COMPOSITIONS OF POSITIVE INTEGERS n VIEWED AS ALTERNATING SEQUENCES OF INCREASING/DECREASING PARTITIONS

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ABSTRACT. Compositions and partitions of positive integers are often studied in separate frameworks where partitions are given by q-series and compositions exhibiting particular patterns are specified by generating functions for these patterns. Here we view compositions as alternating sequences of partitions (i.e., alternating blocks) and obtain results for the asymptotic expectations of the number of such blocks (or parts per block) for different ways of defining the blocks.

1. INTRODUCTION

Compositions and partitions of integers are well studied. For example, partition theory was extensively studied in [6] where the notion of compositions was also introduced. A simple survey of the history of partition theory is contained in [1] and a survey of the study of compositions is contained in [5].

Recently, some studies on compositions and partitions have focused on the relationship between them. In [2], Andrews develops q-series generating functions for certain specific types of compositions which consist of two adjoined partitions (concave compositions). In Section 2.5, Chapter 2 of his book [9], Stanley develops generating functions for another type of "unimodal" composition (consisting of two adjoined blocks of partitions).

This provides some of the motivation for the current study: i.e., presenting more general compositions as sequences of alternating blocks of partitions. However, we allow arbitrary compositions resulting in a decomposition into an arbitrary number of alternating blocks of partitions.

Roughly speaking, any composition of a fixed positive integer n may be viewed as being split up into blocks where each block is either increasing or decreasing. In other words, each block is a partition. (As a preliminary example the composition 1 4 3 2 1 2 3 of 16 may be split into partitions (14) (321) (23).) How compositions are split depends on precisely how the blocks are defined. In Sections 3 - 5 we make three (different) definitions for how to split compositions into alternating blocks and then make use of previous results on subword patterns in compositions (see [4],[5],[8]) to find generating functions for the number of such blocks contained within all the compositions of n.

In Theorem 1 (see Section 6), we find, using the aforementioned three definitions for the blocks, an asymptotic estimate as n tends to infinity for the expected number of partition blocks within an arbitrary composition of n as well as an estimate of the number of parts per block in each case.

2. Preliminaries

We need the following **definitions.**

A **composition** $\sigma = \sigma_1\sigma_2\ldots\sigma_m$ of a positive integer n is an ordered collection of one or more positive integers whose sum is n. Each summand σ_i is called a part of the composition.

A **partition** of a positive integer n is either a non-increasing or non-decreasing sequence of positive integers whose sum is n. (So for example, for the partition $n = a_1 + a_2 + \cdots + a_k$ either $a_1 \geq a_2 \geq \cdots \geq a_k \geq 1$ or $1 \leq a_1 \leq a_2 \leq \cdots \leq a_k$.)

Per the definition found in [7], let $[n] = \{1, 2, \ldots, n\}$ and let $[n]^\ell$ denote the set of words of length ℓ in the alphabet $[n]$. For any word σ in $[n]^m$, let red(σ) denote the member of $[n]^m$ obtained by replacing the smallest letter of σ by 1, replacing all letters corresponding to the second smallest element by 2 and so on. For example, if $\sigma = 42244 \in [4]^5$ then red(σ) = 21122.

We call $\{\text{red}(\sigma)\colon \sigma \in [n]^m,\ 1 \leq m \leq \ell\}$ the set of **subword patterns** in $[n]^\ell$. And we say that there is an occurrence of the pattern $\tau = \sigma_1, \sigma_2, \ldots, \sigma_m$ at index i in the composition (word) $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_s$ if red $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+m-1}) = \tau$, $(i \leq s+1-m)$. **The number of occurrences of the pattern τ in α** is the number of different such indices i satisfying red $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+m-1}) = \tau$. For example, if $\alpha = 4243233$ then there are two occurrences of the pattern $\tau = 212$ (corresponding to 424 and 323) and one occurrence of the pattern $\tau = 11$ (corresponding to 33). Note that we require the letters within a given word corresponding to a pattern to be consecutive.

In accordance with the definitions in [5], an occurrence of any of the patterns

$$\left.\begin{array}{l} 12^{\ell-2}1 \\ 23^{\ell-2}1 \\ 13^{\ell-2}2 \end{array}\right\} \quad \text{integer } \ell \geq 3$$

in an arbitrary composition of n is known as a peak (strict if $\ell = 3$ and weak otherwise). Likewise, any occurrence of the patterns

$$\left.\begin{array}{l} 21^{\ell-2}2 \\ 21^{\ell-2}3 \\ 31^{\ell-2}2 \end{array}\right\} \quad \text{integer } \ell \geq 3$$

is known as a valley (strict if $\ell = 3$ and weak otherwise).

In each of the groups above, the generating function for the second pattern is the same as that for third (as can be seen from the reversal mapping applied to compositions).

We refer to [5], Theorem 4.39, p.120. In these formulae, we ignore the number of parts (i.e. set $y = 1$), and we choose our alphabet set as $A = \mathbb{N}$. We deal with occurrences of the patterns

$$\left.\begin{array}{rcl} \nu_1 &=& 12^{\ell-2}1 \\ \nu_2 &=& 23^{\ell-2}1 \\ \nu_2' &=& 13^{\ell-2}2 \\ \tau_1 &=& 21^{\ell-2}2 \\ \tau_2 &=& 21^{\ell-2}3 \\ \tau_2' &=& 31^{\ell-2}2 \end{array}\right\} \quad \ell \geq 3$$

$$\begin{array}{rcl} \nu_3 &=& 221 \\ \tau_3 &=& 211 \\ \nu_4 &=& 11 \end{array}$$

and obtain generating functions for the number of occurrences of the specified patterns:

$$C_{\mathbb{N}}^{\nu_2'} = C_{\mathbb{N}}^{\nu_2} = \frac{1}{1 - \sum_{i\geq 1} x^i \prod_{j\geq i+1} \left(1 + x^j(q-1) \sum_{\alpha\in\overline{\beta}_{j+1}} x^{\mathrm{ord}(\alpha)}\right)}$$

$$C_{\mathbb{N}}^{\tau_2'} = C_{\mathbb{N}}^{\tau_2} = \frac{1}{1 - \sum_{i\geq 1} x^i \prod_{j=1}^{i-1} \left(1 + x^j(q-1) \sum_{\alpha\in\beta_{j-1}} x^{\mathrm{ord}(\alpha)}\right)},$$

where $\overline{\beta}_{j+1}$ is the set of compositions α with parts in $\{j+1, j+2, \dots\}$ that are order isomorphic to the pattern $1^{\ell-2}$, and β_{j-1} is the set of compositions α with parts in $\{1, 2, \dots, j-1\}$ that are order isomorphic to the same pattern.

In the above formulae x marks the size n of the compositions and q marks the number of occurrences of the pattern under consideration.

So for example, we may write $C_{\mathsf{N}}^{\tau_2}$ as

$$C_{\mathsf{N}}^{\tau_2} = \sum_{n \geq 0} \sum_{b \geq 0} a(n,b) x^n q^b,$$

where $a(n,b)$ is the number of times that compositions of the integer n have exactly b occurrences of the pattern τ_2.

Simplifying the formulae above, we have:

$$\sum_{\alpha \in \overline{\beta}_{j+1}} x^{\mathrm{ord}(\alpha)} = \sum_{r \geq j+1} (x^{\ell-2})^r = \frac{(x^{\ell-2})^{j+1}}{1 - x^{\ell-2}}$$

and

$$\sum_{\alpha \in \beta_{j-1}} x^{\mathrm{ord}(\alpha)} = \sum_{\alpha=1}^{j-1} (x^{\ell-2})^{\alpha} = \frac{x^{\ell-2}(1 - (x^{\ell-2})^{j-1})}{1 - x^{\ell-2}}.$$

So

$$C_{\mathsf{N}}^{\nu_2} = \frac{1}{1 - \sum_{i \geq 1} x^i \prod_{j \geq i+1} \left(1 + x^j (q-1) \cdot \frac{(x^{\ell-2})^{j+1}}{1 - x^{\ell-2}}\right)} \tag{1}$$

and

$$C_{\mathsf{N}}^{\tau_2} = \frac{1}{1 - \sum_{i \geq 1} x^i \prod_{j=1}^{i-1} \left(1 + x^j (q-1) \frac{x^{\ell-2}(1 - x^{(\ell-2)(j-1)})}{1 - x^{\ell-2}}\right)}. \tag{2}$$

$$C_{\mathbb{N}}^{\nu_1} = \frac{1}{1 - \sum_{j\geq 1} \frac{x^j}{1 + x^j(1-q)\frac{(x^{\ell-2})^{j+1}}{1-x^{\ell-2}}}} \tag{3}$$

$$C_{\mathbb{N}}^{\tau_1} = \frac{1}{1 - \sum_{j\geq 1} \frac{x^j}{1 + x^j(1-q)\frac{x^{\ell-2}\left(1-(x^{\ell-2})^{j-1}\right)}{1-x^{\ell-2}}}}. \tag{4}$$

From [5] Theorem 4.3.5, p.115, again setting $y = 1$ and $A = \mathbb{N}$ and also $\ell = 3$, we obtain

$$C_{\mathbb{N}}^{\nu_3} = \frac{1}{1 - \sum_{j\geq 1} x^j \prod_{i\geq j+1} \left(1 - x^{2i}(1-q)\right)} \tag{5}$$

and

$$C_{\mathbb{N}}^{\tau_3} = \frac{1}{1 - \sum_{j\geq 1} x^j \prod_{i=1}^{j-1} \left(1 - x^{2i}(1-q)\right)}. \tag{6}$$

Finally from [5], Table 4.1, pg. 101, we obtain

$$C_{\mathbb{N}}^{\nu_4} = \frac{1}{1 - \sum_{j=1}^{\infty} \frac{x^j}{1-x^j(q-1)}}. \tag{7}$$

The generating function for the total number of occurrences for the associated pattern in all compositions of n is found by differentiating the above formulae with respect to q and setting $q = 1$. We obtain

$$
\begin{aligned}
\left.\frac{\partial C_{\mathrm{N}}^{\nu_1}}{\partial q}\right|_{q=1} &= \sum_{\ell\geq 3}\Big(1-\sum_{j\geq 1}x^j\Big)^{-2}\sum_{j\geq 1}x^j\Big[x^j\sum_{i\geq j+1}(x^{\ell-2})^i\Big] \\
&= \left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{2\ell-2}}{(1-x^{\ell-2})(1-x^{\ell})} \\
\left[\frac{\partial C_{\mathrm{N}}^{\nu_2'}}{\partial q}+\frac{\partial C_{\mathrm{N}}^{\nu_2}}{\partial q}\right]\Bigg|_{q=1} &= 2\left.\frac{\partial C_{\mathrm{N}}^{\nu_2}}{\partial q}\right|_{q=1} \\
&= 2\sum_{\ell\geq 3}\Big(1-\sum_{i\geq 1}x^j\Big)\cdot\sum_{i\geq 1}x^i\sum_{j\geq i+1}\frac{x^j\cdot x^{(\ell-2)(j+1)}}{1-x^{\ell-2}} \\
&= 2\left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{3(\ell-1)}}{(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})} \\
\left.\frac{\partial C_{\mathrm{N}}^{\tau_1}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\sum_{\ell\geq 3}\frac{x^{\ell+2}-x^{2\ell}}{(1-x^2)(1-x^{\ell-2})(1-x^{\ell})} \\
\left[\frac{\partial C_{\mathrm{N}}^{\tau_2'}}{\partial q}+\frac{\partial C_{\mathrm{N}}^{\tau_2}}{\partial q}\right]\Bigg|_{q=1} &= 2\left.\frac{\partial C_{\mathrm{N}}^{\tau_2}}{\partial q}\right|_{q=1} \\
&= 2\left(\frac{1-x}{1-2x}\right)^2\times \\
&\quad \sum_{\ell\geq 3}\frac{x^{\ell+3}-x^{2\ell+1}-x^{2\ell+2}+x^{3\ell}}{(1-x)(1-x^2)(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})} \\
\left.\frac{\partial C_{\mathrm{N}}^{\nu_3}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\cdot\frac{x^5}{(1-x^2)(1-x^3)} \\
\left.\frac{\partial C_{\mathrm{N}}^{\tau_3}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\cdot\frac{x^4}{(1-x)(1-x^3)} \\
\left.\frac{\partial C_{\mathrm{N}}^{\nu_4}}{\partial q}\right|_{q=1} &= \left(\frac{1-x}{1-2x}\right)^2\frac{x^2}{1-x^2}. \qquad (8)
\end{aligned}
$$

3. Blocks that alternate between weakly increasing and weakly decreasing

We first split a composition of n into blocks on the basis that all of the parts in any particular block have the same relationship, namely, $\geq$ or $\leq$, to all the other parts of the same block. More precisely, for the arbitrary composition of n given by

$$n = x_1 + x_2 + \cdots + x_{q_s},$$

where either

$$x_1 \le x_2 \le \cdots \le x_{q_1} > x_{q_1+1} \ge x_{q_1+2} \ge \cdots \ge x_{q_2} < x_{q_2+1} \le \cdots x_{q_s} \quad (9)$$

or

$$x_1 \ge x_2 \ge \cdots \ge x_{q_1} < x_{q_1+1} \le x_{q_1+2} \le \cdots \le x_{q_2} > x_{q_2+1} \ge \cdots x_{q_s}, \quad (10)$$

the blocks are defined to be the s partitions

$$\begin{array}{l} x_1, x_2, \ldots x_{q_1} \\ x_{q_1+1}, x_{q_1+2} \ldots x_{q_2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{q_{s-1}+1}, \ldots x_{q_s} \end{array}$$

derived from (9) or (10), where the value chosen for q_1 is maximal (and, subsequently, for the remaining q_i, $i > 1$). If all parts of the composition are equal, then there is only one block (which we may regard if we like as a weakly increasing partition).

Example 3.1. *The composition 133222 of 13 will be split into blocks (133) and (222) (where we might regard the first as being a weakly increasing partition and the second as weakly decreasing). (1), (33), (222) would not be correct because the first option above has the larger q_1 value.*

With blocks as defined in this section, an occurrence of a peak or a valley, whether weak or strict in either case (i.e., any of the patterns ν_1, ν_2, ν_2', τ_1, τ_2, τ_2'), marks the beginning of a new block.

Hence for any particular composition the total number of blocks = total number of peaks + total number of valleys + 1.

Thus the generating function for the total number of alternating blocks is

$$\begin{aligned} G_1 &= \left[\frac{\partial C_{\mathrm{N}}^{\nu_1}}{\partial q} + \frac{2\partial C_{\mathrm{N}}^{\nu_2}}{\partial q} + \frac{\partial C_{\mathrm{N}}^{\tau_1}}{\partial q} + \frac{2\partial C_{\mathrm{N}}^{\tau_2}}{\partial q}\right]_{q=1} + \left(\frac{1-x}{1-2x}\right) \\ &= \left(\frac{1-x}{1-2x}\right)^2 \times \\ &\quad \times \sum_{\ell\ge 3} \frac{x^{\ell-3}\left(x^5 + x^{\ell+1} - 2x^{\ell+2} - x^{\ell+4} + x^{2\ell} - 2x^{2\ell+1} + 2x^{2\ell+2}\right)}{(1-x)^2(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})} \\ &\quad + \left(\frac{1-x}{1-2x}\right). \end{aligned}$$

The principal pole of the summand in

$$f(x) = \frac{1}{(1-2x)^2} \sum_{\ell \geq 3} \frac{x^{\ell-3}(x^5 + x^{\ell+1} - 2x^{\ell+2} - x^{\ell+4} + x^{2\ell} - 2x^{2\ell+1} + 2x^{2\ell+2}}{(1-x^{\ell-2})(1-x^{\ell-1})(1-x^{\ell})}$$

occurs at $x = \frac{1}{2}$ (with multiplicity 2). Using the asymptotic analysis on page 257 of [3],

$$f(x) \sim \left(\sum_{\ell \geq 3} \frac{(\frac{1}{2})^{\ell-3}\left[(\frac{1}{2})^5 + (\frac{1}{2})^{\ell+1} - 2(\frac{1}{2})^{\ell+2} - (\frac{1}{2})^{\ell+4} + (\frac{1}{2})^{2\ell} - 2(\frac{1}{2})^{2\ell+1} + 2(\frac{1}{2})^{2\ell+2}\right]}{\left(1-(\frac{1}{2})^{\ell-2}\right)\left(1-(\frac{1}{2})^{\ell-1}\right)\left(1-(\frac{1}{2})^{\ell}\right)} \right) \times \left(\frac{1}{1-2x}\right)^2 .$$

Hence

$$[x^n]f(x) \sim (0.136681\ldots)\,(n+1)2^n.$$

The total number of compositions of n is 2^{n-1}. Hence the expected number of blocks as $n \to \infty$ is $(0.273362\ldots)n + 0(1)$. Because the average number of parts of a composition of n is $\frac{n}{2}$, the expected number of parts in the blocks as $n \to \infty$ is $1.82908\ldots$.

Remark. Is the number $1.82908\ldots$ irrational?

4. Blocks that alternate between weakly increasing and strictly decreasing

Consider an arbitrary composition of $n = x_1 + x_2 + \cdots + x_{q_s}$. View the composition in one of the following manners

$$x_1 \leq x_2 \leq \cdots \leq x_{q_1} > x_{q_1+1} > x_{q_1+2} > \cdots > x_{q_2} \leq x_{q_2+1} \leq \cdots \leq x_{q_3} > \cdots x_{q_s}$$

or

$$x_1 > x_2 \cdots > x_{q_1} \leq x_{q_1+1} \leq x_{q_1+2} \leq \cdots \leq x_{q_2} > x_{q_2+1} \cdots > x_{q_3} \leq \cdots x_{q_s}.$$

That is, all equalities are considered as part of a neighbouring (weakly) increasing block. So the compositions are split into partitions

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_{q_1} \\ x_{q_1+1}, x_{q_1+2}, \ldots, x_{q_2} \\ \vdots \\ x_{q_{s-1}+1}, \ldots, x_{q_s} \end{array}\right\} \text{which alternately (weakly) increase or (strictly) decrease.}$$

As in Section 2, the sequence above which is used to determine the blocks is that one with the largest q_1 value.

Example 4.1. *The composition* 133222 *of 13 will be split into blocks (133), (2) and (22) (where we might regard the first and third as weakly increasing and the second as strictly decreasing).*

With blocks as defined in this section, an occurrence of any of the patterns 121, 231, 132, 221 (all peaks) or 212, 213, 312, 211 (all valleys) marks the beginning of a new block. These patterns are identical to those in Section 2 with the restriction that $\ell = 3$ only.

As before, for any particular composition, the total number of blocks = total number of peaks + total number of valleys + 1.

Thus the generating function for the total number of alternating blocks is

$$G_2 = \left\{\left[\frac{\partial C_{\mathrm{N}}^{\nu_1}}{\partial q} + \frac{2\partial C_{\mathrm{N}}^{\nu_2}}{\partial q}\right] + \left[\frac{\partial C_{\mathrm{N}}^{\tau_1}}{\partial q} + \frac{2\partial C_{\mathrm{N}}^{\tau_2}}{\partial q}\right]\left[\frac{\partial C_{\mathrm{N}}^{\nu_3}}{\partial q} + \frac{\partial C_{\mathrm{N}}^{\tau_3}}{\partial q}\right]\right\}\Bigg|_{\substack{\ell=3\\ q=1}} + \left(\frac{1-x}{1-2x}\right).$$

(First square bracket term represents strict peaks; second square bracket terms represent strict valleys and last term represents weak peaks or weak valleys)

$$\begin{aligned} G_2 &= \left(\frac{1-x}{1-2x}\right)^2 \frac{[x^4(1+x^2)] + [x^5+x^6] + [x^4+x^5-2x^6]}{(1-x)(1-x^2)(1-x^3)} + \left(\frac{1-x}{1-2x}\right) \\ &= \left(\frac{1-x}{1-2x}\right)^2 \frac{(1-2x-x^2+x^3+4x^4+3x^5-2x^6)}{(1-x)(1-x^2)(1-x^3)} \\ &= \frac{1}{2} + \frac{2}{3(1-x)} + \frac{1}{7(1-2x)^2} - \frac{13}{98(1-2x)} - \frac{2}{147}\frac{(13-2x-11x^2)}{(1-x^3)}. \end{aligned}$$

Extracting the coefficients of x^n from this equation, we obtain

$$[x^n] = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{48}{98} + \dfrac{2^n}{98} + \dfrac{7\cdot 2^{n+1}}{98} n & \text{for } n \equiv 0 \bmod 3 \text{ but } n \neq 0 \\ \dfrac{68}{98} + \dfrac{2^n}{98} + \dfrac{7\cdot 2^{n+1}}{98} n & \text{for } n \equiv 1 \bmod 3 \\ \dfrac{109}{147} + \dfrac{2^n}{98} + \dfrac{7\cdot 2^{n+1}}{98} n & \text{for } n \equiv 2 \bmod 3 \end{cases}.$$

Since there are 2^{n-1} compositions of integer n, the expected number of blocks is

$$E_1 = \begin{cases} 1 & \text{if } n = 0 \\ \dfrac{48}{98 \cdot 2^{n-1}} + \dfrac{1}{49} + \dfrac{2}{7}n & \text{if } n \equiv 0 \bmod 3 \text{ but } n \neq 0 \\ \dfrac{68}{98 \cdot 2^{n-1}} + \dfrac{1}{49} + \dfrac{2}{7}n & \text{if } n \equiv 1 \bmod 3 \\ \dfrac{109}{147 \cdot 2^{n-2}} + \dfrac{1}{49} + \dfrac{2}{7}n & \text{if } n \equiv 2 \bmod 3 \end{cases}.$$

Asymptotically as $n \to \infty$, the expected number of blocks $= \frac{1}{49} + \frac{2}{7}n$. It is well known that the average number of parts of each composition of n is $\frac{n}{2}$. Therefore the expected number of parts in the blocks forming the compositions of n is $1\frac{3}{4}$ as $n \to \infty$.

5. Blocks that alternate between strictly increasing and strictly decreasing

By analogy with Section 3, view an arbitrary composition $n = x_1 + x_2 + \cdots + x_{q_s}$ in one of the following manners

$$x_1 < x_2 < \cdots < x_{q_1} \geq x_{q_1+1} > x_{q_1+2} > \cdots > x_{q_2} \leq x_{q_2+1} < \cdots < x_{q_3} \geq \cdots x_{q_s}$$

or

$$x_1 > x_2 \cdots > x_{q_1} \leq x_{q_1+1} < x_{q_1+2} < \cdots < x_{q_2} \geq x_{q_2+1} > \cdots > x_{q_3} \leq \cdots x_{q_s}.$$

So the composition is split into partitions

$$\left.\begin{array}{l} x_1, x_2, \ldots, x_{q_1} \\ x_{q_1+1}, x_{q_1+2}, \ldots, x_{q_2} \\ \vdots \\ x_{q_{s-1}+1}, \ldots, x_{q_s} \end{array}\right\} \text{which alternately strictly increase or strictly decrease.}$$

In the event that a composition begins with equalities, the sequence above is chosen so that the first block has one part only.

This definition is a generalization of that used in [2] to define the blocks of the concave compositions.

Examples 5.1. *The composition 22212 is split (2)(2)(21)(2) and not (2)(2)(2)(1)(2). The composition 133222 is split (13)(32)(2)(2).*

With blocks as defined in this section, an occurrence of any of the patterns 121, 231 or 132 (strict peaks) or alternatively 212, 213 or 312 (strict valleys) or 11 (equalities) marks the beginning of a new block.

Thus the generating function for the total number of alternating blocks is

$$\begin{aligned}
G_3 &= \left\{\left[\frac{\partial C_{\mathrm{N}}^{\nu_1}}{\partial q}+\frac{2\partial C_{\mathrm{N}}^{\nu_2}}{\partial q}\right]+\left[\frac{\partial C_{\mathrm{N}}^{\tau_1}}{\partial q}+\frac{2\partial C_{\mathrm{N}}^{\tau_2}}{\partial q}\right]+\left[\frac{\partial C_{\mathrm{N}}^{\nu_4}}{\partial q}\right]\right\}\Bigg|_{\substack{\ell=3\\ q=1}}+\left(\frac{1-x}{1-2x}\right)\\
&= \left(\frac{1-x}{1-2x}\right)^2\left[\frac{1-2x+3x^4+x^5+x^6}{(1-x)(1-x^2)(1-x^3)}\right]\\
&= -\frac{1}{4}+\frac{2}{3(1-x)}+\frac{1(1-x)}{3(1-x^2)}-\frac{5}{294(1-2x)}+\frac{5}{28(1-2x)^2}\\
&\quad+\frac{1(13-2x-11x^2)}{147(1-x^3)}.
\end{aligned}$$

Extracting coefficients of x^n from this expression, we obtain

$$[x^n]=\begin{cases}1 & \text{if } n=0\\ \frac{1}{588}(x(n)+A(n)) & n\neq 0 \text{ as below}\end{cases}$$

where

$$x(n)=\begin{cases}640 & \text{if } n\equiv 0 \bmod 6\ (n\neq 0)\\ 188 & \text{if } n\equiv 1 \bmod 6\\ 544 & \text{if } n\equiv 2 \bmod 6\\ 248 & \text{if } n\equiv 3 \bmod 6\\ 580 & \text{if } n\equiv 4 \bmod 6\\ 152 & \text{if } n\equiv 5 \bmod 6\end{cases} \qquad \text{and } A(n)=95\times 2^n+105\times 2^n n.$$

Since there are 2^{n-1} compositions of the integer n, the expected number of blocks as $n\to\infty$ is $\frac{95}{294}+\frac{5}{14}n$. Since the expected number of parts is $\frac{n}{2}$, the expected number of parts per block is $\frac{7}{5}$.

6. Summary of Results

Above we have proved:

Theorem 1. *For compositions of n split up into blocks of alternately increasing/decreasing partitions, the asymptotic expectation for the number of blocks with corresponding number of parts is:*

Definition of type of decomposition	*Asymptotically expected number of blocks*	*Corresponding number of parts per block*
Section 3 (Weakly increasing/weakly decreasing)	$(0.273362\ldots)n + 0(1)$	$1.82098\ldots$
Section 4 (Weakly increasing/strictly decreasing)	$\frac{2}{7}n + \frac{1}{49}$	$1\frac{3}{4}$
Section 5 (Strictly increasing/strictly decreasing)	$\frac{5}{14}n + \frac{95}{294}$	$1\frac{2}{5}$

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