Packings and coverings of λK_v by the graphs with seven points, seven edges and one 5-circle*

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Abstract. Let λK_v be the complete multigraph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let G be a finite simple graph. A G-packing design (G-covering design) of λK_v, denoted by (v, G, λ)-PD ((v, G, λ)-CD) is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in at most (at least) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design has more (fewer) blocks. There are four graphs with 7 points,7 edges and a 5-circle, denoted by $G_i, i = 1, 2, 3, 4$. In this paper, we have solved the existence problem of the maximum (v, G_i, λ)-PD and the minimum (v, G_i, λ)-CD.

Keywords: G-packing design, G-covering design, G-holey design, G-incomplete design.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. A *t-partite graph* is one whose vertex set can be partitioned

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into t subsets $X_1, X_2, \cdots, X_t$, such that two ends of each edge lie in distinct subsets respectively. Such a partition $(X_1, X_2, \cdots, X_t)$ is called a *t-partition* of the graph. A *complete t-partite graph* with replication λ is a t-partite graph with t-partition $(X_1, X_2, \cdots, X_t)$, in which each vertex of X_i is joined to each vertex of X_j by λ times (where $i \neq j$). Such a graph is denoted by $\lambda K_{n_1, n_2, \cdots, n_t}$ if $|X_i| = n_i$ $(1 \leq i \leq t)$.

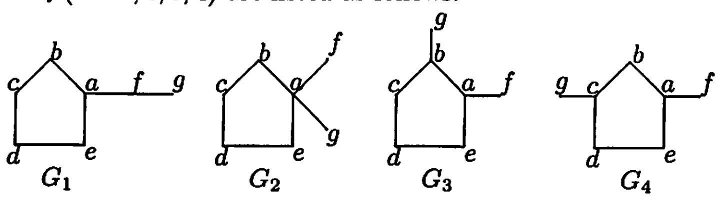
Let G be a finite simple graph. A *G-packing design* (*G-covering design*, *G-design*) of λK_v, denoted by (v, G, λ)-PD ((v, G, λ)-CD, (v, G, λ)-GD), is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in at most (at least, exactly) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be *maximum* (*minimum*) if no other such packing (covering) design has more (fewer) blocks. The number of blocks in a maximum packing designs (minimum covering design), denoted by $p(v, G, \lambda)$ ($c(v, G, \lambda)$), is called the *packing* (*covering*) *number*. It is well known that

$$p(v, G, \lambda) \leq \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor \leq \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil \leq c(v, G, \lambda),$$

where $e(G)$ denotes the number of edges in G, $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (least) integer y such that $y \leq x$ ($y \geq x$). A (v, G, λ)-PD ((v, G, λ)-CD), $(X,\ \mathcal{B})$, is called *optimal* if $|\mathcal{B}|$=$p(v, G, \lambda)$ ($c(v, G, \lambda)$). For convenience, we denote an optimal (v, G, λ)-PD ((v, G, λ)-CD) satisfying $p(v, G, \lambda) = \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor$ ($c(v, G, \lambda) = \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil$) by (v, G, λ)-OPD ((v, G, λ)-OCD). Obviously, there exists a (v, G, λ)-GD if and only if $p(v, G, \lambda) = c(v, G, \lambda)$. So a (v, G, λ)-GD can be regarded as a (v, G, λ)-OPD or a (v, G, λ)-OCD. The *leave graph* $L_\lambda(\mathcal{D})$ of a packing design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. Especially, when $\mathcal{D}$ is optimal, $|L_\lambda(\mathcal{D})|$ is called *leave-edge number* and is denoted by $l_\lambda(v)$. Similarly, the *excess graph* $R_\lambda(\mathcal{D})$ of a covering design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is optimal, $|R_\lambda(\mathcal{D})|$ is called the *repeat-edge number* and denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{D})$, $l_\lambda(v)$, $R_\lambda(\mathcal{D})$ and $r_\lambda(v)$ can be denoted by L_λ, l_λ, R_λ and r_λ briefly. For some graphs, which have less vertices and less edges, the problem of their graph designs, packing designs and covering designs has been researched (see [1-9], [12-18]). Let graph G has six vertices and its edge number not greater than 6. The G-designs, maximum G-packings and minimum G-coverings of λK_v were solved completely by Liang and Yin et al. (see [19-23]).

Let $(X_1, X_2, \cdots, X_t)$ be the t-partition of $\lambda K_{n_1,n_2,\cdots,n_t}$, and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G}=\{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1,n_2,\cdots,n_t}$ can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$HD_\lambda(1^{v-w}w^1)$ is called an *incomplete G-design*, denoted by G-$ID_\lambda(v;w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a (v, G, λ)-GD is a G-$HD_\lambda(1^v)$ or a G-$ID_\lambda(v;w)$ with $w = 0$ or 1. For HD_λ and ID_λ, the subscript can be omitted when $\lambda = 1$.

There are four graphs with 7 points,7 edges and a 5-circle, denoted by $G_i, i = 1, 2, 3, 4$. In this paper, we have solved the existence problem of the maximum (v, G_i, λ)-PD and the minimum (v, G_i, λ)-CD. The existence spectrums of (v, G_i, λ)-GD have been obtained in [10] and [11]. The four graphs G_i $(i = 1, 2, 3, 4)$ are listed as follows.



For convenience, the graphs G_1-G_4 above are denoted by (a, b, c, d, e, f, g).

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, m, λ and nonnegative integer w, if there exist G-$HD_\lambda(h^m)$, G-$ID_\lambda(h + w; w)$ and*

(w,G,λ)-OPD (or $(h+w,G,\lambda)$-OPD), then there exists $(mh+w,G,\lambda)$-OPD with the same leave graph to (w,G,λ)-OPD's (or $(h+w,G,\lambda)$-OPD's). The conclusion still holds by replacing OPD with OCD.

Proof. Let $X=(Z_h\times Z_m)\bigcup W$, where W is a w-set. Suppose there exist

$G\text{-}HD_\lambda(h^m)=(Z_h\times Z_m,\mathcal{A})$,

$G\text{-}ID_\lambda(h+w;w)=((Z_h\times\{i\})\bigcup W,\mathcal{B}_i)$, $i\in Z_m$ or $i\in Z_m\setminus\{0\}$, and

$(w,G,\lambda)\text{-}OPD=(W,\mathcal{C})$ or $(h+w,G,\lambda)\text{-}OPD=((Z_h\times\{0\})\bigcup W,\mathcal{D})$,

then (X,Ω) is a $(mh+w,G,\lambda)$-OPD, where $\Omega=\mathcal{A}\bigcup(\bigcup_{i=0}^{m-1}\mathcal{B}_i)\bigcup\mathcal{C}$ or $\mathcal{A}\bigcup(\bigcup_{i=1}^{m-1}\mathcal{B}_i)\bigcup\mathcal{D}$. Note that

$$|\Omega| = \left\lfloor\frac{\lambda\binom{mh+w}{2}}{e(G)}\right\rfloor=\begin{cases}\frac{\lambda\binom{m}{2}h^2}{e(G)}+m\times\frac{\lambda(\binom{h}{2}+wh)}{e(G)}+\left\lfloor\frac{\lambda\binom{w}{2}}{e(G)}\right\rfloor\\ \frac{\lambda\binom{m}{2}h^2}{e(G)}+(m-1)\times\frac{\lambda(\binom{h}{2}+wh)}{e(G)}+\left\lfloor\frac{\lambda\binom{w+h}{2}}{e(G)}\right\rfloor\end{cases}$$

$$=\begin{cases}|\mathcal{A}|+\sum_{i=0}^{m-1}|\mathcal{B}_i|+|\mathcal{C}|\\ |\mathcal{A}|+\sum_{i=1}^{m-1}|\mathcal{B}_i|+|\mathcal{D}|\end{cases},$$

if $(W,\mathcal{C})$ $(((Z_h\times\{0\})\bigcup W,\mathcal{D}))$ is a (w,G,λ)-OCD ($(h+w,G,\lambda)$-OCD), then a $(mh+w,G,\lambda)$-OCD will be obtained, since the above equation still holds by replacing the symbol $\lfloor\ \rfloor$ by $\lceil\ \rceil$. □

Lemma 2.2 [10] *There exists a G_i-$HD(7^{2t+1})$ for $i=1,2,3,4$.*

Lemma 2.3 [11] *There exist G_i-$ID(7+w;w)$ for $2\le w\le 6$ and $9\le w\le 13$, where $i=1,2,3,4$.*

Lemma 2.4 [9] *Given positive integers v, λ, and μ. Let X be a v-set.*

(1) *Suppose there exist both a (v,G,λ)-OPD $=(X,\mathcal{D})$ (with leave graph $L_\lambda(\mathcal{D})$) and a (v,G,μ)-OPD $=(X,\mathcal{E})$ (with leave graph $L_\mu(\mathcal{E})$). If $|L_\lambda(\mathcal{D})|+|L_\mu(\mathcal{E})|=l_{\lambda+\mu}$, then there exists a $(v,G,\lambda+\mu)$-OPD and its leave graph is just $L_\lambda(\mathcal{D})\bigcup L_\mu(\mathcal{E})$;*

(2) *Suppose there exist both a (v,G,λ)-OCD $=(X,\mathcal{D})$ (with excess graph $R_\lambda(\mathcal{D})$) and a (v,G,μ)-$OCD=(X,\mathcal{E})$ (with excess graph $R_\mu(\mathcal{E})$). If $|R_\lambda(\mathcal{D})|+|R_\mu(\mathcal{E})|=r_{\lambda+\mu}$, then there exists a $(v,G,\lambda+\mu)$-OCD and its*

excess graph is just $R_\lambda(\mathcal{D})\bigcup R_\mu(\mathcal{E})$;

(3) *Suppose there exist both a* (v,G,λ)*-OPD* $=(X,\mathcal{D})$ *(with leave graph* $L_\lambda(\mathcal{D})$*) and a* (v,G,μ)*- OCD* $=(X,\mathcal{E})$ *(with excess graph* $R_\mu(\mathcal{E})$*). If* $L_\lambda(\mathcal{D})\supset R_\mu(\mathcal{E})$ *and* $|L_\lambda(\mathcal{D})|-|R_\mu(\mathcal{E})|=l_{\lambda+\mu}$*, then there exists a* $(v,G,\lambda+\mu)$*-OPD and its leave graph is just* $L_\lambda(\mathcal{D})\backslash R_\mu(\mathcal{E})$;

(4) *Suppose there exist both a* (v,G,λ)*-OCD* $=(X,\mathcal{D})$ *(with excess graph* $R_\lambda(\mathcal{D})$*) and a* (v,G,μ)*-OPD* $=(X,\mathcal{E})$ *(with leave graph* $L_\mu(\mathcal{E})$*). If* $R_\lambda(\mathcal{D})\supset L_\mu(\mathcal{E})$ *and* $|R_\lambda(\mathcal{D})|-|L_\mu(\mathcal{E})|=r_{\lambda+\mu}$*, then there exists a* $(v,G,\lambda+\mu)$*-OCD and its excess graph is just* $R_\lambda(\mathcal{D})\backslash L_\mu(\mathcal{E})$.

3 Packing and covering for $\lambda=1$

The existence spectrums of (v,G_i,λ)-GD are as follows(see Lemmas 3.1,3.2).

Lemma 3.1 [10-11] *For* $i=1,3,4$*, there exist* (v,G_i,λ)*-GD if and only if* $\lambda v(v-1)\equiv 0 \pmod{14}$ *and* $v\geq 7$.

Lemma 3.2 [10-11] *There exist* (v,G_2,λ)*-GD if and only if* $\lambda v(v-1)\equiv 0 \pmod{14}$*,* $v\geq 7$ *and* $(v,\lambda)\neq(7,1)$.

In order to construct the optimal packing designs and covering designs for $\lambda=1$, by Theorem 2.1, Lemma 3.1, Lemma 3.2 and the following tables, we only need to give the constructions of HD, ID and OPD for the pointed orders, where the leave graph of OPD has to be a subgraph of G_i. However, the needed HD and ID have been shown in [10-11], so we only need to construct the OPD listed in the Table 3.1.

(Table 3.1) For G_i, $i=1,2,3,4$

v (*mod* 14)	HD	ID	OPD ($\lambda=1$)
2	7^{2t+1}	(16; 9)	9^*
3	7^{2t+1}	(17; 10)	10
4	7^{2t+1}	(18; 11)	11
5	7^{2t+1}	(19; 12)	12
6	7^{2t+1}	(20; 13)	13
9	7^{2t+1}	(9 ; 2)	9^*
10	7^{2t+1}	(10; 3)	10
11	7^{2t+1}	(11; 4)	11
12	7^{2t+1}	(12; 5)	12
13	7^{2t+1}	(13; 6)	13

$*$: $(9, G_i, 1)$-OPD=G_i-$ID(9;2)$.

Lemma 3.3 *There exist* $(w, G_1, 1)$-OPD *for* $w = 9, 10, 11, 12, 13$.

Proof. Let $(w, G_1, 1)$-OPD=$(X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \bigcup \{a, b\}$,

$(0,4,1,3,6,5,2)$, $(1,a,4,5,6,0,b)$, $(2,0,a,5,b,4,3)$, $(3,b,4,6,2,5,1)$, $(a,2,1,b,6,3,0)$.

$L(\mathcal{B}) = \{(a,b)\}$.

$\underline{w=10}$: $X = Z_6 \bigcup \{a, b, c, d\}$,

$(0,3,b,5,2,1,a)$, $(1,b,4,5,c,3,d)$, $(2,d,4,0,b,1,5)$, $(3,c,a,0,5,2,4)$, $(a,3,4,c,2,d,b)$, $(d,1,4,a,5,0,c)$.

$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

$\underline{w=11}$: $X = Z_5 \bigcup \{a, b, c, d, e, f\}$,

$(0,d,2,4,a,e,1)$, $(1,b,4,3,a,d,f)$, $(2,1,c,0,3,e,4)$, $(3,f,1,4,c,b,0)$, $(4,f,c,2,0,d,3)$, $(a,c,e,b,d,2,f)$, $(f,0,1,3,e,b,2)$

$L(\mathcal{B}) = \{(a,b),(b,c),(c,d),(d,e),(a,e),(a,f)\}$.

$\underline{w=12}$: $X = Z_8 \bigcup \{a, b, c, d\}$,

$(0,1,d,2,b,3,a)$, $(1,7,3,4,b,2,a)$, $(2,3,6,7,c,4,a)$, $(3,d,b,6,c,1,a)$, $(4,c,5,7,d,0,a)$, $(5,d,6,4,1,0,c)$, $(6,0,7,2,5,1,c)$, $(7,b,3,5,4,a,c)$, $(a,6,2,0,d,5,b)$.

$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

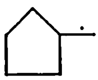
$\underline{w=13}$: $X = Z_{11} \bigcup \{a, b\}$,

$(1,a,3,4,6,0,7)$, $(2,10,3,b,8,9,4)$, $(3,6,9,8,5,1,2)$, $(4,b,10,8,0,5,9)$, $(5,0,10,6,2,a,8)$, $(6,b,9,10,5,7,3)$, $(7,5,b,0,2,10,4)$, $(8,3,9,0,6,1,5)$, $(9,1,4,8,7,a,6)$, $(a,0,3,2,4,7,1)$, $(b,1,10,a,2,7,4)$.

$L(\mathcal{B}) = \{(a,b)\}$. □

Theorem 3.4 *There exist* $(v, G_1, 1)$-OPD *and* $(v, G_1, 1)$-OCD *for* $v \geq 7$.

Proof. By Theorem 2.1, Lemma 2.2, Lemma 2.3 and Lemma 3.3. The leave graphs L_1 for these $OPDs$ are as follows:

v ≡ (mod 7)	2, 6	3, 5	4
L_1	P_2	P_4	

Obviously, each L_1 is a subgraph of the graph G_1. So, the optimal covering designs can be obtained by adding a block containing this L_1. And their excess graph R_1 can be listed in the table:

$v \equiv \pmod 7$	2, 6	3, 5	4
R_1		P_5	P_2

.

□

Lemma 3.5 $p(7, G_2, 1) = 2,\ c(7, G_2, 1) = 4.$

Proof. We know that there is no $(7, G_2, 1)$-GD (see [10]). Therefore, the packing number $p(7, G_2, 1) < 3$ and the covering number $c(7, G_2, 1) > 3$. In fact, there exist a maximum $(7, G_2, 1)$-$PD = (Z_7, \mathcal{B})$ and a minimum $(7, G_2, 1)$-$CD = (Z_7, \mathcal{C})$ as follows:

$$\begin{array}{l}\mathcal{B} = \{(0,5,1,6,4,2,3),\ (2,3,1,0,6,4,5)\},\\ L(\mathcal{B}) = \{(1,2),(1,4),(3,4),(3,5),(3,6),(4,5),(5,6)\};\\ \mathcal{C} = \mathcal{B} \bigcup \{(4,1,2,6,3,0,5),\ (5,0,1,2,4,3,6)\},\\ R(\mathcal{C}) = \{(0,1),(0,4),(0,5),(1,2),(2,4),(2,6),(4,5)\}.\end{array}$$

So, $p(7, G_2, 1) = 2$ and $c(7, G_2, 1) = 4$. □

Lemma 3.6 *There exist* $(w, G_2, 1)$-OPD *for* $w = 9, 10, 11, 12, 13$.

Proof. Let $(w, G_2, 1)$-OPD=$(X, \mathcal{B})$.

$\underline{w = 9}$: $X = Z_7 \bigcup \{a, b\}$,
$(4,0,3,2,1,b,5)$, $(5,1,3,4,2,a,6)$, $(6,b,2,a,0,3,4)$, $(a,3,5,0,1,4,6)$, $(b,0,2,6,1,3,5)$.
$L(\mathcal{B}) = \{(a,b)\}$.

$\underline{w = 10}$: $X = Z_6 \bigcup \{a, b, c, d\}$,
$(0,c,4,b,d,a,1)$, $(1,a,5,2,b,3,c)$, $(2,0,b,3,c,a,4)$, $(3,5,d,4,a,0,2)$, $(4,1,2,d,3,5,0)$, $(5,1,d,a,c,0,b)$.
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

$\underline{w = 11}$: $X = Z_5 \bigcup \{a, b, c, d, e, f\}$,
$(0,d,2,e,c,a,f)$, $(1,f,d,b,0,4,2)$, $(2,f,e,3,b,a,0)$, $(3,0,e,4,c,2,1)$, $(4,b,1,d,3,0,a)$, $(a,c,2,4,d,3,1)$, $(f,b,e,1,c,3,4)$
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d),(d,e),(a,e),(a,f)\}$.

$\underline{w = 12}$: $X = Z_8 \bigcup \{a, b, c, d\}$,
$(0,d,b,6,4,5,c)$, $(1,d,5,2,0,b,4)$, $(2,a,7,1,c,6,3)$, $(3,c,7,2,1,a,5)$, $(4,5,1,6,a,b,2)$, $(5,6,3,4,c,b,a)$, $(6,d,3,b,0,c,7)$, $(7,4,d,2,b,0,5)$, $(a,0,3,7,d,c,1)$.
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

$\underline{w = 13}$: $X = Z_{11} \bigcup \{a, b\}$,

$(1,6,3,4,8,a,10),(2,b,1,7,10,4,3),(3,a,2,6,10,b,1),(4,7,a,0,1,5,b),$
$(5,10,9,2,0,7,1),(6,a,8,10,4,9,7),(7,0,6,5,2,b,3),\ (8,5,b,0,9,6,7),$
$(9,1,2,8,3,5,7),\ (a,10,0,3,5,4,9),\ (b,9,4,0,8,10,6).$
$L(\mathcal{B})=\{(a,b)\}.$ □

Theorem 3.7 *There exist $(v,G_2,1)$-OPD and $(v,G_2,1)$-OCD for $v\geq 8$. And, $p(7,G_2,1)=2$ and $c(7,G_2,1)=4$.*

Proof. It is easy to prove by Theorem 2.1, Lemma 2.2, Lemma 2.3, Lemma 3.5 and Lemma 3.6. Note that the leave graphs L_1 for $(v,G_2,1)$-OPD are same to $(v,G_1,1)$-OPD. Further proof is similar to Theorem 3.4. □

Lemma 3.8 *There exist $(w,G_3,1)$-OPD for $w=9,10,11,12,13$.*

Proof. Let $(w,G_3,1)$-OPD=$(X,\mathcal{B})$.

$\underline{w=9}$: $X=Z_7\bigcup\{a,b\}$,
$(a,0,5,b,4,2,6),\ (a,1,0,4,6,3,5),\ (b,2,3,6,1,0,4),\ (4,3,0,2,5,1,b),$
$(6,5,3,1,2,b,a).$
$L(\mathcal{B})=\{(a,b)\}.$

$\underline{w=10}$: $X=Z_6\bigcup\{a,b,c,d\}$,
$(0,a,5,4,3,d,1),\ (1,3,5,2,0,c,b),\ (2,c,5,b,1,d,3),$
$(3,d,b,4,2,a,1),\ (4,a,2,b,0,1,c),\ (5,d,4,c,0,1,a).$
$L(\mathcal{B})=\{(a,b),(b,c),(c,d)\}.$

$\underline{w=11}$: $X=Z_5\bigcup\{a,b,c,d,e,f\}$,
$(0,c,1,b,d,e,3),\ (0,a,2,1,3,f,d),\ (4,f,c,e,b,3,1),\ (2,f,d,1,e,3,b),$
$(3,d,4,0,b,e,2),\ (4,a,3,f,e,2,1),\ (2,c,4,1,0,b,a)$
$L(\mathcal{B})=\{(a,b),(b,c),(c,d),(d,e),(a,e),(a,f)\}.$

$\underline{w=12}$: $X=Z_8\bigcup\{a,b,c,d\}$,
$(0,a,3,5,4,1,d),\ (1,a,5,0,7,2,4),\ (2,a,7,3,0,b,6),\ (3,d,0,b,6,c,7),$
$(4,d,6,0,c,3,5),\ (5,b,4,2,6,7,1),\ (6,c,2,5,1,4,a),\ (7,c,1,3,2,6,5),$
$(b,d,1,4,7,3,2).$
$L(\mathcal{B})=\{(a,b),(b,c),(c,d)\}.$

$\underline{w=13}$: $X=Z_{11}\bigcup\{a,b\}$,
$(1,a,0,2,10,b,5),(2,a,3,0,1,9,6),(3,b,10,6,4,9,7),(4,b,0,7,a,2,6),$
$(5,1,3,8,10,b,4),(6,2,5,4,0,9,b),(7,3,10,9,1,5,2),(8,4,10,7,2,1,9),$
$(9,5,0,8,b,a,6),\ (7,6,3,5,8,4,1),\ (9,8,a,10,0,7,6).$
$L(\mathcal{B})=\{(a,b)\}.$ □

Theorem 3.9 *There exist $(v,G_3,1)$-OPD and $(v,G_3,1)$-OCD for $v\geq 7$.*

Proof. It is easy to prove by Theorem 2.1, Lemma 2.2, Lemma 2.3 and Lemma 3.8. Note that the leave graphs L_1 for $(v,G_3,1)$-OPD are same to $(v,G_1,1)$-OPD. Further proof is similar to Theorem 3.4. □

Lemma 3.10 *There exist $(w, G_4, 1)$-OPD for $w = 9, 10, 11, 12, 13$.*

Proof. Let $(w, G_4, 1)$-OPD=$(X, \mathcal{B})$.

$\underline{w=9}$: $X = Z_7 \bigcup\{a, b\}$,
$(a,3,0,6,1,5,b)$, $(a,0,1,5,4,2,3)$, $(6,4,2,3,b,a,0)$, $(b,5,3,6,2,1,4)$, $(5,0,4,1,2,6,b)$.
$L(\mathcal{B}) = \{(a,b)\}$.

$\underline{w=10}$: $X = Z_6 \bigcup\{a,b,c,d\}$,
$(0,5,a,1,4,b,3)$, $(1,0,a,4,b,2,d)$, $(2,3,d,0,c,a,5)$, $(3,1,d,2,0,5,b)$, $(4,2,b,5,c,d,3)$, $(5,1,c,3,4,2,a)$.
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

$\underline{w=11}$: $X = Z_5 \bigcup\{a,b,c,d,e,f\}$,
$(0,c,a,3,d,4,1)$, $(f,e,c,3,0,4,2)$, $(1,b,d,4,c,0,a)$, $(2,b,e,1,3,d,4)$, $(3,4,a,2,e,b,0)$, $(4,2,f,d,1,b,3)$, $(f,b,0,2,1,c,e)$
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d),(d,e),(a,e),(a,f)\}$.

$\underline{w=12}$: $X = Z_8 \bigcup\{a,b,c,d\}$,
$(0,3,a,6,b,1,c)$, $(1,5,7,4,6,2,3)$, $(2,7,a,0,6,4,5)$, $(3,4,d,0,2,1,a)$, $(c,1,d,5,3,4,6)$, $(5,2,d,7,0,6,3)$, $(6,3,b,2,c,7,d)$, $(a,1,b,5,4,2,7)$, $(4,0,c,7,1,b,5)$.
$L(\mathcal{B}) = \{(a,b),(b,c),(c,d)\}$.

$\underline{w=13}$: $X = Z_{11} \bigcup\{a,b\}$,
$(1,0,a,2,7,5,3)$, $(2,10,b,8,0,4,1)$, $(3,2,1,10,0,6,8)$, $(4,6,2,9,10,8,5)$, $(5,b,3,7,10,8,1)$, $(6,0,4,3,10,1,a)$, $(7,0,5,4,9,b,3)$, $(8,7,6,5,a,2,b)$, $(9,5,7,a,6,b,4)$, $(a,10,8,3,9,1,6)$, $(b,0,9,1,4,2,8)$.
$L(\mathcal{B}) = \{(a,b)\}$. □

Theorem 3.11 *There exist $(v, G_4, 1)$-OPD and $(v, G_4, 1)$-OCD for $v \geq 7$.*

Proof. It is easy to prove by Theorem 2.1, Lemma 2.2, Lemma 2.3 and Lemma 3.10. Note that the leave graphs L_1 for $(v, G_4, 1)$-OPD are same to $(v, G_1, 1)$-OPD. Further proof is similar to Theorem 3.4. □

3.1 Packings and Coverings for $\lambda > 1$

Lemma 3.12 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $v \equiv 2, 6 \pmod 7$ and $\lambda > 1$ (where $i = 1, 2, 3, 4$).*

Proof. By Lemma 2.4, we have the following table:

λ	1	2	3	4	5	6	
l_λ	1	2	3	4	5	6	$(L_\lambda = L_1 \cup L_{\lambda-1})$
r_λ	6	5	4	3	2	1	$(R_\lambda = R_{\lambda-1} \backslash L_1)$

,

where $L_1 = P_2$ and R_1 is C_5 plus a pendant edge by Theorems 3.4, 3.7, 3.9

and 3.11. □

Lemma 3.13 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $v \equiv 3$, 5 (mod 7) and $\lambda > 1$ (where $i = 1, 2, 3, 4$).*

Proof. By Lemma 2.4, Theorems 3.4, 3.7, 3.9 and 3.11, we have the following table:

λ	1	2	3	4	5	6
l_λ	3	6	2	5	1	4
L_λ	P_4	$L_1 \cup L_1$	$L_1 \backslash R_2$	$L_1 \cup L_3$	$L_3 \backslash R_2$	$L_1 \cup L_5$
r_λ	4	1	5	2	6	3
R_λ	P_5	$R_1 \backslash L_1$	$R_1 \cup R_2$	$R_2 \cup R_2$	$R_2 \cup R_3$	$R_2 \cup R_4$

. □

Lemma 3.14 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for $v \equiv 4$ (mod 7) and $\lambda > 1$ (where $i = 1, 2, 3, 4$).*

Proof. By Lemma 2.4, we have the following table:

λ	1	2	3	4	5	6	
l_λ	6	5	4	3	2	1	$(L_\lambda = L_{\lambda-1} \backslash R_1)$
r_λ	1	2	3	4	5	6	$(R_\lambda = R_1 \cup R_{\lambda-1})$

,

where L_1 is C_5 plus a pendant edge and $R_1 = P_2$ by Theorems 3.4, 3.7, 3.9 and 3.11. □

Theorem 3.15 *There exist (v, G_i, λ)-OPD and (v, G_i, λ)-OCD for any $v \geq 7$ and $\lambda > 1$ (where $i = 1, 2, 3, 4$).*

Proof. By the results of graph design with index $\lambda > 1$ (see Lemma 3.1 and Lemma 3,2), and Lemmas 3.12, 3.13 and 3.14. □

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