$L(j,k)$-Labelings and $L(j,k)$-Edge-Labelings of Graphs *

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Abstract

Let j and k be two positive integers. An $L(j,k)$-labeling of a graph G is an assignment of nonnegative integers to the vertices of G such that the difference between labels of any two adjacent vertices is at least j, and the difference between labels of any two vertices that are at distance two apart is at least k. The minimum range of labels over all $L(j,k)$-labelings of a graph G is called the $\lambda_{j,k}$-number of G, denoted by $\lambda_{j,k}(G)$. Similarly, we can define $L(j,k)$-edge-labeling and $L(j,k)$-edge-labeling number, $\lambda'_{j,k}(G)$, of a graph G. In this paper, we show that if G is $K_{1,3}$-free with maximum degree Δ then $\lambda_{j,k}(G) \leq k\lfloor\Delta^2/2\rfloor + j\Delta - 1$ except that G is a 5-cycle and $j = k$. Consequently we obtain an upper bound for $\lambda'_{j,k}(G)$ in terms of the maximum degree of $L(G)$, where $L(G)$ is the line graph of G. This improves the upper bounds for $\lambda'_{2,1}(G)$ and $\lambda'_{1,1}(G)$ given by Georges and Mauro [Ars Combinatoria 70 (2004), 109-128]. As a corollary we show that Griggs and Yeh' conjecture that $\lambda_{2,1}(G) \leq \Delta^2$ holds for all $K_{1,3}$-free graphs and hence holds for all line graphs. We also investigate the upper bound for $\lambda'_{j,k}(G)$ for $K_{1,3}$-free graphs G.

Keywords: $L(j,k)$-labeling, $L(j,k)$-edge-labeling, $K_{1,3}$-free graph, Line graph.

1 Introduction

Let j and k be two positive integers. An $L(j,k)$-*labeling* of a graph G is an integer assignment f to the vertices of G such that if $uv \in E(G)$ then $|f(u) - f(v)| \geq j$; and if $d(u,v) = 2$ then $|f(u) - f(v)| \geq k$. Elements of the image of f are called labels. The *span* of f, $s(f)$, is the difference

*Project 10671033 supported by NSFC, Project XJ0607230 supported by Southeast University Science Foundation.

between the maximum and minimum labels used by f. The minimum span taken over all $L(j,k)$-labelings of a graph G, denoted by $\lambda_{j,k}(G)$, is called the *$L(j,k)$-labeling number* of G. An $L(j,k)$-labeling of a graph G with minimum span is called a *$\lambda_{j,k}$-labeling.*

Motivated from the channel assignment problem introduced by Hale [12], Griggs and Yeh [11] first proposed and studied the $L(2,1)$-labeling of a graph. Since then the $\lambda_{2,1}$-numbers of graphs have been studied extensively, see [2,6,9,11,20,21]. And $L(j,k)$-labelings were also investigated in many papers, see [5–8,16].

We say that two edges e_1 and e_2 are adjacent (at distance 1) if and only if they share a common end vertex. Two edges e_1 and e_2 are at distance 2 if and only if they are not adjacent and there exists an edge adjacent to both e_1 and e_2. Analogous to the above definition of $L(j,k)$-labeling, an *$L(j,k)$-edge-labeling* of a graph G is a function f from $E(G)$ to nonnegative integers such that $|f(e_1)-f(e_2)| \geq j$ if e_1 and e_2 are adjacent, and $|f(e_1)-f(e_2)| \geq k$ if e_1 and e_2 are at distance 2. The *$L(j,k)$-edge-labeling number* of a graph G, denoted by $\lambda'_{j,k}(G)$, is the minimum span over all $L(j,k)$-edge-labelings of G. Without loss of generality, we may assume that the minimum label used by an $L(j,k)$-edge-labeling of a graph G is always 0.

From the above definitions, it is easy to see that an $L(j,k)$-edge-labeling of a graph G corresponds to an $L(j,k)$-labeling of $L(G)$, the line graph of G. Thus $\lambda'_{j,k}(G) = \lambda_{j,k}(L(G))$. The $L(j,k)$-edge-labeling of a graph was studied by Georges and Mauro in [10].

Let $\Delta(G)$ and $\Delta_L(G)$ denote the maximum degree of G and $L(G)$, respectively. Among others, Georges and Mauro proved the following results.

Theorem 1.1 *[10] Suppose G is a graph with maximum degree $\Delta \geq 1$. Let Δ_L be the maximum degree of the line graph $L(G)$. Then*

$$2(\Delta-1) \leq \lambda'_{2,1}(G) \leq \Delta_L(\Delta+2) \leq 2\Delta^2+2\Delta-4.$$

Furthermore, if G is Δ-regular, then $2\Delta \leq \lambda'_{2,1}(G) \leq \frac{\Delta_L^2}{2} + 3\Delta_L$.

Theorem 1.2 *[10] Let G be a graph with maximum degree $\Delta \geq 1$. Let Δ_L be the maximum degree of the line graph $L(G)$. Then*

$$\Delta_L \leq \lambda'_{1,1}(G) \leq \Delta_L\Delta \leq 2\Delta^2-2\Delta.$$

Furthermore, if G is Δ-regular, then $\Delta_L \leq \lambda'_{1,1}(G) \leq \frac{\Delta_L^2}{2} + \Delta_L$.

Corollary 1.3 *[10] If H is a graph such that $H = L(G)$ for some graph G with $\delta(G) \geq 4$, then $\lambda_{2,1}(H) \leq \Delta^2(H)$.*

Theorem 1.4 *[10] For $n \geq 2$, $\lambda'_{2,1}(K_2) = 0$, $\lambda'_{2,1}(K_3) = 4$, $\lambda'_{2,1}(K_4) = 7$, and $\lambda'_{2,1}(K_n) = n(n-1)/2 - 1$ for $n \geq 5$.*

A wheel of length n, W_n, is obtained by adding a new vertex adjacent to all vertices in C_n.

Theorem 1.5 *[10] $\lambda'_{2,1}(W_3) = \lambda'_{2,1}(W_4) = 7$, $\lambda'_{2,1}(W_5) = 9$, and $\lambda'_{2,1}(W_n) = 2n - 2$ for $n \geq 6$.*

Let $T_\infty(\Delta)$ denote the infinite tree with each vertex having degree Δ.

Theorem 1.6 *[10] Let T be a tree with maximum degree $\Delta \geq 3$. Then*

$$2\Delta - 2 \leq \lambda'_{2,1}(T) \leq \lambda'_{2,1}(T_\infty(\Delta)) \leq 2\Delta + 3.$$

Theorem 1.7 *[10]*

$$\lambda'_{2,1}(T_\infty(\Delta)) = \begin{cases} 2\Delta + 1, & \text{if } \Delta = 3, 4; \\ 2\Delta + 2, & \text{if } \Delta = 5; \\ 2\Delta + 3, & \text{if } \Delta \geq 6. \end{cases}$$

Griggs and Yeh [11] made the following conjecture.

Conjecture 1.8 *For any graph G with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$.*

Almost all papers concerning $L(j,k)$-labeling of a graph deal with the case that $j \geq k$. In this paper, we shall allow that $j \leq k$. When we are studying the $L(j,k)$-edge-labeling of a graph, we shall always assume that all graphs we considered have no loops. However we allow multiple edges in a graph.

In this paper, we show that if G is $K_{1,3}$-free with maximum degree Δ then $\lambda_{j,k}(G) \leq k\lfloor \Delta^2/2 \rfloor + j\Delta - 1$ except that G is a 5-cycle and $j = k$. Consequently, we have that except G is a 5-cycle and $j = k$, $\lambda'_{j,k}(G) \leq k\lfloor \Delta_L^2/2 \rfloor + j\Delta_L - 1$. This improves the upper bounds for $\lambda'_{2,1}(G)$ and $\lambda'_{1,1}(G)$ given by Georges and Mauro in [10]. As a corollary we show that Conjecture 1.8 holds for all $K_{1,3}$-free graphs and hence holds for all line graphs. This improves Corollary 1.3. We also investigate the upper bound for $\lambda'_{j,k}(G)$ for $K_{1,3}$-free graphs G.

The cardinality of any finite set S shall be denoted by $|S|$. The complement graph of G shall be denoted by $\overline{G}$.

2 The Upper bound for $\lambda_{j,k}(G)$ for $K_{1,3}$-free graphs

Let G be a graph and let ω be a function which assigns each edge of G a positive integer, i.e., $\omega : E(G) \to \mathbf{N}$. An assignment $f : V(G) \to \mathbf{N}$ of colors to the vertices of G is proper if $|f(u) - f(v)| \geq \omega(uv)$ for each $uv \in E(G)$. A *weighted degree* $deg_\omega(v)$ of a vertex v of G is the sum of the weights of the edges incident with v. The *maximum weighted degree* $\Delta_\omega(G)$ is the largest $deg_\omega(v)$, where $v \in V(G)$. Define $\chi_\omega(G)$ to be the smallest number for which there is a proper assignment f such that $1 \leq f(v) \leq \chi_\omega(G)$ for all $v \in V(G)$.

The inequality $\chi_\omega(G) \leq \Delta_\omega + 1$ was recently proved by McDiarmid in [17–19]. In [15], the authors proved an analogue of Brooks' theorem as follows:

Lemma 2.1 *Let G be a 2-connected graph and let ω be a function which assigns to the edges of G positive integers. If $\chi_\omega(G) = \Delta_\omega + 1$, then the weighted degree of each vertex of G is equal to $\Delta_\omega(G)$ and one of the following holds:*

- *G is an odd cycle and all its edges have the same weights.*
- *G is a complete graph and all its edges have the same weights.*

For any fixed positive integer k, the kth power of a graph G is the graph G^k whose vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{xy | 1 \leq d_G(x, y) \leq k, x, y \in V(G)\}$, where $d_G(x, y)$ is the distance between the vertices x and y in the graph G.

Let j and k be any two positive integers. Suppose G is a graph and the square graph of G is G^2. Define a function π from $E(G^2)$ to $\mathbf{N}$ as: $\pi(e) = j$ if $e \in E(G)$ and $\pi(e) = k$ if $e \in E(G^2) \setminus E(G)$. Then a proper weighted channel assignment $f : V(G^2) \to \mathbf{N}$ with respective to the weight π is equivalent to an $L(j, k)$-labeling of G. Note that the only difference between an $L(j, k)$-labeling and a proper weighted channel assignment is that the label 0 can be used in an $L(j, k)$-labeling. Therefore $\chi_\pi(G^2) = \lambda_{j,k}(G) + 1$.

A graph is called $K_{1,3}$-free if it contains no induced subgraph $K_{1,3}$. Let $ex(p, K_3)$ be the maximal number of edges in a graph of order p not containing K_3. It is well known that $ex(p, K_3) = \lfloor p^2/4 \rfloor$ and the only extremal graph is $K_{\lfloor p/2 \rfloor, \lceil p/2 \rceil}$.

Theorem 2.2 *Let G be a simple graph and let Δ be the maximum degree of G. Suppose $\Delta \geq 2$. If G is $K_{1,3}$-free then, except the case that G is a 5-cycle and $j = k$, we have $\lambda_{j,k}(G) \leq k\lfloor \Delta^2/2 \rfloor + j\Delta - 1$.*

Proof. Without loss of generality, we may assume that G is connected. Let π be the weighted function for G^2 defined as above. Since $\lambda_{j,k}(G) = \chi_\pi(G^2) - 1$, it suffices to show that $\chi_\pi(G^2) \leq k\lfloor\Delta^2/2\rfloor + j\Delta$. Let x be any vertex of G. Denote by $N(x)$ the set of vertices adjacent to x. Let $t = |N(x)| \leq \Delta$. Since G is $K_{1,3}$-free, the complement of $G[N(x)]$ contains no K_3. Thus the complement of $G[N(x)]$ has at most $\lfloor t^2/4\rfloor$ edges and hence $G[N(x)]$ has at least $\binom{t}{2} - \lfloor t^2/4\rfloor$ edges. Let $N_2(x)$ denote the set of vertices distance 2 away from x. Then $|N_2(x)| \leq t(\Delta-1)-2[\binom{t}{2} - \lfloor t^2/4\rfloor] = t\Delta + 2\lfloor t^2/4\rfloor - t^2 \leq t\Delta - t^2/2 \leq \Delta^2/2$. It follows that, for any vertex in G, there are at most Δ vertices adjacent to it and at most $\lfloor\Delta^2/2\rfloor$ vertices distance two away from it. Thus $\Delta_\pi(G^2) \leq k\lfloor\Delta^2/2\rfloor + j\Delta$. If $\chi_\pi(G^2) \geq k\lfloor\Delta^2/2\rfloor+j\Delta+1$, then $\Delta_\pi(G^2) = k\lfloor\Delta^2/2\rfloor+j\Delta$ and $\chi_\pi(G^2) = \Delta_\pi(G^2)+1$. Since the second power of a connected graph with maximum degree at least 2 is always 2-connected, G^2 together with π is one of the forms described in Lemma 2.1, i.e., G^2 is either an odd cycle or a complete graph with all its edges having the same weight.

Except for P_3 and K_3, there is no graph whose second power is a cycle, where P_3 is a path with three vertices. For $G = P_3$, we have $\Delta_\pi(G^2) = \max\{2j, j+k\} < k\Delta^2/2 + j\Delta$, a contradiction. For $G = K_3$, one can get a similar contradiction.

The remaining case is that G^2 is a complete graph and all its edges having the same weight. This is the case only when G is a complete graph or G is not complete but G^2 is and $j = k$. First suppose G is a complete graph. Let n be the number of vertices of G, then $\Delta_\pi(G^2) = j(n-1)$ and $\Delta = n - 1$. Since we assume $\Delta \geq 2$, it is easy to see that $\Delta_\pi(G^2) = j(n-1) < k\lfloor\Delta^2/2\rfloor + j\Delta$, a contradiction.

Now suppose G is not complete but G^2 is and $j = k$. If $\Delta = 2$, G must be P_3, C_3 or C_4 and the theorem holds for these graphs clearly when $j = k$. So we assume that $\Delta \geq 3$. Let n be the number of vertices of G. Since G^2 is complete and $j = k$, $\Delta_\pi(G^2) = (n-1)j$. Next we show that $\Delta_\pi(G^2) < k\lfloor\Delta^2/2\rfloor+j\Delta$ and thus get a contradiction. Suppose to the contrary that $\Delta_\pi(G^2) = k\lfloor\Delta^2/2\rfloor+j\Delta$. Since $j = k$ and $\Delta_\pi(G^2) = (n-1)j$, we have $n = \Delta^2/2 + \Delta + 1$ if Δ is even and $n = (\Delta^2-1)/2 + \Delta + 1$ if Δ is odd.

Since G^2 is complete, any pair of vertices of G are at distance at most 2 in G. It follows that, for any vertex x of G, $|N(x)| + |N_2(x)|$ equals $\Delta^2/2 + \Delta$ if Δ is even and $(\Delta^2-1)/2 + \Delta$ if Δ is odd. Let x be a vertex of degree Δ in G. Since $n = |V(G)| = \lfloor\Delta^2/2\rfloor + \Delta + 1$ and G is of diameter 2, the edge number of the induced subgraph $G[N(x)]$ is at most $\frac{1}{2}(\Delta(\Delta-1) - \lfloor\Delta^2/2\rfloor)$. This implies that $\bar{G}[N(x)]$ (the complement of $G[N(x)]$) has at least $\frac{1}{2}\lfloor\Delta^2/2\rfloor = \lfloor\Delta^2/4\rfloor$ edges. Since G is $K_{1,3}$-free, $\bar{G}[N(x)]$ contains no K_3 and thus it has at most $\lfloor\Delta^2/4\rfloor$ edges. It follows that $\bar{G}[N(x)]$ has exactly $\lfloor\Delta^2/4\rfloor$ edges and $\bar{G}[N(x)] \cong K_{\lfloor\Delta/2\rfloor,\lceil\Delta/2\rceil}$ and each vertex in $N(x)$ has degree Δ. Consequently $G[N(x)]$ must be the disjoint union of $K_{\lfloor\Delta/2\rfloor}$ and $K_{\lceil\Delta/2\rceil}$. So far we have shown that if a vertex x is of degree Δ in G then each vertex in $N(x)$ also has degree Δ and $G[N(x)]$ is the disjoint union of $K_{\lfloor\Delta/2\rfloor}$ and $K_{\lceil\Delta/2\rceil}$. And this clearly implies that G is Δ-regular and for any vertex x, $G[N(x)]$ is the disjoint union of $K_{\lfloor\Delta/2\rfloor}$ and $K_{\lceil\Delta/2\rceil}$.

We first deal with the case that $\Delta \geq 4$ is even. Let x be a vertex of G. Then $G[N(x)]$ is the disjoint union of two cliques. Let $A_x = \{x_1, x_2, \ldots, x_{\Delta/2}\}$, $B_x = \{y_1, y_2, \ldots, y_{\Delta/2}\}$, and $N(x) = A_x \cup B_x$. We may assume that $G[A_x]$ and $G[B_x]$ are two cliques of G. For $i = 1, 2, \ldots, \Delta/2$, denote by A_{x_i} the vertex set $N(x_i) \setminus (A_x \cup \{x\})$ and by A_{y_i} the vertex set $N(y_i) \setminus (B_x \cup \{x\})$. Clearly $|A_{x_i}| = |A_{y_i}| = \Delta/2$ for $i = 1, 2, \ldots, \Delta/2$. Since $|N(x)| + |N_2(x)|$ equals $\Delta^2/2 + \Delta$, all these sets are pairwise disjoint. And each of these sets induces a clique of order $\Delta/2$. Let y be any vertex in $\bigcup_{i=1}^{\Delta/2} A_{y_i}$. Since $d(x_i, y) \leq 2$ for each $i = 1, 2, \ldots, \Delta/2$, y must be adjacent to some vertex in A_{x_i}. It follows that any vertex $y \in A_{y_1}$ is nonadjacent to any vertex $y' \in A_{y_2}$. Since $d(y, y') = 2$, there is some vertex z in $\bigcup_{i=1}^{\Delta/2} A_{x_i}$ such that zy and zy' are edges of G. Suppose $z \in A_{x_i}$. Then $\{z, x_i, y, y'\}$ induces a $K_{1,3}$, a contradiction to our assumption.

We now suppose that $\Delta \geq 3$ is odd. Let x be a vertex of G. Then $G[N(x)]$ is the disjoint union of two cliques. Let $A_x = \{x_1, x_2, \ldots, x_{(\Delta-1)/2}\}$, $B_x = \{y_1, y_2, \ldots, y_{(\Delta+1)/2}\}$, and $N(x) = A_x \cup B_x$. Suppose $G[A_x]$ and $G[B_x]$ are cliques of G. For $i = 1, 2, \ldots, (\Delta-1)/2$, denote by A_{x_i} the vertex set $N(x_i) \setminus (A_x \cup \{x\})$ and for $i = 1, 2, \ldots, (\Delta+1)/2$ denote by A_{y_i} the vertex set $N(y_i) \setminus (B_x \cup \{x\})$. Clearly $|A_{x_i}| = (\Delta+1)/2$ for $i = 1, 2, \ldots, (\Delta-1)/2$ and $|A_{y_i}| = (\Delta-1)/2$ for $i = 1, 2, \ldots, (\Delta+1)/2$. Since $|N(x)| + |N_2(x)|$ equals $(\Delta^2-1)/2 + \Delta$, all these sets are pairwise disjoint. And each of these sets induces a clique of order $(\Delta+1)/2$ or $(\Delta-1)/2$. Let y be any vertex in A_{x_1}. Since $d(y_i, y) \leq 2$ for each $i = 1, 2, \ldots, (\Delta+1)/2$, y must be adjacent to some vertex in A_{y_i}. It follows that $d(y) \geq \Delta + 1$, a contradiction. ∎

If G is a 5-cycle and $j = k$ then we clearly have $\lambda_{j,j}(C_5) = 4j$.

3 The Upper bound for $\lambda'_{j,k}(G)$

Since line graphs are $K_{1,3}$-free, the following theorem follows from Theorem 2.2 immediately.

Theorem 3.1 *Let G be a simple or multiple graph and let Δ_L be the maximum degree of its line graph. Suppose $\Delta_L \geq 2$. Except the case that G is a 5-cycle and $j = k$, we have $\lambda'_{j,k}(G) \leq k\lfloor \Delta_L^2/2 \rfloor + j\Delta_L - 1$.*

Corollary 3.2 *Let G be a simple graph with maximum degree $\Delta \geq 2$. If G is $K_{1,3}$-free then $\lambda_{2,1}(G) \leq \lfloor \Delta^2/2 \rfloor + 2\Delta - 1$.*

It is easy to check that the inequality $\lfloor \Delta^2/2 \rfloor + 2\Delta - 1 \leq \Delta^2$ holds for all $\Delta \geq 3$. In the case $\Delta = 2$, G is the disjoint union of paths and cycles, so $\lambda_{2,1}(G) \leq \Delta^2$. Thus Conjecture 1.8 holds for all $K_{1,3}$-free graphs and hence for all line graphs. This is an improvement of Corollary 1.3.

The following corollary improves the upper bounds $\Delta_L^2/2 + 3\Delta_L$ and $2\Delta^2 + 2\Delta - 4$ for $\lambda'_{2,1}(G)$ in Theorem 1.1.

Corollary 3.3 *Let G be a simple or multiple graph with maximum degree $\Delta \geq 2$ and let $\Delta_L \geq 2$ be the maximum degree of its line graph. Then $\lambda'_{2,1}(G) \leq \lfloor \Delta_L^2/2 \rfloor + 2\Delta_L - 1 \leq 2\Delta^2 - 3$.*

Next we apply Theorem 3.1 to the case $j = k = 1$ and to strong chromatic index of graphs.

Corollary 3.4 *Let G be a graph with maximum degree $\Delta \geq 2$. Let Δ_L be the maximum degree of the line graph $L(G)$. Then $\lambda'_{1,1}(G) \leq \lfloor \Delta_L^2/2 \rfloor + \Delta_L - 1$.*

Corollary 3.4 improves the upper bound $\Delta_L^2/2 + \Delta_L$ for $\lambda'_{1,1}(G)$ in Theorem 1.2.

A *strong matching* in a graph G is an induced subgraph of G that forms a matching. A *strong edge coloring* of a graph G is an edge coloring of G such that each color class is a strong matching. The *strong chromatic index* of a graph G, denoted by $s\chi'(G)$, is the smallest number of colors in a strong edge coloring of G. It is not difficult to see that a strong edge coloring of a graph G is an $L(1,1)$-edge labeling of G. Note that we use 0 in an $L(1,1)$-edge labeling of a graph, it is clear that $s\chi'(G) = \lambda'_{1,1}(G) + 1$ for any graph G. Thus Corollary 3.4 implies the following corollary.

Corollary 3.5 *Let G be a graph with maximum degree $\Delta \geq 2$. Let Δ_L be the maximum degree of the line graph $L(G)$. If G is not isomorphic to a 5-cycle, then $s\chi'(G) \leq \lfloor \Delta_L^2/2 \rfloor + \Delta_L \leq 2\Delta^2 - 2\Delta$.*

It was conjectured by Erdős and Nešetřil that $s\chi'(G) \leq 5\Delta^2/4$ if Δ is even and $\leq 5\Delta^2/4 - \Delta/2 + 1/4$ if Δ is odd, where Δ is the maximum degree of G. The conjecture is clearly true for $\Delta \le 2$. The case $\Delta = 3$ was settled independently by Andersen [1] and by Horák, Qing, and Trotter [13]. They showed that $s\chi'(G) \le 10$ for graphs with maximum degree 3. Horák [14] showed that $s\chi'(G) \le 23$ for graphs with maximum degree 4. And recently, Cranston [3] showed that $s\chi'(G) \le 22$ for graphs with maximum degree 4. The conjecture is unsolved for $\Delta \ge 4$.

If $\Delta = 3$ then $\Delta_L \le 4$ and Corollary 3.5 gives the upper bound 12 which is just 2 bigger than the best known upper bound 10. If $\Delta = 4$ then $\Delta_L \le 6$ and Corollary 3.5 gives the upper bound 24 which is also just 2 bigger than the best known upper bound 22. In particular, when $\Delta_L = 3$ the upper bound 7 given by Corollary 3.5 is the best possible. This can be seen from the following defined graph H_1. H_1 is the graph obtained from a 5-cycle by adding a new vertex and joining it to two nonadjacent vertices of the 5-cycle. Then H_1 has 7 edges and any two edges of it are at distance at most 2. Thus $s\chi'(H_1) = 7$.

Faudree etc. in [4] asked a problem: is $s\chi'(G) \le 7$ if G is a graph with $d_G(x) + d_G(y) \le 5$ for any edge xy of G? Note that if $d_G(x) + d_G(y) \le 5$ for any edge xy of G then $\Delta_L \le 3$. Therefore the upper bound 7 given by Corollary 3.5 with $\Delta_L = 3$ answers the problem ask by Faudree etc..

For the rest of the paper, we shall improve the upper bound provided in Theorem 3.1, if G is $K_{1,3}$-free.

Suppose G is a graph which may have multiple edges. For an edge e of G, denote by $d_G(e)$ the number of edges which are at distance 1 from e, and $d_G^2(e)$ the number of edges which are distance 1 or 2 away from e. Clearly $\Delta_L(G) = \max\{d_G(e) | e \in E(G)\}$. Denote by $\mu(e)$ the multiplicity of e.

For any positive integer z, define

$$\theta(z) = \begin{cases} k(3z^2/8 + z/2) + jz, & \text{if } z \text{ is even;} \\ k[3(z^2-1)/8 + (z-1)/2] + jz, & \text{if } z \text{ is odd.} \end{cases}$$

Theorem 3.6 *Let G be a simple or multiple graph and let Δ_L be the maximum degree of its line graph. Suppose $\Delta_L \ge 2$. If G is $K_{1,3}$-free then, except the case that G is a 5-cycle and $j = k$, we have $\lambda'_{j,k}(G) \le \theta(\Delta_L) - 1$.*

Proof. Without loss of generality, we may assume that G is connected. Let π be the weighted function for $(L(G))^2$ defined as in the previous section. Since $\lambda'_{j,k}(G) = \lambda_{j,k}(L(G)) = \chi_\pi((L(G))^2) - 1$, it suffices to show that $\chi_\pi((L(G))^2) \le \theta(\Delta_L)$. Let xy be any edge of G. Let $d_G(x) = a + \mu(xy)$ and $d_G(y) = b + \mu(xy)$ for some $a, b \ge 0$. Then $d_G(xy) = a + b + \mu(xy) - 1 \le \Delta_L$ and $a + b \le \Delta_L$. Let $A = N_G(x) \setminus \{y\} = \{x_1, x_2, \ldots, x_a\}$ and $B =$

$N_G(y) \setminus \{x\} = \{y_1, y_2, \ldots, y_b\}$. Clearly $|A| = a$ and $|B| = b$. Let e_A (e_B, respectively) denote the number of edges incident with at least one vertex in A (B, respectively) but not incident with x (y, respectively). And let y_A (y_B, respectively) denote the number of edges in $G[A]$ ($G[B]$, respectively).

Since G is $K_{1,3}$-free, $\overline{G}[A]$ contains no K_3. It follows that $\overline{G}[A]$ has at most $\lfloor a^2/4 \rfloor$ edges and hence $G[A]$ contains at least $\binom{a}{2} - \lfloor a^2/4 \rfloor$ edges. That is $y_A \geq \binom{a}{2} - \lfloor a^2/4 \rfloor$. Noting that for each $i = 1, 2, \ldots, a$, $d_G(xx_i) \leq \Delta_L$, and the edge xx_i is adjacent to the edge xy and the edges xx_j with $j \neq i$, we know that each vertex x_i is incident with at most $\Delta_L - a$ edges in the form of ux_i with $u \notin A$. Therefore we have

$$\begin{aligned} e_A &\leq a(\Delta_L - a) - y_A \\ &\leq a(\Delta_L - a) - a(a-1)/2 + \lfloor a^2/4 \rfloor \\ &= a\Delta_L + a/2 + \lfloor a^2/4 \rfloor - 3a^2/2. \end{aligned}$$

Similarly, $e_B \leq b\Delta_L + b/2 + \lfloor b^2/4 \rfloor - 3b^2/2$. It follows that

$$\begin{aligned} & d_G^2(xy) - d_G(xy) \leq e_A + e_B \\ \leq\ & (a+b)\Delta_L + (a+b)/2 + (\lfloor a^2/4 \rfloor + \lfloor b^2/4 \rfloor) - 3(a^2+b^2)/2 \\ \leq\ & \begin{cases} (a+b)\Delta_L + (a+b)/2 - 5(a^2+b^2)/4, & \text{if } a+b \text{ is even;} \\ (a+b)\Delta_L + (a+b)/2 - 5(a^2+b^2)/4 - 1/4, & \text{if } a+b \text{ is odd.} \end{cases} \\ \leq\ & \begin{cases} 3\Delta_L^2/8 + \Delta_L/2, & \text{if } \Delta_L \text{ is even;} \\ 3(\Delta_L^2 - 1)/8 + (\Delta_L - 1)/2, & \text{if } \Delta_L \text{ is odd.} \end{cases} \end{aligned}$$

It follows that, for any vertex in $L(G)$, there are at most Δ_L vertices adjacent to it and at most $3\Delta_L^2/8 + \Delta_L/2$ (or $3(\Delta_L^2 - 1)/8 + (\Delta_L - 1)/2$) vertices distance two away from it. So if $\chi_\pi((L(G))^2) \geq \theta(\Delta_L) + 1$, then $\Delta_\pi((L(G))^2) = \theta(\Delta_L)$ and $\chi_\pi((L(G))^2) = \Delta_\pi((L(G))^2) + 1$. Since the second power of a connected graph with maximum degree at least 2 is always 2-connected, $(L(G))^2$ together with π is one of the forms described in Lemma 2.1, i.e., $(L(G))^2$ is either an odd cycle or a complete graph with all its edges having the same weight.

Except for P_3 and K_3, there is no graph whose second power is a cycle, where P_3 is a path with three edges. For $G = P_3$, we have $\Delta_\pi((L(G))^2) = \max\{2j, j+k\} < k(3\Delta_L^2/8 + \Delta_L/2) + j\Delta_L$, a contradiction. For $G = K_3$, one can get the same contradiction.

The remaining case is that $(L(G))^2$ is a complete graph and all its edges having the same weight. This is the case only when $L(G)$ is a complete graph or $L(G)$ is not complete but $(L(G))^2$ is and $j = k$. First suppose $L(G)$ is a complete graph. Let m be the number of vertices of $L(G)$, then

$\Delta_\pi((L(G))^2) = j(m-1)$ and $\Delta_L = m-1$. Since we assume $\Delta_L \geq 2$, m must be greater than or equal to 3. Thus $\Delta_\pi = j(m-1) < \theta(\Delta_L)$, a contradiction.

Now suppose $L(G)$ is not complete but $(L(G))^2$ is and $j = k$. Let m be the number of edges of G (that is the number of vertices of $L(G)$). Since $(L(G))^2$ is complete and $j = k$, $\Delta_\pi((L(G))^2) = (m-1)j$. Next we show that $\Delta_\pi((L(G))^2) < \theta(\Delta_L)$ and thus get a contradiction. Suppose to the contrary that $\Delta_\pi = \theta(\Delta_L)$. Since $j = k$ and $\Delta_\pi((L(G))^2) = (m-1)j$, we have $m = 3\Delta_L^2/8 + 3\Delta_L/2 + 1$ if Δ_L is even and $m = 3(\Delta_L^2 - 1)/8 + (3\Delta_L - 1)/2 + 1$ if Δ_L is odd. When $\Delta_L = 2$, G must be P_3, C_3 or C_4 and the theorem holds for these graphs clearly when $j = k$. So we assume that $\Delta_L \geq 3$. Since $(L(G))^2$ is complete, any pair of edges of G are at distance at most 2. It follows that, for any edge xy of G, $d_G(xy) + d_G^2(xy)$ equals $3\Delta_L^2/8 + 3\Delta_L/2$ if Δ_L is even and $3(\Delta_L^2 - 1)/8 + (3\Delta_L - 1)/2$ if Δ_L is odd. From the calculation in the beginning of the proof, we know that this is the case only when $a = b = \Delta_L/2$ if Δ_L is even and only when $\{a, b\} = \{(\Delta_L - 1)/2, (\Delta_L + 1)/2\}$ if Δ_L is odd, and that the multiplicity of any edge xy is 1. Therefore, if Δ_L is even then G is simple and $(\Delta_L/2 + 1)$-regular. Let n be the vertex number of G. Then $(\Delta_L/2 + 1)n = 2(3\Delta_L^2/8 + 3\Delta_L/2 + 1)$. This implies that $n = 3\Delta_L/2 + 3 - 2/(\Delta_L + 2)$. But this is impossible since n is an integer, a contradiction. If Δ_L is odd then G is simple and bipartite. Let $V_1 = \{v | d_G(v) = (\Delta_L + 1)/2,\ v \in V(G)\}$ and $V_2 = \{v | d_G(v) = (\Delta_L + 3)/2,\ v \in V(G)\}$. Then $|V_1|(\Delta_L + 1)/2 = |V_2|(\Delta_L + 3)/2$. It follows that $2|V_2|(\Delta_L + 3)/2 = 2m = 2[3(\Delta_L^2 - 1)/8 + (3\Delta_L - 1)/2 + 1]$. Therefore $|V_2| = \Delta_L - (\Delta_L^2 - 1)/[4(\Delta_L + 3)]$. Note that Δ_L is odd, it is not difficult to check that $(\Delta_L^2 - 1)/[4(\Delta_L + 3)]$ can not be an integer. This is a contradiction since $|V_2|$ is an integer. ∎

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