{-1} \in xy^{-1}.$$

Proof. If $x = e$, then $xy = ey^{-1} = \{e, y^{-1}\}$. Per Proposition 4.9, $e / y = \{e, y^{-1}\}$. Hence, $ey^{-1} = e / y$. If $y = e$, then $xy^{-1} = xe = \{e, x\}$. Per Proposition 4.1, $x / e = x$. Thus, $xe = x / e \cup \{e\}$. Now, suppose that $x \neq e$ and that $y \neq e$. Then, $x \neq y$ implies $e \notin xy^{-1}$. Per 4.9, $y^{-1} \in e / y$. Thus, $xy^{-1} \subseteq x(e / y)$. Per Proposition 3.7.i, $x(e / y) \subseteq (xe) / y$. Thus, $xy^{-1} \subseteq (xe) / y = \{x, e\} / y = x / y \cup e / y = x / y \cup \{e, y^{-1}\}$. Since $e \notin xy^{-1}$, it follows that $xy^{-1} \subseteq x / y \cup \{y^{-1}\}$. On the other hand, $x / y \subseteq x / (e / y^{-1}) \subseteq xy^{-1} / e = xy^{-1}$. Therefore, if $y^{-1} \in xy^{-1}$, then

$xy^{-1} = x / y \cup \{y^{-1}\}$ is true. If, on the other hand, $y^{-1} \notin xy^{-1}$, then $xy^{-1} = x / y$ is true.

Per duality:

Proposition 4.11. *In a fortified transposition H_V -group, if $x \neq y$:*

$y^{-1}x = y \setminus x \cup \{y^{-1}\}$ if $y^{-1} \notin y^{-1}x$ and $y^{-1}x = y \setminus x$ if $y^{-1} \in y^{-1}x$.

5. Sub- H_V -groups

Let H be an H_V -group. A semisub- H_V -group of H is a non-void subset h of H , such that $ab \subseteq h$ for each $a, b \in h$. h is a sub- H_V -group, if and only if $ah = ha = h$ for each $a \in h$. It is obvious that the intersection of two semisub- H_V -groups or sub- H_V -groups of H can be the empty set. Also, when the intersection of two sub- H_V -groups of H is non-void, then this intersection is not always a sub- H_V -group of H . However, it is always a semisub- H_V -group of H . A subset k of H is a right closed (respectively a left closed) subset, if $a, b \in k$ implies $a/b \subseteq k$ (respectively $b \setminus a \subseteq k$). k is called closed, if it is closed both from the right and from the left.

Proposition 5.1. *If h is both a semisub- H_V -group of an H_V -group H and a closed subset of H , then it is a sub- H_V -group of H .*

Proof. The validity of the reproduction axiom must be proven. So, let x be an element of h . Since h is a semisub- H_V -group of H , inclusions $xh \subseteq h$ and $hx \subseteq h$ are valid. Next, let y be an arbitrary element of h . Then, $x \setminus y = \{t \in H \mid y \in xt\}$ is a subset of h . Hence, there exists $t \in H$, such that $y \in xt \subseteq xh$. Thus, $h \subseteq xh$. Per duality, $hx \subseteq h$. Therefore, $xh = hx = h$, QED.

Following the Krasner-Mittas terminology for subhypergroups [23, 28], the following definitions are established for sub- H_V -groups:

Definition 5.1. *A sub- H_V -group h of a H_V -group H is called closed from the right in H (respectively from the left), if $ah \cap h = \emptyset$ for each $a \in H-h$ (respectively $ha \cap h = \emptyset$). h is called closed, if it is closed both from the right and from the left.*

Definition 5.2. *A sub- H_V -group h of a H_V -group H is called invertible from the right in H (respectively from the left), if $ah \cap a'h = \emptyset$ for each $a, a' \in H$ with $ah \neq a'h$ (respectively $ha \neq ha'$ implies that $ha \cap ha' = \emptyset$). h is called invertible, if it is invertible both from the right and the left.*

Examples of such sub- H_V -group may emerge by utilizing Construction Method I. Indeed, one can easily see that Construction Method I, via the proper definition of set I_{ab} , transforms closed or invertible subhypergroups to closed or invertible sub- H_V -group. For example, if h is a closed or invertible subhypergroup, then we choose $I_{ab} \subseteq h$ for all $a, b \in h$.

Proposition 5.2. *A sub- H_V -group h of an H_V -group H is closed from the right in H (respectively from the left), if and only if the non-void intersection $ah \cap h$, implies that $a \in h$, $a \in H$ (respectively $ha \cap h \neq \emptyset$).*

Proof. If h is closed from the right and if $ah \cap h \neq \emptyset$, then $a \in H-h$. Thus, $a \in h$. Conversely now, if $a \in H-h$, then $a \notin h$ and thus $ah \cap h = \emptyset$. Therefore, h is closed from the right.

Proposition 5.3. *Let a be an arbitrary element of a closed sub- H_V -group k of any H_V -group H . Then:*

- i. $k = k/a = a/k$ and
- ii. $k = a \setminus k = k \setminus a$.

Proof. (i) Obviously $k/a \subseteq k$. Now, let $x \in k$; then, $xa \subseteq k$ and so $x \in k/a$. Hence, $k \subseteq k/a$ and, therefore, $k = k/a$. Similarly, it can be shown that $k = a/k$. Thus, (i) is valid. (ii) is the dual of (i).

The following are true in a manner analogous to what has been proven in hypergroups for the closed sub-hypergroups (see [13, 15]):

Proposition 5.4. *Let h be a sub- H_V -group of any H_V -group H and suppose that $a/b \subseteq h$ (respectively $b \setminus a \subseteq h$) for each $a, b \in h$. Then, h is a right closed sub- H_V -group (respectively left closed).*

Proof. Suppose that $xh \cap h \neq \emptyset$ for some $x \in H$. Then, there are $a, b \in h$ such that $a \in xb$; hence, $x \in a/b$. Since $a/b \subseteq h$, it follows that $x \in h$; QED.

Proposition 5.5. *In every right closed (respectively left closed) sub- H_V -group h of any H_V -group H , $a/b \subseteq h$ is valid (respectively $b \setminus a \subseteq h$) for each $a, b \in h$.*

Proof. Suppose that h is right closed in H and $a, b \in h$. Then, $x \in a/b$ yields $a \in xb$, which implies that $h \cap xh \neq \emptyset$. Thus, since h is right closed, $x \in h$. Therefore, $a/b \subseteq h$.

Per Propositions 5.3 and 5.4:

Proposition 5.6. *A sub- H_V -group of a H_V -group is closed, if and only if $b \setminus a \subseteq h$ and $a/b \subseteq h$ for each $a, b \in h$.*

Next, the following are valid for the semisub- H_V -groups and the closed sub- H_V -group of an H_V -group:

Proposition 5.7. *Let h_1 and h_2 be two semisub- H_V -groups (respectively closed sub- H_V -group) of a H_V -group of a H_V -group H . If $h_1 \cap h_2$ is non-void, then it is a semisub- H_V -group (respectively closed sub- H_V -group) of H .*

Proof. Let $x, y \in h_1 \cap h_2$; then, xy is a subset of both h_1 and h_2 . Thus, $xy \in h_1 \cap h_2$ and, therefore, $h_1 \cap h_2$ is a semisub- H_V -group of H . Now, if h_1 and

h_2 are closed sub- H_V -groups of H , then x/y and $y \setminus x$ are also subsets of $h_1 \cap h_2$. Therefore, per Proposition 5.1, $h_1 \cap h_2$ is a closed sub- H_V -group of H .

Corollary 5.1. *The set of the semisub- H_V -groups (respectively closed sub- H_V -group), which contains a non-void subset E of H , is a complete lattice.*

Proposition 5.8. *If h is a sub- H_V -group invertible from the right in an H_V -group H , then:*

- i. $x \in xh$, for every $x \in H$ and
- ii. $x \in yh$ implies that $xh = yh$ and that $y \in xh$.

P r o o f. (i) Since $H = Hh$, there exists an element z in H such that $x \in zh$. Therefore, $xh \subseteq zhh = zh$. Thus, $xh \cap zh \neq \emptyset$ and, since h is invertible, it follows that $xh = zh$. Therefore, x belongs to xh .

(ii) $x \in yh$ implies that $xh \subseteq yhh = yh$. Thus, $xh \cap yh \neq \emptyset$ and, since h is invertible, it follows that $xh = yh$. Next, from (i) it follows that $y \in yh$ and, since $yh = xh$, it follows that $y \in xh$.

Remark 5.1. Duality gives analogous results for invertible from the left sub- H_V -groups in a H_V -group.

Corollary 5.2. *If h is a sub- H_V -group invertible from the right (respectively from the left) in an H_V -group H , then h is closed from the right (respectively from the left) in H .*

Proposition 5.9. *If h is a sub- H_V -group in an H_V -group H , then:*

- i. h is invertible from the right, if and only if $y \setminus x \approx h$ implies that $x \setminus y \approx h$,
- ii. h is invertible from the left, if and only if $x/y \approx h$ implies that $y/x \approx h$.

P r o o f. (i) Let h be invertible from the right in H and let $y \setminus x \approx h$. Then, $x \in yh$. Thus, per Proposition 5.8.ii, $y \in xh$; therefore $x \setminus y \approx h$. Conversely, suppose that $xh \approx yh$ for some $x, y \in H$. Let $z \in xh \cap yh$. Then, $zh \subseteq xh \cap yh$ is valid. Next, $z \in xh$ yields $x \setminus z \approx h$, which (because of the supposition) implies that $z \setminus x \approx h$; therefore, $x \in zh$. Hence, $xh \subseteq zh$ and, consequently, $zh = xh$. Similarly, $zh = yh$ and, therefore, $xh = yh$. (ii) follows from the principle of duality.

Appendix

In [32], a Mathematica package is developed, which utilize the two basic properties, associativity and reproductivity, as the means of testing whether a hypergroupoid is a hypergroup or not. In this package, if we substitute the following lines in the `AssociativityTest[]`

```
test = Union[Flatten[Union[Extract[LookupTable1,
Distribute[{LookupTable1[[i, j]], {k}], List]]]] ==
Union[Flatten[UnionExtract[LookupTable1,
Distribute[{{i}, LookupTable1[[j, k]]], List]]]];
```

with:

```
test = Intersection[Union[Flatten[Union[Extract[LookupTable1,
Distribute[{LookupTable1[[i, j]], {k}], List]]]],
Union[Flatten[Union[Extract[LookupTable1,
Distribute[{i}, LookupTable1[[j, k]], List]]]]] != { }
```

then weak associativity is checked and H_V -groups emerge, thus supplying examples of structures such as the ones dealt with in this paper. The enumeration of these structures is the subject of a forthcoming paper.

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