Type I Codes over GF(4)

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Abstract

It was shown by Gaborit el al. [10] that a Euclidean self-dual code over $GF(4)$ with the property that there is a codeword whose Lee weight $\equiv 2 \pmod 4$ is of interest because of its connection to a binary singly-even self-dual code. Such a self-dual code over $GF(4)$ is called Type I. The purpose of this paper is to classify all Type I codes of lengths up to 10 and extremal Type I codes of length 12, and to construct many new extremal Type I codes over $GF(4)$ of

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lengths from 14 to 22 and 34. As a byproduct, we construct a new extremal singly-even self-dual binary [36, 18, 8] code, and a new extremal singly-even self-dual binary [68, 34, 12] code with a previously unknown weight enumerator W_2 for $\beta = 95$ and $\gamma = 1$.

Key Words. Binary self-dual code, Euclidean self-dual code over $GF(4)$.

1 Introduction

We briefly review basic definitions. A *linear* $[n,k]$ *code* C over $GF(4)$ is a k-dimensional vector subspace of $GF(4)^n$, where $GF(4)$ is the Galois field with four elements $0, 1, \omega$, and $\overline{\omega}$ satisfying $\overline{\omega} = \omega^2$ and $\overline{\omega} = 1 + \omega$. The *Hamming weight* $wt_H(\mathbf{x})$ of $\mathbf{x} \in GF(4)^n$ is the number of nonzero components of $\mathbf{x}$. Let $n_0(\mathbf{x}), n_\omega(\mathbf{x}), n_{\overline{\omega}}(\mathbf{x})$, and $n_1(\mathbf{x})$ be the number of 0's, ω's, $\overline{\omega}$'s, and 1's in a vector $\mathbf{x} \in GF(4)^n$, respectively. The *Lee weight* $wt_L(\mathbf{x})$ of $\mathbf{x} \in GF(4)^n$ is defined as $2n_1(\mathbf{x}) + n_\omega(\mathbf{x}) + n_{\overline{\omega}}(\mathbf{x})$. Note that $wt_L(0) = 0, wt_L(1) = 2, wt_L(\omega) = 1$, and $wt_L(\overline{\omega}) = 1$. Thus the Lee weight $wt_L(\mathbf{x})$ of $\mathbf{x} \in GF(4)^n$ is the rational sum of the Lee weights of all the coordinates of $\mathbf{x}$. The *minimum Lee weight* d_L (resp. *minimum Hamming weight* d_H) of C is the smallest Lee (resp. Hamming) weight among all non-zero codewords of C.

Two codes C_1 and C_2 are *(permutation) equivalent* if there exists a coordinate permutation sending C_1 onto C_2 [2],[10]. The *(permutation) automorphism group* PAut of C is the set of all coordinate permutations preserving C. The *direct sum* of two codes C_1 and C_2 is $C_1 \oplus C_2 = \{(u,v) | u \in C_1 \text{ and } v \in C_2\}$. C^n denotes the direct sum of n copies of C. If $\mathcal{D}$ is equivalent to $C_1 \oplus C_2$, it is called *decomposable*, otherwise *indecomposable*. The *complete weight enumerator* $cwe_C(a,b,c,d)$ of C is

$$\sum_{\mathbf{c} \in C} a^{n_0(\mathbf{c})} b^{n_\omega(\mathbf{c})} c^{n_{\overline{\omega}}(\mathbf{c})} d^{n_1(\mathbf{c})}.$$

The *Lee weight enumerator* of C is defined as

$$\sum_{\mathbf{c} \in C} y^{wt_L(\mathbf{c})} = cwe_C(1, y, y, y^2).$$

The *Gray map* ϕ from $GF(4)^n$ to $GF(2)^{2n}$, first appeared in [17, pp. 508] and then in [10], is defined as

$$\phi(\omega\mathbf{x} + \overline{\omega}\mathbf{y}) = (\mathbf{x}, \mathbf{y}) \text{ for } \mathbf{x}, \mathbf{y} \in GF(2)^n,$$

where $(\mathbf{x}, \mathbf{y})$ is the binary vector of length $2n$.

The *Euclidean inner product* is defined as $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + \cdots + x_ny_n \in GF(4)$, for two vectors $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ in $GF(4)^n$. The *dual code* $C^\perp$ of C is defined as

$$C^\perp = \{\mathbf{x} \in GF(4)^n | \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{y} \in C\}.$$

If $C = C^\perp$, then C is called a *(Euclidean) self-dual* code. A Euclidean self-dual code over $GF(4)$ is called *Type II* if the Lee weight of every codeword is divisible by 4 and *Type I* if there is a codeword whose Lee weight $\equiv 2 \pmod 4$ [2],[10]. We remark that a Euclidean self-dual code over $GF(4)$ can have a codeword of odd Hamming weight even though all codewords have even Lee weights.

It was shown by Gaborit et al. [10] that if C is a Euclidean self-orthogonal code over $GF(4)$, then $\phi(C)$ is a binary self-orthogonal code. So C is a Type I (resp. Type II) code over $GF(4)$ if and only if $\phi(C)$ is a singly-even (resp. doubly-even) binary self-dual code. As a binary self-dual code contains all one vector $\mathbf{1}$, any Euclidean self-dual code over $GF(4)$ contains all one vector. There has been a classification of Type II codes of lengths 4, 8, and 12. It is known that there are only one Type II code of length 4 and exactly two Type II codes of length 8 [10], and that there are exactly seven Type II codes of length 12, one of which is extremal [2]. Several examples of extremal Type I codes are in [2],[10].

Our paper is the first attempt to classify Type I codes over $GF(4)$. We classify all Type I (and Type II) codes of lengths up to 10 and extremal Type I (and Type II) codes of length 12, and construct many new extremal Type I codes over $GF(4)$ of lengths from 14 to 22 and 34. We also give their corresponding binary singly-even self-dual codes whenever possible. As a byproduct, we construct a new extremal singly-even self-dual binary $[36, 18, 8]$ code with a previously unknown group order and a new extremal singly-even self-dual binary $[68, 34, 12]$ code with a previously unknown weight enumerator W_2 for $\beta = 95$ and $\gamma = 1$ [14]. We also prove that a Euclidean self-dual $[12, 6]$ code over $GF(4)$ with minimum Hamming weight 6 is unique; it is permutation equivalent to the extended quadratic residue $[12, 6]$ code over $GF(4)$.

We summarize the currently known status of extremal or optimal (with respect to Lee weight) Euclidean self-dual codes over $GF(4)$ of even lengths n ($2 \le n \le 22$) and $n = 34$ in Table 1. Here $d_L(I)$ and $d_L(II)$ denote the highest minimum Lee weight of Type I and Type II codes, respectively. The number of Type I codes and that of Type II codes are separated by ; and entries without reference are obtained from this paper. A period indicates that the list of codes is complete. The column with (d_H; no.) gives the number of Euclidean self-dual codes with highest minimum Hamming weight d_H of lengths $n \le 12$ and the last column with d_H for our codes gives the minimum Hamming weight of our Type I codes. The attainable Hamming weight of our Euclidean self-dual codes over $GF(4)$ is better than the Pless-Pierce bound [20] for $8 \leq n \leq 20$ and $n = 32$, and slightly weaker than the Table 6 of [9] for $n \geq 14$.

2 Preliminaries and Methods

The following lemmas are straightforward by the definition of the Gray map.

Lemma 2.1 ([10]). *The Gray map ϕ is a $GF(2)$-linear isometry from $(GF(4)^n$, Lee distance) onto $(GF(2)^{2n}$, Hamming distance) where the Lee distance of two codewords* $\mathbf{x}$ *and* $\mathbf{y}$ *is the Lee weight of* $\mathbf{x} - \mathbf{y}$. *The Lee weight enumerator of a code C over $GF(4)$ is the same as the Hamming weight enumerator of $\phi(C)$.*

Lemma 2.2 ([10]). *If C_1 and C_2 are equivalent Euclidean self-dual codes over $GF(4)$, then $\phi(C_1)$ and $\phi(C_2)$ are equivalent. The converse is not true.*

We now give an upper bound for the minimum Lee weights of self-dual codes over $GF(4)$ by using Rains' bound [22] for binary self-dual codes.

Lemma 2.3 ([10]). *Let $d_L(I,n)$ and $d_L(II,n)$ be the highest minimum Lee weights of a Type I code and a Type II code, respectively, of length n. Then*

$$d_L(I,n) \leq 4\left\lfloor \frac{n}{12} \right\rfloor + 4 \quad (n \equiv 0 \pmod 2) \tag{1}$$

$$d_L(II,n) \leq 4\left\lfloor \frac{n}{12} \right\rfloor + 4 \quad (n \equiv 0 \pmod 4). \tag{2}$$

A Type I (resp. Type II) code of length n satisfying the above bound is called *extremal*. An *optimal* Type I code has the highest minimum Lee weight among all Type I codes of that length.

We now give a building-up construction method of Euclidean self-dual codes over $GF(4)$ from smaller length self-dual codes.

Theorem 2.4 (Building-up). *Let $G_0 = (L|R) = (l_i|r_i)$ be a generator matrix (may not be in standard form) of a Euclidean self-dual code C_0 over $GF(4)$ of length $2n$. where l_i and r_i are rows of $n \times n$ matrices L and R respectively for $1 \leq i \leq n$. Let $\mathbf{x} = (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n})$ be a vector in $GF(4)^{2n}$ with $\mathbf{x} \cdot \mathbf{x} = 1$. Suppose that $y_i := (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n}) \cdot (l_i|r_i)$ for $1 \leq i \leq n$ under the Euclidean inner product. Then the following matrix*

$$G = \left[\begin{array}{cc|cccccc} 1 & 0 & x_1 & \cdots & x_n & x_{n+1} & \cdots & x_{2n} \\ \hline y_1 & y_1 & & & & & & \\ \vdots & \vdots & & L & & & R & \\ y_n & y_n & & & & & & \end{array}\right]$$

generates a Euclidean self-dual code C over $GF(4)$ of length $2n+2$.

Proof. This is a modified construction of Hermitian self-dual codes over $GF(4)$ in [16]. □

Using Theorem 2.4 we can prove the following.

Theorem 2.5. *Any Euclidean self-dual code C over $GF(4)$ of length $2n$ with minimum Hamming weight $d_H > 2$ is obtained from some Euclidean self-dual code C_0 of length $2n-2$ (up to equivalence) by the construction in Theorem 2.4.*

Proof. The proof is similar to that of [15, Theorem 2]. We omit the details. □

Corollary 2.6. *Any Euclidean self-dual code C over $GF(4)$ of length $2n$ with minimum Lee weight $d_L \geq 4$ is obtained from some Euclidean self-dual code C_0 of length $2n-2$ (up to equivalence) by the construction in Theorem 2.4.*

Proof. We note that if C has minimum Lee weight $d_L \geq 4$ then it has minimum Hamming weight $d_H > 2$. The reason is that if $d_H \leq 2$ then there are at most two nonzero positions in any codeword of Hamming weight 2. To have minimum Lee weight $d_L \geq 4$, such codewords should have two 1's and 0's in the rest of coordinates. Then since C is linear we have a codeword with two ω's and 0's in the rest of coordinates. Then the codeword has Lee weight 2. a contradiction. Hence the corollary follows from Theorem 2.5. □

When a Euclidean self-dual code C over $GF(4)$ of length $2n$ has minimum Hamming weight 2. we can decompose it as in the case of a binary self-dual code with minimum weight 2 [23].

Theorem 2.7 (Decomposition). *If C is a Euclidean self-dual code over $GF(4)$ of length $2n$ with minimum Hamming (also Lee) weight 2, then C is permutation equivalent to the direct sum of i_2 and C', where i_2 is the repetition code with generator matrix [1 1] and C' is a Euclidean self-dual code over $GF(4)$ of length $2n-2$.*

3 Equivalence between Euclidean codes over $GF(4)$

We recall that two Euclidean codes C_1 and C_2 of length n over $GF(4)$ are equivalent if there is a permutation of coordinates which sends C_1 onto C_2. We associate to such a permutation of length n a permutation of length $2n$

as follows because a direct checking of equivalence of two codes over $GF(4)$ seems to be hard.

Let $\beta : GF(4) \to GF(2)^2$ be defined as $\beta(0) = (0,0)$, $\beta(1) = (1,1)$, $\beta(\omega) = (1,0)$, and $\beta(\overline{\omega}) = (0,1)$. For $\mathbf{x} = (x_1, \cdots, x_n) \in GF(4)^n$ we define $\beta(\mathbf{x}) = (\beta(x_1), \cdots, \beta(x_n))$. If two Euclidean codes C_1 and C_2 over $GF(4)$ are equivalent, then clearly $\beta(C_1)$ and $\beta(C_2)$ are equivalent. Let T_n be the permutation group on $2n$ elements generated by $\alpha_1 = (1\ 3\ 5 \cdots 2n-1)(2\ 4\ 6 \cdots 2n)$ and $\alpha_2 = (1\ 3)(2\ 4)$. Then T_n is isomorphic to S_n (the symmetric group on n elements). We observe that given a Euclidean code C over $GF(4)$ of length n and its binary image $\beta(C) = \mathcal{B}$, the permutations of coordinates of C correspond to the permutations of $\mathcal{B}$ generated by α_1 and α_2. Thus we have the following proposition.

Proposition 3.1. *Let C be a Euclidean code over $GF(4)$ of length n associated to the binary code $\beta(C) = \mathcal{B}$. Then $PAut(C) = Aut(\mathcal{B}) \cap T_n$.*

Lemma 3.2 ([8]). *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be binary codes with a permutation P such that $\mathcal{B}_1 P = \mathcal{B}_2$. A permutation Q satisfies $\mathcal{B}_1 Q = \mathcal{B}_2$ if and only if $Q \in Aut(\mathcal{B}_1)P$, a right coset of $Aut(\mathcal{B}_1)$ in the symmetric group on the length of $\mathcal{B}_1$.*

Using Lemma 3.2 we have a way to check equivalence as follows and this was implemented in Magma.

Proposition 3.3. *Let C_1 and C_2 be Euclidean codes over $GF(4)$ of length n associated to binary codes $\beta(C_1) = \mathcal{B}_1$ and $\beta(C_2) = \mathcal{B}_2$ of length $2n$. Suppose P is a permutation on $2n$ elements such that $\mathcal{B}_1 P = \mathcal{B}_2$. C_1 and C_2 are equivalent as Euclidean codes over $GF(4)$ if and only if $Aut(\mathcal{B}_1)P \cap T_n \neq \emptyset$.*

4 Classification of Type I codes of lengths up to 12

We give the mass formula for Type I codes over $GF(4)$ of length n.

Proposition 4.1. *Let $N(n)$ be the number of Type I codes over $GF(4)$ of length n. Then $N(2) = 1$. $N(4) = 3$ and for any even $n \geq 6$,*

$$N(n) = (4^{\frac{n}{2}-1} - 1) \prod_{i=1}^{\frac{n}{2}-2} (4^i + 1) \text{ if } n \equiv 0 \pmod 4, \tag{3}$$

$$N(n) = \prod_{i=1}^{\frac{n}{2}-1} (4^i + 1) \text{ if } n \equiv 2 \pmod 4. \tag{4}$$

Proof. The first equality (3) and $N(4) = 3$ follow by subtracting the mass formula for Type II codes in [10] from the mass formula for Euclidean (Type I or Type II) self-dual codes in [23]. The second equality (4) and $N(2) = 1$ are just the mass formula for Euclidean (Type I or Type II) self-dual codes in [23]. □

A complete classification of binary self-dual codes of lengths ≤ 24 was given in [19],[21]. Using information there we can classify all Type I codes over $GF(4)$ of lengths up to 10 and extremal Type I codes of length 12. We observe the following lemma which is used in determining indecomposable codes.

Lemma 4.2. *If the Gray image of a self-dual code C over $GF(4)$ is indecomposable, so is C. If the Gray image of a self-dual code C over $GF(4)$ is decomposable and each component of the image is also the image of a smaller code over $GF(4)$, then C is decomposable.*

4.1 Lengths 2 and 4

It is clear that there is only one Type I code of length 2 whose generator matrix is [1 1] . Its binary image is equivalent to C_2^2 [19]. We now consider $n = 4$. There is a unique Euclidean self-dual Type I code $C_{4,1}$ of length 4. This is a cyclic code with generator (1010). We have verified it by using the mass formula as follows. The group order of $C_{4,1}$ is 8 as in Table 2, where A_i denotes the number of codewords with Lee weight i, |PAut| denotes the order of the permutation automorphism group of the corresponding code C over $GF(4)$, $\phi(C)$ is the Gray image of C, and 'de' means decomposable and 'in' indecomposable. Hence we check that $4!/|\mathrm{PAut}(C_{4,1})| = 3 = N(4)$. There is a unique Type II code [10], denoted by $C_{4,2}$ here. It is generated by $\{(10\omega\bar{\omega}), (01\bar{\omega}\omega)\}$. We remark that $C_{4,2}$ is a Reed-Solomon [4, 2] code over $GF(4)$ with $d_L = 4$ and $d_H = 3$.

4.2 Length 6

Using Theorem 2.4 with generator matrices of $C_{4,1}$ and $C_{4,2}$, we obtain three inequivalent codes, denoted by $C_{6,1}, C_{6,2}$, and $C_{6,3}$. $C_{6,1}$ is a cyclic code with generator (100100). $C_{6,2}$ is generated by $\{(100100), (0100\omega\bar{\omega}), (0010\bar{\omega}\omega)\}$ and $C_{6,3}$ by $\{(100\omega\bar{\omega}0), (0101\bar{\omega}\bar{\omega}). (001\omega1\omega)\}$. We remark that the code $C_{6,3}$ is equivalent to C_6 in [2]. We compute the Lee weight distribution and group order in Table 2. We checked that

$$\frac{6!}{2^4 \cdot 3} + \frac{6!}{2^3 \cdot 3} + \frac{6!}{2 \cdot 3^2} = 85 = N(6).$$

It is easy to see that $d_L = 4$ if and only if $d_H = 3$ or 4. The following is proved.

Theorem 4.3. *There are exactly three Euclidean self-dual Type I codes of length 6, one of which is an extremal* $[6, 3]$ *code with* $d_L = 4$ *and* $d_H = 3$.

4.3 Length 8

We denote a generator matrix of a code C by $G(C)$. One Type I $[8, 4]$ code with $d_L = 4$ was given in [2]. We obtain six inequivalent Type I codes by using Theorem 2.4 with $G(C_{6,1})$ with $\mathbf{x} = (000\omega\omega 1), (000001)$, or $(0000\bar{\omega}\omega)$ and $G(C_{6,2})$ with $\mathbf{x} = (000111), (000\omega\omega 1)$, or (000001). We also construct two inequivalent Type II codes by the same method with $G(C_{6,1})$ with $\mathbf{x} = (000111)$ and $G(C_{6,3})$ with $\mathbf{x} = (000\bar{\omega}\omega 0)$. We denote these eight codes by $C_{8,1}, \cdots, C_{8,8}$ in the displayed order. We compute the Lee weight distribution and group order in Table 2. There are exactly two Type II codes of length 8 up to permutation-equivalence [10]. The codes $C_{8,7}$ and $C_{8,8}$ are such. We check that for the six Type I codes

$$\frac{8!}{32} + \frac{8!}{384} + \frac{8!}{36} + \frac{8!}{192} + \frac{8!}{18} + \frac{8!}{96} = 5355 = N(8).$$

Hence the following is proved.

Theorem 4.4. *There are exactly six Euclidean self-dual Type I codes of length 8, three of which are indecomposable and extremal* $[8, 4]$ *codes with* $d_L = 4$. *There are exactly two Type I codes of length* 8 *with* $d_H = 4$ *and one Type II code of length* 8 *with* $d_H = 4$.

4.4 Length 10

It is known [2] that there is a Type I $[10, 5]$ code with $d_L = 4$ whose Gray image is M_{20}. We obtain five inequivalent Type I $[10, 5]$ codes with $d_L = 4$, denoted by $C_{10,1}, \cdots, C_{10,5}$ from $C_{8,1}$ with

$$\mathbf{x} = (00000111), (0000\bar{\omega}\omega 11), (00001\bar{\omega}\omega 1), (0000\bar{\omega}0\bar{\omega}1), (0000\bar{\omega}\omega\omega\omega).$$

Similarly four such codes $C_{10,6}, \cdots, C_{10,9}$ are obtained from $C_{8,3}$ with $\mathbf{x} = (00000111)$, $C_{8,4}$ with $\mathbf{x} = (00001101)$, and $C_{8,5}$ with $\mathbf{x} = (00001\bar{\omega}\omega 1)$, or $(000000\bar{\omega}\omega)$, respectively. We compute the Lee weight distribution and group order in Table 3. We note that any Type I $[10, 5]$ code with $d_L = 2$ is a direct sum of $[1\ 1]$ and one of the eight codes in Section 4.3 by Theorem 2.7. There are eight inequivalent such codes, denoted by $\mathcal{D}_{10,i} = [1\ 1] \oplus C_{8,i}$ for $1 \le i \le 8$. The group orders of $\mathcal{D}_{10,i}$ $(1 \le i \le 8)$ are 64, 3840, 144, 384, 36, 576, 2688, and 576, respectively. We check that

$$\sum_{i=1}^{9} \frac{10!}{|\mathrm{PAut}(C_{8,i})|} + \sum_{i=1}^{8} \frac{10!}{|\mathrm{PAut}(\mathcal{D}_{8,i})|} = 1419925 = N(10).$$

Hence the following is proved.

Theorem 4.5. *There are exactly nine Type I extremal* $[10,5]$ *codes with* $d_L = 4$*, eight of which are indecomposable and four of which have* $d_H = 4$*. There are exactly eight Type I* $[10.5]$ *codes with* $d_L = 2$.

4.5 Length 12

We want to classify Type I codes of length 12. Since binary self-dual $[24k, 12k, 4k+4]$ codes are doubly-even [22], there is no extremal Type I $[12,6]$ code with $d_L = 8$. One Type I $[12,6]$ code with $d_L = 6$ was given in [2]. We note that Euclidean self-dual $[12,6]$ codes with $d_L = 6$ cannot be built from decomposable self-dual $[10,5]$ codes with $d_L = 2$. So by considering all possibilities of x with the nine $[10,5]$ codes $C_{10,i}$ $(1 \le i \le 9)$, we prove that there are exactly two Type I $[12,6]$ codes with $d_L = 6$, denoted by $C_{12,1}$ and $C_{12,2}$ up to equivalence. In fact $C_{12,1}$ and $C_{12,2}$ can be obtained from $C_{10,2}$ with $\mathbf{x} = (\bar{\omega}\bar{\omega}\omega\bar{\omega}111111)$ and $(\bar{\omega}1\omega\omega\omega11111)$, respectively. There exists a unique Type II $[12.6]$ code with $d_L = 8$ (cf. [2],[10]). This is permutation equivalent to the extended quadratic residue code of length 11 over $GF(4)$. We can reconstruct it from $C_{10,2}$ with $\mathbf{x} = (0\omega\omega\bar{\omega}1\bar{\omega}1111)$. See Table 3 for the Lee weight distribution and group order of these codes. Also by considering $C_{10,1}$ with all possibilities for x we obtain exactly 26 self-dual $[12,6]$ codes with $d_L = 4$ (available from the authors), 25 of which are Type I. We further obtain exactly 17 more inequivalent Type I $[12,6]$ codes with $d_L = 4$ and two more Type II $[12,6]$ codes with $d_L = 4$ from $C_{10,i}$ for $i = 2, \cdots .6$. Due to the computational problem by Magma, we stop considering more $[12,6]$ codes with $d_L = 4$.

Theorem 4.6. *There are exactly two Type I optimal* $[12,6]$ *codes with* $d_L = 6$*, both of which are indecomposable and have* $d_H = 5$*. There are at least* 42 *Type I* $[12,6]$ *codes with* $d_L = 4$.

Corollary 4.7. *A Euclidean self-dual* $[12,6]$ *code with* $d_H = 6$ *over* $GF(4)$ *is unique; it is permutation equivalent to the extended quadratic residue code of length* 11 *over* $GF(4)$.

Proof. If such a code C exists, then $d_L = 6$ or 8 by the fact that $d_H \le d_L$. Since there is no Type I $[12.6]$ code with $d_L = 6$ and $d_H = 6$ by Theorem 4.6, C should have $d_L = 8$ and be of Type II. Thus C is equivalent to the extended quadratic residue of length 11 by the uniqueness of a Type II $[12,6]$ code with $d_L = 8$. □

Remark 4.8. This corollary completes the entry $n = 12$ in Table 6 of [9] as only one code.

5 New extremal Type I codes of lengths $n \geq 14$

For lengths $n \geq 14$ we are mainly interested in extremal codes. For example, for $n = 16$ there are at least $\left(\prod_{i=1}^{7}(4^i + 1)\right)/16! \approx 4670$ Euclidean Type I or Type II self-dual codes over $GF(4)$.

5.1 Length 14

For the length 14 we construct as many optimal Type I codes of that length as possible. In Section 4.5 we showed that there are two optimal Type I $[12, 6]$ codes with $d_L = 6$ and one extremal Type II $[12, 6]$ code with $d_L = 8$. By attempting all possibilities of $\mathbf{x}$ with $\mathcal{C}_{12,2}$ we obtain exactly 25 inequivalent Type I $[14, 7]$ codes with $d_L = 6$. Further these codes have $d_H = 5$. We remark that one Type I $[14, 7]$ code with $d_L = 6$ was given in [2]. On the other hand there are 21 inequivalent Type I $[14, 7]$ codes with $d_L = 6$ from $\mathcal{C}_{12,1}$ and two inequivalent Type I $[14, 7]$ codes with $d_L = 6$ from $\mathcal{C}_{12,3}$. These are all equivalent to one of the 25 $[14, 7]$ codes ($d_L = 6$) from $\mathcal{C}_{12,2}$. Since self-dual $[14, 7]$ codes with $d_L = 6$ can also come from self-dual $[12, 6]$ codes with $d_L = 4$, it is possible to have more Type I $[14, 7]$ codes with $d_L = 6$. We state our result in the following and omit the detail.

Theorem 5.1. *There are at least 25 Type I optimal* $[14, 7]$ *codes with* $d_L = 6$ *and* $d_H = 5$. *There exist at least six* $[14, 7]$ *codes* ($d_L = 6$) *with trivial automorphism group.*

5.2 Length 16

It is known [2] that there exist at least one extremal Type I $[16, 8]$ code with $d_L = 8$ and at least four extremal Type II $[16, 8]$ codes with $d_L = 8$. Using the Type I $[14, 7]$ codes with $d_L = 6$, we construct five extremal Type I $[16, 8]$ codes with $d_L = 8$ and four extremal Type II $[16, 8]$ codes with $d_L = 8$. The Gray images of the five extremal Type I $[16, 8]$ codes ($d_L = 8$) produce two singly-even binary self-dual $[32, 16, 8]$ codes which are two of the three such codes in [4] or [8, Table 5]. We omit their generator matrices. Table 4 gives the Lee weight distribution and group order of these codes. Again using Type I $[14, 7]$ codes ($d_L = 6$), we construct at least 605 inequivalent Type I $[16, 8]$ codes with $d_L = 6$, most of them have a trivial automorphism. It is interesting to note that the Gray images of some of these $[16, 8]$ codes have an automorphism group of order 2. It is known that if there is a rigid binary self-dual code of length 32, then it is a self-dual $[32, 16, 6]$ code [18]. So it is possible to have a rigid binary self-dual code of length 32 which is the Gray image of some rigid Type I $[16,8]$ code over $GF(4)$ with $d_L = 6$.

Theorem 5.2. *There are at least five Type I extremal* $[16,8]$ *codes with* $d_L = 8$ *and* $d_H = 6$ *with distinct Hamming weight distribution. Their Gray images produce two singly-even binary self-dual* $[32,16,8]$ *codes. There are at least four Type II extremal* $[16,8]$ *codes with* $d_L = 8$ *and* $d_H = 6$ *whose Gray images are the quadratic residue* $[32,16,8]$ *code* q_{32}, *C84, or C85.*

5.3 Length 18

In [2], the first extremal Type I $[18,9]$ code ($d_L = 8$) was given. It has the Lee weight enumerator

$$W_{18,1}(y) = 1 + 225y^8 + 2016y^{10} + 9555y^{12} + \cdots$$

which is one of the two weight enumerators of extremal singly-even self-dual $[36,18,8]$ codes in [4]. Using D_{16} [2] with many possibilities of $\mathbf{x}$, we construct five extremal Type I $[18,9]$ codes ($d_L = 8$), all of which have the above weight enumerator. See Table 6 for the generators. Here vectors in the second column correspond to the right eight coordinates, the left half being 0's. In particular, the Gray image of $C_{18,4}$ gives a new extremal singly-even self-dual binary $[36,18,8]$ code with previously unknown automorphism group of order $384 = 2^7 \cdot 3$. It was shown [4],[13],[15] that there are at least 14 inequivalent singly-even self-dual binary [36,18,8] codes. We summarize our results as follows.

Theorem 5.3. *There are at least five inequivalent Type I* $[18,9]$ *codes* ($d_L = 8$) *over* $GF(4)$ *with* $W_{18,1}$ *and at least* 15 *inequivalent singly-even self-dual binary* $[36,18,8]$ *codes.*

5.4 Length 20

In [2], the first extremal Type I $[20,10]$ code with $d_L = 8$, D_{20}, was given. It is a pure double circulant code. It has the Lee weight enumerator

$$1 + 285y^8 + 1024y^{10} + 11040y^{12} + \cdots .$$

Using two codes with generator matrices $K_{18,1}$ and $K_{18,2}$ (see Table 5) with many possible vectors $\mathbf{x}$, we construct five new extremal Type I $[20,10]$ codes with $d_L = 8$, all of which have previously unknown Lee weight enumerators. See Table 6 for such codes. The possible Lee weight enumerator of a Type I $[20,10]$ code with $d_L = 8$ is as follows [4].

$$W_{20}(y) = 1 + (125 + 16\beta)y^8 + (1664 - 64\beta)y^{10} + \cdots .$$

The code D_{20} has the weight enumerator for $\beta = 10$. The five codes have Lee weight enumerators for

$$\beta = 0, 1, 2, 3, \text{ or } 4.$$

We summarize our results as follows.

Theorem 5.4. *There are at least six inequivalent Type I* $[20, 10]$ *codes with* $d_L = 8$ *and* W_{20} *for* $\beta = 0, 1, 2, 3, 4,$ *or* 10.

5.5 Length 22

In [10, Table II], one extremal Type I $[22, 11]$ code with $d_L = 8$ was given. The possible Lee weight enumerator of an extremal Type I $[22, 11]$ code ($d_L = 8$) is as follows [4].

$$W_{22,1}(y) \quad = \quad 1 + (44 + 4\beta)y^8 + (976 - 8\beta)y^{10} + \cdots$$

or

$$W_{22,2}(y) \quad = \quad 1 + (44 + 4\beta)y^8 + (1232 - 8\beta)y^{10} + \cdots .$$

Using D_{20} [2] with many possibilities of $\mathbf{x}$ we construct 12 new extremal Type I $[22, 11]$ codes ($d_L = 8$) with weight enumerator $W_{22,2}$ for

$$\beta = 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \text{ or } 22.$$

See Table 6 for such codes. We summarize our results as follows.

Theorem 5.5. *There are at least 12 inequivalent Type I* $[22, 11]$ *codes with* $d_L = 8$ *having distinct Lee weight enumerators.*

5.6 Length 34

There are two possible Lee weight enumerators of an extremal Type I $[34, 17]$ code over $GF(4)$ with $d_L = 12$ according to [6] as follows.

$$W_{34,1}(y) \quad = \quad 1 + (442 + 4\beta)y^{12} + (10864 - 8\beta)y^{14} + \cdots$$

or

$$W_{34,2}(y) \quad = \quad 1 + (442 + 4\beta)y^{12} + (14960 - 8\beta - 256\gamma)y^{14} + \cdots .$$

An extremal Type I $[34, 17]$ code with $d_L = 12$, $d_H = 10$, and $W_{34,1}$ for $\beta = 104$ was in [10, Table II]. By D_{32} [2] with $\mathbf{x} = (0 \cdots 0\omega\overline{\omega}\overline{\omega}1\overline{\omega}\omega00\omega1111111)$ of length 32, we construct an extremal Type I $[34, 17]$ code with $d_L = 12$ and $d_H = 9$ with the Lee weight enumerator $W_{34,2}$ for $\beta = 95$ and $\gamma = 1$. This weight enumerator is previously unknown (see [24], [14] and references therein). We summarize our results as follows.

Theorem 5.6. *There exists an extremal Type I [34, 17] code over $GF(4)$ with $d_L = 12$, $d_H = 9$ and the Lee weight enumerator $W_{34,2}$ for $\beta = 95$ and $\gamma = 1$. Its Gray image is a new singly-even self-dual binary extremal [68, 34, 12] code with weight enumerator $W_{34,2}$ for $\beta = 95$ and $\gamma = 1$, where $W_{34,2} := W_2$ in the notation of [14].*

6 Conclusion and open problems

We have classified Euclidean Type I codes over $GF(4)$ of lengths up to 10 and extremal Type I codes of length 12, and constructed many new extremal Type I codes of lengths from 14 to 22 and 34 efficiently by building-up smaller self-dual codes. As a byproduct, we construct a new extremal singly-even self-dual binary [36, 18, 8] code (recently classified in [7]) and a new extremal singly-even self-dual binary [68, 34, 12] code with a previously unknown weight enumerator. We also prove that a Euclidean self-dual [12, 6] code with $d_H = 6$ over $GF(4)$ is unique; it is permutation equivalent to the extended quadratic residue code of length 11 over $GF(4)$. (We remark that this result also follows from the fact that there is a unique linear [12, 6, 6] code over $GF(4)$. recently done by Gulliver et. al. [11].)

There are other interesting lengths $n \geq 14$ for which the existence of extremal codes is not known. We mention some problems here.

1. In [2], an example of Type I [24, 12] code over $GF(4)$ with $d_L = 8$ was given. The existence of an optimal Type I [24, 12] code with $d_L = 10$ is not known even though there exist several binary self-dual [48, 24, 10] codes.

2. The existence of an extremal Type I [28, 14] code over $GF(4)$ with $d_L = 12$ is interesting since if so then its Gray image will be an extremal singly-even self-dual [56, 28, 12] code whose existence is open.

3. In [2], a Type I [36, 18] code over $GF(4)$ with $d_L = 12$ was given. So far the existence of an optimal Type I [36, 18] code over $GF(4)$ with $d_L = 14$ is not known. If exists, it will imply an optimal binary self-dual [72, 36. 14] code whose existence is a long standing open problem.

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Table 1: The highest minimum Lee weight of a Euclidean self-dual code over $GF(4)$ of length n and its attainable Hamming weight

n	$d_L(I)$	$d_L(II)$	no. of codes	(d_H; no.)	d_H for our codes
2	2		1.	(2; 1)	
4	2	4	1. ; 1.[2]	(3; 1)	
6	4		1.	(3; 1)	
8	4	4	3. ; 2.[10]	(4; 3)	
10	4		9.	(4; 4)	
12	6	8	2. ; 1.[2]	(6; 1)	
14	6		≥ 25		$d_H = 5$
16	8	8	≥ 5 ; ≥ 5[2]		$d_H = 6$
18	8 .		≥ 5		$d_H = 6$
20	8	8	≥ 6 ; ≥ 1[2]		$d_H = 6$
22	8		≥ 12		$d_H = 6$
34	12		≥ 2		$d_H = 9$

Table 2: Lee weight distribution and group order of all Type I codes of lengths $n = 4, 6, 8$

codes C	A_0	A_2	A_4	A_6	A_8	|PAut|	$\phi(C)$	(in)de.	d_H
$C_{4,1}$	1	4	6	4	1	2^3	C_2^4	de	2
$C_{4,2}$	1		14		1	$2^2 \cdot 3$	A_8	in	3
$C_{6,1}$	1	6	15	20	15	$2^4 \cdot 3$	C_2^6	de	2
$C_{6,2}$	1	2	15	28	15	$2^3 \cdot 3$	$C_2^2 \oplus A_8$	de	2
$C_{6,3}$	1		15	32	15	$2 \cdot 3^2$	B_{12}	in	3
$C_{8,1}$	1		12	64	102	32	F_{16}	in	4
$C_{8,2}$	1	8	28	56	70	384	C_2^8	de	2
$C_{8,3}$	1	2	16	62	94	36	$C_2^2 \oplus B_{12}$	de	2
$C_{8,4}$	1		12	64	102	192	F_{16}	in	4
$C_{8,5}$	1		12	64	102	18	F_{16}	in	3
$C_{8,6}$	1	4	20	60	86	96	$C_2^4 \oplus A_8$	de	2
$C_{8,7}$	1		28		198	1344	A_8^2	de	3
$C_{8,8}$	1		28		198	288	A_8^2	de	4

Table 3: Lee weight distribution and group order of extremal Type I codes of lengths $n = 10, 12$

codes C	A_4	A_6	A_8	A_{10}	A_{12}	|PAut|	$\phi(C)$	(in)de.	d_H
$C_{10,1}$	13	64	242	384		48	S_{20}	in	4
$C_{10,2}$	5	80	250	352		10	M_{20}	in	4
$C_{10,3}$	5	80	250	352		24	M_{20}	in	4
$C_{10,4}$	13	64	242	384		18	S_{20}	in	3
$C_{10,5}$	9	72	246	368		12	R_{20}	in	3
$C_{10,6}$	9	72	246	368		72	R_{20}	in	4
$C_{10,7}$	17	56	238	400		504	L_{20}	in	3
$C_{10,8}$	9	72	246	368		81	R_{20}	in	3
$C_{10,9}$	29	32	226	448		216	$A_8 \oplus B_{12}$	de	3
$C_{12,1}$		64	375	960	1296	12	Z_{24}	in	5
$C_{12,2}$		64	375	960	1296	10	Z_{24}	in	5
$C_{12,3}$			759		2576	660	Golay Code	in	6

Table 4: Lee weight distribution and group order of extremal Type I or Type II codes of length $n = 16$ with $d_H = 6$

codes C	A_8	A_{10}	A_{12}	|PAut|	$\phi(C)$ or ref.	$\|\mathrm{Aut}(\phi(C))\|$	Type
$C_{16,1}$	364	2048	6720	8	[4],[8]	$2^{12} \cdot 3 \cdot 7$	I
$C_{16,2}$	364	2048	6720	14	[4],[8]	$2^{12} \cdot 3 \cdot 7$	I
$C_{16,3}$	364	2048	6720	128	[4],[8]	$2^{15} \cdot 3^2$	I
$C_{16,4}$	364	2048	6720	16	[4],[8]	$2^{12} \cdot 3 \cdot 7$	I
$C_{16,5}$	364	2048	6720	16	[4],[8]	$2^{12} \cdot 3 \cdot 7$	I
$C_{16,6}$	620		13888	24	$C85(f_2^{16})$	$2^9 \cdot 3^2 \cdot 5$	II
$C_{16,7}$	620		13888	16	$C85(f_2^{16})$	$2^9 \cdot 3^2 \cdot 5$	II
$C_{16,8}$	620		13888	8	$C81(q_{32})$	$2^5 \cdot 3 \cdot 5 \cdot 31$	II
$C_{16,9}$	620		13888	14	$C84(f_4^8)$	$2^{12} \cdot 3 \cdot 7$	II

Table 5: Generator matrices $K_{18,1}$ and $K_{18,2}$

$$K_{18,1} = \begin{bmatrix} 1000000000\omega\overline{\omega}111111 \\ 0100000001 1\omega 0\overline{\omega}11\overline{\omega\omega} \\ 00100000011\omega\omega 1\overline{\omega\omega}11 \\ 0001000000\overline{\omega}\omega\omega 11\omega 00 \\ 000010000\overline{\omega}1\overline{\omega}1\overline{\omega}11\overline{\omega}1 \\ 000001000\overline{\omega}1\overline{\omega\omega\omega}\omega 10\overline{\omega} \\ 0000001000\overline{\omega}\omega 00\omega\omega 11 \\ 000000010100\overline{\omega}\omega 0\omega 0\overline{\omega} \\ 0000000010\overline{\omega}\omega 11\omega 00\omega \end{bmatrix}, K_{18,2} = \begin{bmatrix} 1000000000001\overline{\omega}1\omega 11 \\ 010000000\overline{\omega}\omega 1\omega 00\omega 10 \\ 001000000\omega\overline{\omega}0110\overline{\omega\omega}0 \\ 000100000\omega\omega 10\overline{\omega}\omega\omega 0\overline{\omega} \\ 00001000010\omega\overline{\omega\omega\omega}1\omega\omega \\ 00000100010\omega 1\overline{\omega}0110 \\ 000000100\overline{\omega}1\overline{\omega\omega\omega\omega}\omega 10 \\ 00000001010\overline{\omega\omega}1\omega\omega\omega\overline{\omega} \\ 0000000010\overline{\omega}11\omega 00\omega\omega \end{bmatrix}$$

Table 6: New extremal Type I codes of lengths $n = 18, 20, 22$, all with $d_L = 8$ and $d_H = 6$

codes C	x with left half 0's	β	|PAut|	$\|Aut(\phi(C))\|$	using	$W(y)$
$C_{18,1}$	$0\overline{\omega}0\omega1111$		1	$2 \cdot 3$	D_{16} [2]	$W_{18,1}$
$C_{18,2}$	$0\overline{\omega\omega}\omega\omega111$		3	$2 \cdot 3$	D_{16}	$W_{18,1}$
$C_{18,3}$	$101\overline{\omega\omega}111$		2	$2^3 \cdot 3$	D_{16}	$W_{18,1}$
$C_{18,4}$	$001\overline{\omega}1\omega11$		1	$2^7 \cdot 3$	D_{16}	$W_{18,1}$
$C_{18,5}$	$\overline{\omega\omega}\omega01\omega11$		1	$2^3 \cdot 3$	D_{16}	$W_{18,1}$
$C_{20,1}$	$\omega0\overline{\omega\omega}\omega\overline{\omega}\omega11$	1	1	1	$K_{18,1}$	W_{20}
$C_{20,2}$	$110\omega10\omega11$	2	1	1	$K_{18,1}$	W_{20}
$C_{20,3}$	$\overline{\omega\omega\omega}1\omega\overline{\omega}\omega11$	3	1	1	$K_{18,1}$	W_{20}
$C_{20,4}$	$111\overline{\omega}10\omega11$	4	1	1	$K_{18,1}$	W_{20}
$C_{20,5}$	$\omega\omega\omega\omega11111$	0	1	1	$K_{18,2}$	W_{20}
$C_{22,1}$	$\overline{\omega}0\omega\omega\overline{\omega}\omega\omega010$	10	1	1	D_{20} [2]	$W_{22,2}$
$C_{22,2}$	$1\omega\overline{\omega}10\omega010\overline{\omega}$	12	1	1	D_{20}	$W_{22,2}$
$C_{22,3}$	$\omega\overline{\omega}1001\omega0\overline{\omega}1$	13	1	1	D_{20}	$W_{22,2}$
$C_{22,4}$	$\overline{\omega\omega}\omega101001\omega$	14	1	1	D_{20}	$W_{22,2}$
$C_{22,5}$	$\omega\omega0\omega0\overline{\omega\omega}01\omega$	15	1	1	D_{20}	$W_{22,2}$
$C_{22,6}$	$\overline{\omega}0\omega1\overline{\omega}0\omega\omega0\omega$	16	1	1	D_{20}	$W_{22,2}$
$C_{22,7}$	$0\overline{\omega}1\overline{\omega}001\overline{\omega\omega}1$	17	1	1	D_{20}	$W_{22,2}$
$C_{22,8}$	$0101\omega\omega1101$	18	1	1	D_{20}	$W_{22,2}$
$C_{22,9}$	$\overline{\omega}0\omega\overline{\omega}0\overline{\omega}1\overline{\omega}0\omega$	19	1	1	D_{20}	$W_{22,2}$
$C_{22,10}$	$\overline{\omega\omega}00\overline{\omega\omega}0\overline{\omega}1\overline{\omega}$	20	1	1	D_{20}	$W_{22,2}$
$C_{22,11}$	$11\overline{\omega}0\overline{\omega\omega}010\overline{\omega}$	21	1	1	D_{20}	$W_{22,2}$
$C_{22,12}$	$\overline{\omega}1\overline{\omega}000\omega1\omega1$	22	2	4	D_{20}	$W_{22,2}$