THE SIGNED k-DOMINATION NUMBERS IN GRAPHS

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ABSTRACT. For any integer $k \geq 1$, a *signed (total) k-dominating function* is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{w \in N[v]} f(w) \geq k$ ($\sum_{w \in N(v)} f(w) \geq k$) for every $v \in V(G)$, where $N(v) = \{u \in V(G) | uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The minimum of the values of $\sum_{v \in V(G)} f(v)$, taken over all signed (total) k-dominating functions f, is called the *signed (total) k-domination number* and is denoted by $\gamma_{kS}(G)$ ($\gamma^t_{kS}(G)$, resp.). In this paper, several sharp lower bounds of these numbers for general graphs are presented.

1. INTRODUCTION

All graphs considered in this paper are finite and undirected without loops or multiple edges. For a general reference on graph theory, the reader is directed to [1].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighbourhood* $\{u \in V(G) | uv \in E(G)\}$ and the *closed neighbourhood* $\{v\} \cup \{u \in V(G) | uv \in E(G)\}$ of a vertex $v \in V(G)$ are denoted by $N(v)$ and $N[v]$, respectively. For a subset $S \subseteq V(G)$, $\deg_S(v)$ denotes the number of vertices in S adjacent to v. In particular, $\deg_{V(G)}(v) = \deg(v)$, the degree of v in G. For disjoint subsets S and T of vertices, we use $E(S, T)$ for the set of edges between S and T, and let $e(S, T) = |E(S, T)|$. The subgraph of G induced by S is denoted by $G[S]$. The complete graph on n vertices and its complement are denoted by K_n and $\overline{K}_n$, respectively. Let $x : V(G) \to \mathbb{R}$ be a real-valued function. We write $x(S)$ for $\sum_{v \in S} x(v)$ for $S \subseteq V(G)$.

Domination in graphs is well studied in graph theory. The literature on this subject has been detailed in the two books [7, 8]. The signed domination has been broadly studied in, for instance, [2, 3, 4, 5, 9, 10, 11, 12, 13, 15].

Let $k \geq 1$ be an integer and let G be a graph with minimum degree $k - 1$. A *signed k-dominating function* (SkDF) is a function $f : V(G) \to \{-1, 1\}$ satisfying $\sum_{w \in N[v]} f(w) \geq k$ for every $v \in V(G)$. The minimum of the values of $\sum_{v \in V(G)} f(v)$, taken over all signed k-dominating functions f, is called the *signed k-domination number* and is denoted by $\gamma_{kS}(G)$. For a graph G with minimum degree k, the *signed total k-dominating function* (STkDF) and the *signed total k-domination number*, denoted $\gamma^t_{kS}(G)$, can be defined analogously by changing 'closed' neighbourhood in the definition of signed k-domination number to 'open' neighbourhood. As assumption $\delta(G) \geq k - 1$ ($\delta(G) \geq k$, resp.) is clearly necessary, *we will always assume that when we discuss $\gamma_{kS}(G)$ ($\gamma^t_{kS}(G)$, resp.) all graphs involved satisfy $\delta(G) \geq k - 1$ ($\delta(G) \geq k$, resp.).*

Key words and phrases. signed k-dominating function; signed total k-dominating function; signed k-domination number; signed total k-domination number.

In the special case when $k = 1$, $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ are the *signed domination number* and the *signed total domination number* investigated in [2, 3, 4, 15] and [5, 9, 14], respectively. If f maps to $\{0, 1\}$ rather than $\{-1, 1\}$, then $\gamma_{kS}(G)$ is the *k-tuple domination number* introduced by Harary and Haynes in [6].

We establish lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ for bipartite graphs and general graphs in terms of their orders. We present lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ for general graphs in terms of their orders and sizes. We also establish lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ for r-regular graphs.

Throughout this paper, if f is a SkDF or STkDF of G, then we let P and Q denote the sets of those vertices of G which are assigned (under f) the values 1 and -1, respectively, and we let $p = |P|$ and $q = |Q|$. Therefore, $f(V(G)) = p - q = 2p - n$.

2. Lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$

In this section, we first present lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ for general graphs in terms of their orders. Given a positive integer k. We define two families $\mathcal{F}$ and $\mathcal{F}'$ of graphs as follows.

For $t \geq 1$, let $a = (k+1)t$ and $b = (k+1)t^2 - kt$, and let $F_{k,t}$ be the set of graphs of order $n = a + b = (k+1)t^2 + t$ obtained from the disjoint union of K_a and $\overline{K}_b$ by adding edges between $V(K_a)$ and $V(\overline{K}_b)$ so that each vertex in $\overline{K}_b$ joined to exactly $k+1$ vertices in K_a, and each vertex in K_a joined to exactly $(k+1)t - k$ vertices in $\overline{K}_b$. For $t \geq 2$, let $a' = kt$ and $b' = kt^2 - (k+1)t$, and let $F'_{k,t}$ be the set of graphs of order $n' = a' + b' = kt^2 - t$ obtained from the disjoint union of $K_{a'}$ and $\overline{K}_{b'}$ by adding edges between $V(K_{a'})$ and $V(\overline{K}_{b'})$ so that each vertex in $\overline{K}_{b'}$ joined to exactly k vertices in $K_{a'}$, and each vertex in $K_{a'}$ joined to exactly $kt - k - 1$ vertices in $\overline{K}_{b'}$. Let $\mathcal{F} = \bigcup_{t\geq 1} F_{k,t}$ and $\mathcal{F}' = \bigcup_{t\geq 2} F'_{k,t}$.

Theorem 1. *If G is a graph of order n, then*

(1) $\gamma_{kS}(G) \geq -1 - n + \sqrt{4n(k+1)+1}$;

(2) $\gamma^t_{kS}(G) \geq 1 - n + \sqrt{4nk+1}$.

The equality in (1) holds if $G \in \mathcal{F}$; and the equality in (2) holds if $G \in \mathcal{F}'$.

Proof. We only prove (1), as (2) can be proved similarly. Let f be a SkDF such that $\gamma_{kS}(G) = f(V(G))$. Then $\gamma_{kS}(G) = |P| - |Q| = 2p - n$. Notice that every vertex in Q must be adjacent to at least $k+1$ vertices in P. By the pigeonhole principle, there exists a vertex v in P adjacent to at least $(k+1)|Q|/|P| = (k+1)(n-p)/p$ vertices in Q. Thus,

$$\begin{aligned} k &\leq f(N[v]) \\ &\leq |P| - (k+1)(n-p)/p \\ &= p - (k+1)(n-p)/p. \end{aligned}$$

i.e.,

$$p^2 + p - (k+1)n \geq 0.$$

Solving the above inequality for p, we obtain that

$$p \geq \frac{1}{2}\left(-1 + \sqrt{4n(k+1)+1}\right).$$

Therefore, $\gamma_{kS}(G) = 2p - n \geq -1 - n + \sqrt{4n(k+1)+1}$.

Suppose that $G \in \mathcal{F}$. Then $G \in F_{k,t}$ for some $t \geq 1$. Thus, G has order $n = (k+1)t^2 + t$, $a = (k+1)t$ and $b = (k+1)t^2 - kt$. Assigning the value 1 to each vertex in K_n, and -1 to all other vertices, we define a SkDF f of G satisfying $f(V(G)) = (k+1)t - ((k+1)t^2 - kt) = -(k+1)t^2 + (2k+1)t = -1 - n + \sqrt{4n(k+1)+1}$. Thus, $\gamma_{kS}(G) \le -1 - n + \sqrt{4n(k+1)+1}$. Consequently, $\gamma_{kS}(G) = -1 - n + \sqrt{4n(k+1)+1}$. □

Secondly, we establish lower bounds of $\gamma_{kS}(G)$ and $\gamma_{kS}^t(G)$ for general graphs in terms of their orders and sizes.

Theorem 2. *If G is a graph of order n and size m, then*

(1) $\gamma_{kS}(G) \ge \frac{1}{k+2}((2k+1)n - 2m)$;

(2) $\gamma_{kS}^t(G) \ge 2n - 2m/k$.

The equality in (1) holds if $G \in \mathcal{F}$; and the equality in (2) holds if $G \in \mathcal{F}'$.

Proof. We only prove (1), as (2) can be proved similarly. Let f be a SkDF such that $\gamma_{kS}(G) = f(V(G))$. Then $\gamma_{kS}(G) = |P| - |Q| = 2p - n$. As each vertex in Q must be adjacent to at least $k+1$ vertices in P,

$$e(P,Q) \ge (k+1)q = (k+1)(n-p).$$

Notice that for each vertex v of P, $\deg_P(v) \ge \deg_Q(v) + k - 1$. Thus,

$$(k+1)(n-p) \le e(P,Q) = \sum_{v\in P} \deg_Q(v) \le \sum_{v\in P} (\deg_P(v) - k + 1).$$

i.e.,

$$(k+1)(n-p) \le 2|E(G[P])| - (k-1)p.$$

So,

$$|E(G[P])| \ge \frac{1}{2}((k+1)n - 2p).$$

Thus,

$$\begin{aligned} m &\ge |E(G[P])| + e(P,Q) \\ &\ge \frac{1}{2}((k+1)n - 2p) + (k+1)(n-p). \end{aligned}$$

Hence,

$$p \ge \frac{1}{k+2}\left(\frac{3(k+1)n}{2} - m\right).$$

It turns out that

$$\gamma_{kS}(G) \ge \frac{1}{k+2}((2k+1)n - 2m).$$

To see this bound is sharp, let $G \in \mathcal{F}$. Thus, $G \in F_{k,t}$ for some t and has order $n = (k+1)t^2 + t$ and size $m = (k+1)t((k+1)t - k) + \frac{1}{2}(k+1)t((k+1)t - 1)$. As seen in the proof of Theorem 1, $\gamma_{kS}(G) = -(k+1)t^2 + (2k+1)t = \frac{1}{k+2}((2k+1)n - 2m)$. □

Thirdly, we present lower bounds of $\gamma_{kS}(G)$ and $\gamma_{kS}^t(G)$ for general bipartite graphs in terms of their orders. Given a positive integer k. We define two families $\mathcal{H}$ and $\mathcal{H}'$ of bipartite graphs as follows.

For $t \ge 1$, let $a = (k+1)t$ and $b = c = (k+1)t^2 - (k-1)t$, and let $H_{k,t}$ be the set of graphs of order $n = 2a + b + c = 2(k+1)t^2 + 4t$ obtained from the disjoint union of $K_{a,a}$ with the partite sets X and Y, $\overline{K_b}$ and $\overline{K_c}$ by adding edges between X and $V(\overline{K_b})$, and edges between Y and $V(\overline{K_c})$, so that each vertex in $\overline{K_b}$ joined to exactly $k+1$ vertices in X, each vertex in X joined to exactly $(k+1)t - k + 1$ vertices in $\overline{K}_b$, each vertex in $\overline{K}_c$ joined to exactly $k+1$ vertices in Y, and each vertex in Y joined to exactly $(k+1)t-k+1$ vertices in $\overline{K}_c$; let $a' = kt$ and $b' = c' = kt^2 - kt$, and let $H'_{k,t}$ be the set of graphs of order $n' = 2a' + b' + c' = 2kt^2$ obtained from the disjoint union of $K_{a',a'}$ with the partite sets X' and Y', $\overline{K}_{b'}$ and $\overline{K}_{c'}$ by adding edges between X' and $V(\overline{K}_{b'})$, and edges between Y' and $V(\overline{K}_{c'})$, so that each vertex in $\overline{K}_{b'}$ joined to exactly k vertices in X', each vertex in X' joined to exactly $kt-k$ vertices in $\overline{K}_{b'}$, each vertex in $\overline{K}_{c'}$ joined to exactly k vertices in Y', and each vertex in Y' joined to exactly $kt-k$ vertices in $\overline{K}_{c'}$. Let $\mathcal{H} = \bigcup_{t\geq 1} H_{k,t}$ and $\mathcal{H}' = \bigcup_{t\geq 1} H'_{k,t}$.

Theorem 3. *If G is a bipartite graph of order n, then*

(1) $\gamma_{kS}(G) \geq -4 - n + 2\sqrt{2n(k+1)+4}$;

(2) $\gamma^t_{kS}(G) \geq -n + 2\sqrt{2kn}$.

The equality in (1) holds if $G \in \mathcal{H}$; and the equality in (2) holds if $G \in \mathcal{H}'$.

Proof. We only prove (2), as (1) can be proved similarly. Let f be a STkDF of G such that $\gamma^t_{kS}(G) = f(V(G))$. Let X and Y be the partite sets of G. Further, let X^+ and X^- be the sets of vertices in X that are assigned the value +1 and -1 (under f), respectively. Let Y^+ and Y^- be defined analogously. Then $P = X^+ \cup Y^+$ and $Q = X^- \cup Y^-$. For convenience, let $|X^+| = a, |X^-| = s, |Y^+| = b$ and $|Y^-| = t$. Hence, $\gamma^t_{kS}(G) = a + b - s - t = 2(a+b) - n$.

Every vertex in Y^- must be adjacent to at least k vertices in X^+. Therefore, by the pigeonhole principle, there is a vertex v in X^+ adjacent to at least $k|Y^-|/|X^+| = kt/a$ vertices in Y^-. Hence,

$$k \leq f(N(v)) \leq |Y^+| - k|Y^-|/|X^+| = b - kt/a.$$

i.e.,

$$ab \geq k(a+t). \tag{2.1}$$

By a similar argument, one may show that

$$ab \geq k(b+s). \tag{2.2}$$

Adding (2.1) and (2.2), we obtain that

$$2ab \geq k(s+t+a+b) = kn. \tag{2.3}$$

By the fact that $2ab \leq (a+b)^2/2$, together with (2.3), we have that

$$a + b \geq \sqrt{2kn}.$$

Thus, $\gamma^t_{kS}(G) = 2(a+b) - n \geq -n + 2\sqrt{2kn}$.

Suppose that $G \in \mathcal{H}$. Thus, $G \in H_{k,t}$ for some $t \geq 1$. Note that G has order $n = 2kt^2$. Assigning 1 to the $2kx$ vertices of $K_{a,a}$, and -1 to all other vertices, we define a STkDF f of G satisfying $f(V(G)) = 4kt - 2kt^2 = -n + 2\sqrt{2kn}$. Hence, $\gamma^t_{kS}(G) \leq -n + 2\sqrt{2kn}$. It follows that $\gamma^t_{kS}(G) = -n + 2\sqrt{2kn}$. □

Remark 4. *The following table shows the lower bounds on γ_{2S} and γ^t_{2S} of trees of order 10, 20 and 30 given in Theorems 1, 2 and 3, respectively.*

	Bounds on γ_{2S} given in			Bounds on γ^t_{2S} given in		
	Thm 1	Thm 2	Thm 3	Thm 1	Thm 2	Thm 3
$n = 10$	0	8	2	0	11	3
$n = 20$	-5	16	-1	-6	21	-2
$n = 30$	-12	23	-6	-13	31	-8

Finally, we present lower bounds of $\gamma_{kS}(G)$ and $\gamma^t_{kS}(G)$ for r-regular graphs in terms of their orders.

Theorem 5. *If G is r-regular graph of order n, then*

(1)

$$\gamma_{kS}(G) \geq \begin{cases} \frac{(k+1)n}{r+1} & k \equiv r \text{ (mod 2)}; \\ \frac{kn}{r+1} & \text{otherwise}; \end{cases}$$

(2)

$$\gamma^t_{kS}(G) \geq \begin{cases} \frac{kn}{r} & k \equiv r \text{ (mod 2)}; \\ \frac{(k+1)n}{r} & \text{otherwise}. \end{cases}$$

The lower bounds in Theorem 5 are sharp, as will follow from Corollary 6.

Proof of Theorem 5. We only prove (1), as (2) can be proved similarly. Let f be a SkDF such that $\gamma_{kS}(G) = f(V(G))$. As G is a r-regular graph,

$$\sum_{v \in V(G)} f(N[v]) = (r+1)f(V(G)). \tag{2.4}$$

We discuss the following two cases.

Case 1. $k \equiv r$ (mod 2).

Note that, in this case, $|N[v]| = r + 1 \not\equiv k$ (mod 2) for each $v \in V(G)$. So, $f(N[v]) \geq k + 1$ for each $v \in V(G)$. By (2.4), it follows that

$$(r+1)f(V(G)) \geq (k+1)n.$$

Hence, $\gamma_{kS}(G) \geq \frac{(k+1)n}{r+1}$.

Case 2. $k \not\equiv r$ (mod 2).

As for each $v \in V(G)$, $f(N[v]) \geq k$.

$$\sum_{v \in V(G)} f(N[v]) \geq kn.$$

By (2.4), it follows that $(r+1)f(V(G)) \geq kn$. Hence, $\gamma_{kS}(G) \geq \frac{kn}{r+1}$. □

The following Corollary is immediate from Theorem 5.

Corollary 6. *Let $k \geq 1$ be an integer. For any integer $n \geq k$, we have*

(1)

$$\gamma_{kS}(K_n) = \begin{cases} k & n \equiv k \text{ (mod 2)}; \\ k+1 & \text{otherwise}; \end{cases}$$

(2)

$$\gamma^t_{kS}(K_{n,n}) = \begin{cases} 2k & n \equiv k \text{ (mod 2)}; \\ 2(k+1) & \text{otherwise}. \end{cases}$$

Proof. We prove (1) first. As K_n is an $(n-1)$-regular graph, by Theorem 5, it suffices to show that

$$\gamma_{kS}(K_n) \leq \begin{cases} k & n \equiv k \pmod 2; \\ k+1 & \text{otherwise}; \end{cases}$$

We discuss the following two cases.

Case 1. $n \equiv k \pmod 2$.

Assigning 1 to each of $(n+k)/2$ vertices, and -1 to the remaining $(n-k)/2$ vertices, we produce a SkDF f of K_n such that $f(V(K_n)) = k$. Hence, $\gamma_{kS}(K_n) \leq k$.

Case 2. $n \not\equiv k \pmod 2$.

Assigning 1 to each of $(n+k+1)/2$ vertices, and -1 to the remaining $(n-k-1)/2$ vertices, we produce a SkDF f of K_n such that $f(V(G)) = k+1$. Hence, $\gamma_{kS}(K_n) \leq k+1$.

We now prove (2). By Theorem 5, it suffices to show that

$$\gamma^t_{kS}(K_{n,n}) \leq \begin{cases} 2k & n \equiv k \pmod 2; \\ 2(k+1) & \text{otherwise}; \end{cases}$$

Let X and Y be the partite sets of $K_{n,n}$. We discuss the following two cases.

Case 1. $n \equiv k \pmod 2$.

Assigning 1 to each of $(n+k)/2$ vertices in X and each of $(n+k)/2$ vertices in Y, and -1 to the remaining vertices, we produce a STkDF f of $K_{n,n}$ such that $f(V(K_{n,n})) = 2k$. Hence, $\gamma^t_{kS}(K_{n,n}) \leq 2k$.

Case 2. $n \not\equiv k \pmod 2$.

Assigning 1 to each of $(n+k+1)/2$ vertices in X and each of $(n+k+1)/2$ vertices in Y, and -1 to the remaining vertices, we produce a STkDF f of $K_{n,n}$ such that $f(V(K_{n,n})) = 2(k+1)$. Hence, $\gamma^t_{kS}(K_{n,n}) \leq 2(k+1)$. □

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