Packings and coverings for four particular graphs each with six vertices and nine edges ($\lambda = 1$) *

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Abstract: Let λK_v be the complete multigraph with v vertices. Let G be a finite simple graph. A *G-design* G-$GD_\lambda(v)$ (*G-packing* G-$PD_\lambda(v)$, *G-covering* G-$CD_\lambda(v)$) of λK_v is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined exactly (at most, at least) in λ blocks. In this paper, we will discuss the maximum packing designs and the minimum covering designs for four particular graphs each with six vertices and nine edges.

Key words: G-packing, G-covering, Holey G-design.

AMS Classification: 05B07.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is an undirected graph with v vertices, where any two distinct vertices x and

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y are joined by λ edges $\{x, y\}$. Let G be a finite simple graph. A *G-design G-$GD_\lambda(v)$* (*G-packing G-$PD_\lambda(v)$*, *G-covering G-$CD_\lambda(v)$*) of λK_v is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined exactly (at most, at least) in λ blocks. A packing (covering) is said to be *maximum* (*minimum*) if no other such packing (covering) of the same order has more (fewer) blocks. The number of blocks in a maximum packing (minimum covering), denoted by $p(v, G, \lambda)$ ($c(v, G, \lambda)$), is called the *packing* (*covering*) number. Obviously,

$$p(v, G, \lambda) \le U(v, G, \lambda) = \left\lfloor \frac{\lambda v(v-1)}{2|E(G)|} \right\rfloor \le \left\lceil \frac{\lambda v(v-1)}{2|E(G)|} \right\rceil = V(v, G, \lambda) \le c(v, G, \lambda), \quad (*)$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (least) integer y such that $y \le x$ ($y \ge x$). A G-$PD_\lambda(v)$ (G-$CD_\lambda(v)$) is called *optimal*, denoted by G-$OPD_\lambda(v)$ (G-$OCD_\lambda(v)$), if the left (right) equality in $(*)$ holds. Obviously, there exists a G-$GD_\lambda(v)$ if and only if $p(v, G, \lambda) = c(v, G, \lambda)$. So a G-$GD_\lambda(v)$ can be regarded as a G-$OPD_\lambda(v)$ or a G-$OCD_\lambda(v)$.

The *leave* $L_\lambda(\mathcal{P})$ of a packing G-$PD_\lambda(v) = (V, \mathcal{P})$ is a subgraph of λK_v and its edges are the supplement of $\mathcal{P}$ in λK_v. When $\mathcal{P}$ is maximum, $|L_\lambda(P)|$ is called *leave-edges number* and is denoted by $l_\lambda(v)$. Similarly, the *excess* $R_\lambda(\mathcal{C})$ of a covering G-$CD_\lambda(v) = (V, \mathcal{C})$ is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{C}$. When $\mathcal{C}$ is minimum, $|R_\lambda(C)|$ is called *excess-edges number* and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{P})$ and $l_\lambda(v)$ ($R_\lambda(\mathcal{P})$ and $r_\lambda(v)$) can be denoted by L_λ and l_λ (R_λ and r_λ) briefly.

Let $X = \bigcup_{i=1}^{t} X_i$ be the vertex set of $K_{n_1, \cdots, n_t}$, a complete multi-partite graph consisting of t parts with size $n_1, \cdots, n_t$, where these sets X_i, $1 \le i \le t$, are disjoint and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G} = \{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1, n_2, \cdots, n_t}$, a t-partite graph with replication λ, can be decomposed into edge-disjoint subgraphs $\mathcal{A}$, called by *block*, each of which is isomorphic to G, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the type of the holey G-design. Usually, the type is denoted by exponential form, for example, the type

$n_1^{k_1} n_2^{k_2} \cdots n_m^{k_m}$ denotes n_1 occurrences of k_1, n_2 occurrences of k_2, $\cdots$, n_m occurrences of k_m. A G-$HD_\lambda(1^{v-w}w^1)$ is called *incomplete G-design*, denoted by G-$ID_\lambda(v,w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. For $\lambda = 1$, the index λ in GD_λ (or $HD_\lambda, ID_\lambda, OPD_\lambda, OCD_\lambda$) is often omitted.

Suppose, in graph G, there are t_i vertices with degree m_i, where m_i are distinct integers, $1 \le i \le p$. For a graph design (or packing, or covering) $(X, \mathcal{B})$, the block-set consists of b blocks, each vertex u in the vertex-set X need to join with $n(u)$ other vertices. All vertices in X are divided into s classes $C_1, \cdots, C_s$, according to different $n(u)$, $|C_u| = c_u, 1 \le u \le s$. The vertex u occurs exactly in $x_i(u)$ blocks as m_i-degree vertices, $1 \le i \le p$. Define

$$m_1^{t_1} m_2^{t_2} \cdots m_p^{t_p} \text{— the } \textit{degree-type} \text{ of } \textit{graph } G,$$

$$m_1^{x_1(u)} m_2^{x_2(u)} \cdots m_p^{x_p(u)} \text{— the } \textit{degree-distribution} \text{ of the vertex } u.$$

Obvious, $|V(G)| = \sum_{i=1}^{p} t_i$ and $|E(G)| = \frac{1}{2}\sum_{i=1}^{p} m_i t_i$. We have the *degree-equations* as follows:

$$\begin{cases} \sum_{i=1}^{p} m_i x_i(u) = n(u) \\ \sum_{i=1}^{p} x_i(u) \le b, \ x_i(u) \ge 0 \end{cases},$$

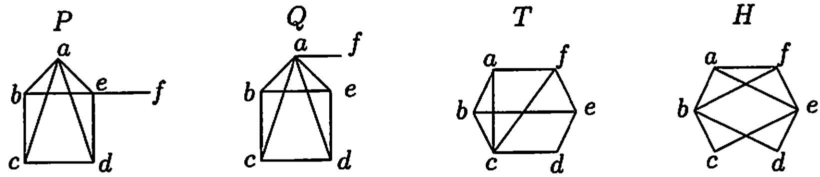
where $u \in C_u, 1 \le u \le s$. Obviously, each solution $(x_1(u), \ \cdots, \ x_p(u))$ of the above system gives a possible block distribution degree-type of the vertex u. For each $u \in C_u$, suppose there are $q(u)$ such solutions $(x_{1,j}(u), \cdots, x_{p,j}(u))$, $1 \le j \le q(u)$. Then, we have the *degree-type distribution equations* as follows:

$$\begin{cases} \sum_{u=1}^{s} \sum_{j=1}^{q(u)} x_{i,j}(u) y_j(u) = b t_i, \ 1 \le i \le p \\ \sum_{j=1}^{q(u)} y_j(u) = c_u, \ 1 \le u \le s \end{cases}.$$

Any solution $(y_1(u), \ y_2(u), \ \cdots, \ y_q(u))$ of the above system corresponds to a kind of degree-type distribution of all vertices on X in all blocks: the block distribution degree-type of $y_j(u)$ vertices is

$$m_1^{x_{1,j}(u)} m_2^{x_{2,j}(u)} \cdots m_p^{x_{p,j}(u)}, \ 1 \le u \le s, \ 1 \le j \le q(u).$$

In this paper, we will discuss the maximum packing designs and the minimum covering designs of four graphs with six vertices and nine edges for $\lambda = 1$, which are listed as follows. For convenience, as a block in design, each graph may be denoted by (a, b, c, d, e, f) according to the following vertex-labels.



Lemma 1.1[13] *For graph $G \in \{P, Q, T\}$ or $\{H\}$,*

$$\text{there exists } G\text{-}GD_\lambda(v) \iff \begin{cases} v \geq 6 \\ \lambda v(v-1) \equiv 0 \pmod{18} \\ (\lambda, v) \neq (1,9) \text{ (or } (1,9), (3,6)) \end{cases} .$$

Lemma 1.2[8] *Given positive integers h, m, ω and λ, where $m > 1$, suppose there exist G-$ID_\lambda(h+\omega, \omega)$ and G-$HD(h^m)$.*

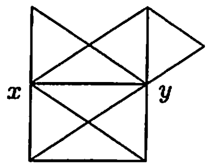
(1). If there exists G-$OPD_\lambda(\omega)$ or G-$OPD_\lambda(h+\omega)$, then there exists G-$OPD_\lambda(mh+\omega)$.

(2). If there exists G-$OCD_\lambda(\omega)$ or G-$OCD_\lambda(h+\omega)$, then there exists G-$OCD_\lambda(mh+\omega)$.

2 Existence for small orders

Lemma 2.1 (1). *$p(6, P, 1) = 1$ and $c(6, P, 1) = 2$;*
(2). *$p(7, P, 1) = 1$ and $c(7, P, 1) = 3$;*
(3). *$p(8, P, 1) = 2$ and $c(8, P, 1) = 4$;*
(4). *$p(9, P, 1) = 3$ and $c(9, P, 1) = 5$.*

Proof. Below, the symbol L (or R) represents the leave (or excess).
(1). A P-$OPD(6)$: $(0, 1, 2, 3, 4, 5)$. Its leave is a subgraph of P, so P-$OCD(6)$ can be obtained.
(2). P-$OCD(7)$: $(1, 2, 3, 4, 5, 6), (0, 2, 4, 5, 6, 1), (3, 5, 6, 4, 1, 0)$,
$R = \{02, 04, 14, 34, 56, 45\}$.
The packing number $p(7, P, 1) \leq U(7, P, 1) = 2$. The *degree-type* of P is $1^1 3^3 4^2$ and the four 4°-vertices form a 4-cycle. If $p(7, P, 1) = 2$, then $P \subseteq K_7 \setminus P = G$, where G is as follows.



But, the 4°-vertex of P can only be chosen from x or y in G. However, there is no 4-cycle in the subgraph $G\backslash\{x\}$ and $G\backslash\{y\}$. Therefore, $p(7, G, 1) = 1$.
(3). Suppose that $p(8, P, 1) = U(8, P, 1) = 3$, then there exists a P-$OPD(8)$ with three blocks and $|R| = 1$. In the packing, six vertices have degree 7 and two vertices have degree 6. By the degree-type of P, the solutions of the degree-equations $x_1 + 3x_2 + 4x_3 = 7$ (or 6) are expressed in the following table.

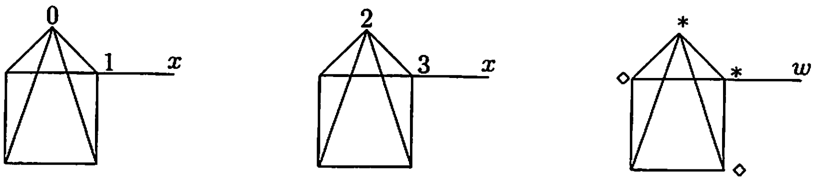
7	1	3	4
a	0	1	1
b	1	2	0

6	1	3	4
c	2	0	1
d	0	2	0

The degree-type distribution equation is expressed by the following matrix:

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 0 & 1 & 2 & 0 & 3 \\ 1 & 2 & 0 & 2 & 9 \\ 1 & 0 & 1 & 0 & 6 \\ 1 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 1 & 2 \end{array} \right) \end{array}$$

It has the unique solution: $a = 5$, $b = c = d = 1$. Denote the vertices with degree-types $3^1 4^1$ (for $a = 5$) by $0, 1, 2, 3, 4$, and others by x ($1^2 4^1$, for $b = 1$), y (3^2, for $c = 1$) and w ($1^1 3^2$, for $d = 1$). First, arrange x, x, w in three 1°-positions. Next, arrange $0, 1, 2, 3, 4$ and x in six 4°-positions. Without loss of generality, the distribution can be as follows:



where two $*$ are 4 and x. Further, two positions $\diamond$ must be arranged by 0 and 2, respectively. Thus, the vertex 4 must appear in 3°-position in first or second block, which is impossible. So, $p(8, P, 1) \leq 2$. We have

$p(8, P, 1) = 2 : (1,2,3,4,5,6), (0,2,4,6,7,1),$

$L = \{(0,1),(0,3),(0,5),(1,6),(2,6),(3,5),(3,6),(3,7),(4,7),(5,7)\}.$

$c(8, P, 1) = 4 : (1,2,3,4,5,6), (0,2,4,6,7,1), (3,5,7,2,6,1), (1,7,4,5,0,3),$

$R = \{(0,7),(1,4),(1,5),(1,7),(2,3),(2,7),(5,6),(4,5)\}.$

(4). By Lemma 1.1, there exists no P-$GD(9)$. So, $p(9, P, 1) \leq 3$ and $c(9, P, 1) \geq 5$. But, we have

$p(9, P, 1) = 3 : (1,4,5,2,3,6), (6,8,2,4,7,1), (0,3,8,5,7,2),$

$L = \{(0,1),(0,2),(0,4),(0,6),(1,6),(1,8),(3,5),(4,8),(5,6)\}.$

$c(9, P, 1) = 5 : (1,4,5,2,3,6), (6,8,2,4,7,1), (0,3,8,5,7,2),$

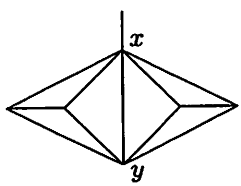
$(0,1,2,4,6,5), (8,6,1,4,3,5),$

$R = \{(1,2),(2,4),(6,4),(6,8),(4,8),(3,8),(1,4),(3,4),(3,6)\}.$ ■

Lemma 2.2 (1). *$p(6, Q, 1) = 1$ and $c(6, Q, 1) = 2$;*
(2). *$p(7, Q, 1) = 1$ and $c(7, Q, 1) = 3$;*
(3). *$p(8, Q, 1) = 2$ and $c(8, Q, 1) = 4$;*
(4). *$p(9, Q, 1) = 3$ and $c(9, Q, 1) = 5$;*
(5). *$p(11, Q, 1) = 5$ and $c(11, Q, 1) = 7$;*
(6). *$p(14, Q, 1) = 9$ and $c(14, Q, 1) = 11$.*

Proof. (1). A Q-$OPD(6)$: $(0,1,2,3,4,5)$. Its leave is a subgraph of Q, so Q-$OCD(6)$ can be obtained.

(2). Suppose that $p(7, Q, 1) = U(7, Q, 1) = 2$. The graph $K_7 \setminus Q$ is as follows.



Clearly, the graph Q has a 5°-vertex A, which is adjacent to a 4-cycle. However, in the graph $K_7 \setminus Q$, A can only be x or y. And, after deleting x or y, the subgraph of $K_7 \setminus Q$ has no 4-cycle. So, there is no subgraph Q in $K_7 \setminus Q$. Therefore, $p(7, Q, 1) = 1$ and the block is $(0,1,2,3,4,5)$. Furthermore, we have $c(7, Q, 1) = 3 : (0,1,2,3,4,5), (6,5,1,3,4,0), (2,5,3,0,4,6)$. Its excess is $R = \{(0,2),(0,3),(0,4),(2,3),(3,4),(4,5)\}$.

(3). Suppose that $p(8, Q, 1) = U(8, Q, 1) = 3$, then $|R| = 1$. From the degree-type $1^1 3^4 5^1$ of graph Q, we have the corresponding degree-equations and the degree-type distribution equation:

7	1	3	5
a	2	0	1
b	1	2	0

6	1	3	5
c	1	0	1
d	0	2	0

$$\begin{array}{cccccc} a & b & c & d & & \\ \end{array}$$
$$\begin{pmatrix} 2 & 1 & 1 & 0 & \vdots & 3 \\ 0 & 2 & 0 & 2 & \vdots & 12 \\ 1 & 0 & 1 & 0 & \vdots & 3 \\ 1 & 1 & 0 & 0 & \vdots & 6 \\ 0 & 0 & 1 & 1 & \vdots & 2 \end{pmatrix}$$

But the sum of the third and the forth equations contradict with the first equation, which shows that no solutions exist. So, $p(8, Q, 1) \leq 2$. And,

$p(8, Q, 1) = 2 :\ (0, 1, 2, 3, 4, 5),\ (7, 1, 3, 5, 6, 0),$

$L = \{(2,4),(1,5),(2,5),(4,5),(0,6),(2,6),(3,6),(4,6),(2,7),(4,7)\};$

$c(8, Q, 1) = 4 :\ (0, 1, 2, 3, 4, 5),\ (7, 1, 3, 5, 6, 0),$

$(2, 4, 6, 1, 5, 7),\ (0, 3, 4, 7, 6, 1),$

$R = \{(0,1),(0,3),(0,4),(0,7),(1,2),(1,6),(3,4),(6,7)\}.$

(4). By Lemma 1.1, there exists no Q-$GD(9)$. So, $p(9, Q, 1) \leq 3$ and $c(9, Q, 1) \geq 5$.

$p(9, Q, 1) = 3 :\ (0, 1, 2, 3, 4, 5),\ (5, 1, 3, 6, 7, 2),\ (8, 6, 2, 7, 4, 5),$

$L = \{(0,6),(0,7),(1,6),(1,8),(2,4),(3,7),(3,8),(4,5),(0,8)\};$

$c(9, Q, 1) = 5 :\ (0, 1, 2, 3, 4, 5), (5, 1, 3, 6, 7, 2), (8, 6, 2, 7, 4, 5),$

$(7, 6, 1, 8, 0, 3), (8, 5, 0, 2, 4, 3),$

$R = \{(0,2),(0,5),(0,8),(5,8),(1,7),(2,8),(4,8),(6,7),(7,8)\}.$

(5). Suppose that $p(11, Q, 1) = U(11, Q, 1) = 6$, which contains six blocks and $|R| = 1$. The degree-type of the graph Q is $1^1 3^4 5^1$. There are nine 10°-vertices and two 9°-vertices in the packing design. The corresponding degree-equations are expressed by the tables:

10	1	3	5
a	0	0	2
b	2	1	1
c	5	0	1
d	1	3	0
e	4	2	0

9	1	3	5
f	1	1	1
g	4	0	1
h	0	3	0
i	3	2	0

The degree-type distribution equation is expressed by the following matrix:

$$\begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array}$$

$$\begin{pmatrix} 0 & 2 & 5 & 1 & 4 & 1 & 4 & 0 & 3 & \vdots & 6 \\ 0 & 1 & 0 & 3 & 2 & 1 & 0 & 3 & 2 & \vdots & 24 \\ 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & \vdots & 6 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \vdots & 9 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \vdots & 2 \end{pmatrix}$$

It has only one solution: $a = 3,\ d = 6,\ h = 2,\ b = c = e = f = g = i = 0$. Denote the vertices with degree-types 5^2 (for $a = 3$) by x, y, z, then the 5-degree positions in all six blocks must be x, x, y, y, z, z. Each vertex beside x, y, z appears in at most three blocks, since the unique 5-degree vertex joins to others in Q. However, there are $d = 12$ vertices with degree-type $1^1 3^3$, each of which must appear in four blocks. This contradiction shows to exist no Q-$OPD(11)$, so $p(11, Q, 1) \leq 5$. Below, give a maximum Q-$PD(11)$ and a Q-$OCD(11)$:

$p(11, Q, 1) = 5:$ $(5,0,2,1,3,7), (6,10,9,8,7,5), (9,0,4,2,7,1),$
$(1,0,6,4,8,7), (10,3,4,5,8,0),$

$L = \{(1,10), (2,3), (2,6), (2,8), (2,10), (3,6), (3,7), (3,9), (4,7), (5,9)\};$

$c(11, Q, 1) = 7:$ $(5,3,1,8,6,10), (9,0,1,5,8,2), (2,3,8,10,9,0),$
$(4,0,2,1,3,5), (10,1,6,3,7,0), (7,0,5,2,6,1), (4,6,8,7,9,10),$

$R = \{(0,2), (1,3), (1,5), (1,7), (2,9), (3,6), (5,8), (6,8)\}$.

(6). Suppose that $p(14, Q, 1) = U(14, Q, 1) = 10$, which contains ten blocks and $|R| = 1$. The degree-type of the graph Q is $1^1 3^4 5^1$. There are twelve $13°$-vertices and two $12°$-vertices in the packing design. The corresponding degree-equations are:

12	1	3	5
a	2	0	2
b	1	2	1
c	4	1	1
d	7	0	1
e	0	4	0
f	3	3	0
g	6	2	0
h	9	1	0

13	1	3	5
i	0	1	2
j	3	0	2
k	2	2	1
l	5	1	1
m	8	0	1
n	1	4	0
p	4	3	0
q	7	2	0

The degree-type distribution equation is:

$$\begin{array}{cccccccccccccccc} a & b & c & d & e & f & g & h & i & j & k & l & m & n & p & q \end{array}$$

$$\left(\begin{array}{cccccccccccccccccc} 2 & 1 & 4 & 7 & 0 & 3 & 6 & 9 & 0 & 3 & 2 & 5 & 8 & 1 & 4 & 7 & \vdots & 10 \\ 0 & 2 & 1 & 0 & 4 & 3 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 4 & 3 & 2 & \vdots & 40 \\ 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & \vdots & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \vdots & 12 \end{array}\right)$$

It's solutions are:

$$\begin{cases} b = -2a - c - d + k + l + m + 2n + 2p + 2q - 14 \\ e = a - f - g - h - k - l - m - 2n - 2p - 2q + 16 \\ i = c + 2d + f + 2g + 3h + l + 2m + p + 2q + 4 \\ j = -c - 2d - f - 2g - 3h - k - 2l - 3m - n - 2p - 3q + 8 \end{cases}.$$

It is easy to know $i \le 5$ (degree-type $3^1 5^2$) by the structure of the graph. But, the third equation implies $i \ge 4$. Below, it will be proved that $i \ne 4, 5$. Therefore, there exists no Q-$OPD(14)$.

1. If $i = 5$, then this five vertices with degree-types $3^1 5^2$ will occupy all 5°-positions and five 3°-positions of ten blocks. It is easy to see that at most nine edges between this five vertices appear, which is a contradiction since there are ten edges between five vertices.

2. If $i = 4$, then this four vertices x, y, z, u of degree-type $3^1 5^2$ will occupy eight 5°-positions and four 3-degree positions of ten blocks. Since $\binom{4}{2} = 6$, we need to generate six edges between the 12 positions. There are only two possible forms: $C_3 \cup C_3$ and $(K_4 \setminus P_2) \cup P_2'$, where P_2 and P_2' are different edges. However, both can not form a K_4, which is a contradiction.

Thus, there exists no Q-$OPD(14)$, i.e., $p(14, Q, 1) \le 9$. Below, give a maximum Q-$PD(14)$ and a Q-$OCD(14)$:

$p(14, Q, 1) = 9:$ $(4,6,10,7,12,9), (7,0,5,2,6,1), (12,0,8,3,10,1),$
$(6,5,8,11,13,3), (1,0,9,5,11,6), (8,1,10,2,13,4),$
$(3,2,9,7,11,5), (4,0,2,1,3,5), (9,13,12,11,10,8),$

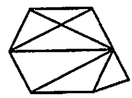
$L = \{(0,13), (2,12), (3,13), (4,11), (4,13),$
$(5,10), (5,12), (6,9), (7,8), (7,13)\};$

$c(14, Q, 1) = 11:$ $(1,0,2,4,3,7), (9,4,7,12,8,0), (13,2,3,7,10,12),$
$(10,0,8,3,9,1), (0,4,5,6,1,2), (10,4,5,11,6,12),$

$(11,0,13,4,12,7),(12,1,5,3,6,2),(7,0,5,2,6,8),$
$(13,5,8,6,9,1),(11,1,8,2,9,3),$
$R=\{(0,1),(0,2),(0,5),(0,6),(0,9),(1,4),(1,6),(4,5)\}.$ ■

Lemma 2.3 (1). *$p(6,T,1)=1$ and $c(6,T,1)=3$;*
(2). *$p(7,T,1)=1$ and $c(7,T,1)=3$;*
(3). *$p(7,T,1)=1$ and $c(7,T,1)=3$;*
(4). *$p(7,T,1)=1$ and $c(7,T,1)=3$.*

Proof. (1). A T-$OPD(6)$: $(0,1,2,3,4,5)$. Its leave is a subgraph of T, so T-$OCD(6)$ can be obtained.
(2). Note that T is a hexagon with two diameters and one chord. However, there is no such hexagon in the following graph $K_7\setminus T$:



Thus, we can not arrange other block in $K_7\setminus T$, so $p(7,T,1)=1$. Furthermore, $c(7,T,1)=3:$ $(0,1,2,3,4,5),(1,3,6,4,0,5),(2,4,5,3,0,6),$
$R=\{(0,5),(5,4),(0,3),(2,5),(5,6),(0,4)\}.$
(3). Suppose $p(8,T,1)=U(8,T,1)=3$, then $L=1$. The degree type of graph T is $2^1 3^4 4^1$. The corresponding degree-equations and the degree-type distribution equation are:

7	2	3	4
a	0	1	1
b	2	1	0

6	2	3	4
c	1	0	1
d	0	2	0
e	3	0	0

$$\begin{pmatrix} a & b & c & d & e & & \\ 0 & 2 & 1 & 0 & 3 & \vdots & 3 \\ 1 & 1 & 0 & 2 & 0 & \vdots & 12 \\ 1 & 0 & 1 & 0 & 0 & \vdots & 3 \\ 1 & 1 & 0 & 0 & 0 & \vdots & 6 \\ 0 & 0 & 1 & 1 & 1 & \vdots & 2 \end{pmatrix}.$$

From the second and fourth equations we get $d=3$, which contradicts with $c+d+e=2$. Thus, the equation system has no solution, $p(8,T,1)<3$. We have $p(8,T,1)=2:(0,1,2,3,4,5),(0,3,6,5,1,7),$
$L=\{(0,4),(2,4),(3,5),(1,6),(2,6),(4,6),(2,7),(3,7),(4,7),(5,7)\}.$
Adding two new blocks $(2,6,4,5,3,7),(0,4,7,5,6,1)$, we can get T-$OCD(8)$ and $R=\{(0,1),(0,7),(1,7),(4,7),(4,5),(4,6),(5,6),(3,6)\}.$
(4). By Lemma 1.1, T-$GD(9)$ not exists, so $p(9,T,1)\le 3$, $c(9,T,1)\ge 5$.

However, $p(9,T,1)=3:(0,1,2,3,4,5),(0,3,6,5,1,8),(2,6,4,0,7,8)$,

$$L=\{(3,5),(1,6),(1,7),(2,7),(3,7),(4,7),(5,7),(3,8),(5,8)\}.$$

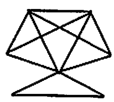
Adding two blocks $(5,3,7,4,0,2)$ and $(1,6,3,8,5,7)$, we can get T-$OCD(9)$ and $R=\{(0,2),(0,3),(0,4),(1,3),(2,5),(3,6),(3,7),(5,6),(5,7)\}$. ■

Lemma 2.4 (1). $p(6,H,1)=1$ *and* $c(6,H,1)=3$;
(2). $p(7,H,1)=1$ *and* $c(7,H,1)=3$;
(3). $p(8,H,1)=2$ *and* $c(8,H,1)=3$;
(4). $p(9,H,1)=3$ *and* $c(9,H,1)=5$;
(5). $p(11,H,1)=5$ *and* $c(11,H,1)=7$.

Proof. (1). A H-$OPD(6)$ can be taken as $(0,1,2,3,4,5)$. However, its leave is not a subgraph of H. Thus $c(6,H,1)\geq 3$. We have

$$c(6,H,1)=3:(0,1,2,3,4,5),(2,1,0,4,5,3),(1,0,3,2,5,4).$$
$$R=\{(0,4),(0,4),(0,1),(0,1),(1,5),(1,5),(4,5),(4,5),(0,3),(0,3),(0,2),(3,5)\}.$$

(2). Obviously, H is not a subgraph of the following graph $K_7\setminus H$, which implies $p(7,H,1)=1$.



$$p(7,H,1)=1:\ (0,1,2,3,4,5),$$
$$L=\{(0,2),(0,3),(0,6),(1,4),(1,6),(2,3),(2,5),(2,6),(3,5),(3,6),(4,6),(5,6)\}.$$
$$c(7,H,1)=3:\ (0,1,2,3,4,5),(0,6,1,5,3,2),(3,4,2,1,5,6),$$
$$R=\{(1,3),(1,5),(3,4),(3,5),(4,5),(2,4)\}.$$

(3). H-$OPD(8)$ on the set Z_8 contains three block $(0,1,2,3,4,5),(1,6,0,5,7,4)$ and $(6,2,0,5,3,7)$. Obviously, its leave is a subgraph of H, so H-$OCD(8)$ can be obtained.

(4). By Lemma 1.1, there exists no H-$GD(9)$, so $p(9,H,1)\leq 3$, $c(9,H,1)\geq 5$. And, $p(9,H,1)=3:(0,1,2,3,4,5),(1,6,0,5,7,4),(6,2,0,5,3,8)$.

$$L=\{(2,3),(2,7),(3,7),(6,7),(0,8),(1,8),(4,8),(5,8),(7,8)\}.$$

Adding two blocks $(1,4,2,6,7,8)$ and $(2,8,0,5,7,3)$, we can obtain $c(9,H,1)=5$, and $R=\{(1,4),(1,7),(4,6),(2,4),(2,8),(3,8),(2,7),(0,7),(5,7)\}$.

(5). A detailed proof is listed in Appendix A, which is presented on our website: http://qdkang.hebtu.edu.cn (as electronic results, in Online). By that appendix, $p(11,H,1)<6$. And,

$p(11,H,1)=5:(0,2,4,5,3,1),(0,5,1,7,6,4),(0,8,1,2,9,7),$
$(1,7,2,3,10,4),(3,8,4,5,9,6),$
$L=\{(0,10),(2,3),(2,6),(5,6),(5,10),$
$(6,10),(7,10),(8,9),(8,10),(9,10)\}.$

Adding two blocks $(8,10,0,7,1,9)$ and $(10,5,0,3,2,6)$, we can obtain $c(11,H,1)=7$, and $R=\{(0,1),(0,2),(0,5),(1,7),(1,8),(1,9),(2,10),(3,5)\}$. ∎

3 Holey designs of particular orders

Lemma 3.1 *P-$HD(9^4)$, P-$HD(9^{2t+1})$ and P-$HD(18^{t+2})$ exist for $t \geq 1$.*

Proof. A P-$HD(9^4)$ on the set $Z_9 \times Z_4$ is listed as follows, module $(9,-)$.

$(0_0,0_1,2_2,1_1,0_2,4_0),(0_0,2_1,6_2,3_1,8_2,4_0),(0_0,3_2,3_3,5_1,4_3,2_0),$
$(0_1,7_2,1_3,2_0,0_3,7_1),(0_0,5_3,1_2,6_3,7_2,4_3),(0_0,0_3,4_1,1_3,6_1,7_0).$

For the other designs see [13]. ∎

Lemma 3.2 *There exist Q-$HD(9^t)$ and Q-$HD(18^t)$ for $t \geq 3$.*

Proof. By [13], for $t \geq 3$, there exist Q-$HD(9^t)$ $(t \neq 6,8)$ and Q-$HD(18^t)$. We only need to construct:

<u>Q-$HD(9^6)$</u> on $Z_9 \times Z_6$. Take the following base blocks module $(9,-)$.

$(0_0,7_5,0_2,8_5,2_2,3_1),(0_0,1_2,0_3,3_2,1_3,4_1),(0_3,5_4,1_0,7_4,1_1,0_5),$
$(8_5,1_4,8_0,8_4,0_1,4_2),(0_0,3_5,0_1,1_5,1_1,5_1),(5_2,4_1,6_0,0_3,4_4,8_1),$
$(0_0,4_2,6_3,5_2,8_4,3_4),(0_0,1_4,2_5,5_4,4_5,7_4),(0_4,4_2,4_5,2_2,5_5,8_2),$
$(0_0,2_1,6_2,8_1,7_2,6_1),(0_0,4_3,5_5,7_3,6_5,2_3),(1_3,0_1,7_5,1_1,5_5,5_0),$
$(0_1,0_2,5_3,2_2,2_4,3_2),(0_4,8_3,2_1,6_3,2_5,3_2),(0_1,2_3,5_4,3_3,3_4,7_3).$

<u>Q-$HD(9^8)$</u> on $Z_9 \times Z_8$. Take the following base blocks module $(9,-)$.

$(0_0,1_1,2_2,2_1,4_2,0_4),(0_1,6_2,4_3,7_2,3_6,2_4),(0_2,4_3,4_4,2_3,5_4,1_3),$
$(0_3,4_4,7_5,5_4,5_5,5_0),(0_4,4_5,7_6,5_5,5_6,6_1),(0_5,4_6,0_7,5_6,7_7,0_1),$
$(0_6,6_7,2_3,7_7,1_0,6_4),(1_7,2_5,4_1,0_0,3_1,8_3),(0_0,5_1,8_3,6_1,6_3,2_4),$
$(0_1,4_2,7_4,5_2,5_4,4_4),(0_2,0_3,6_5,5_3,8_5,5_5),(8_3,6_4,5_6,0_1,7_6,1_0),$
$(0_4,1_0,0_7,8_5,5_7,2_0),(0_5,6_6,5_0,8_6,6_0,3_1),(2_6,2_7,0_1,3_7,2_1,5_4),$
$(3_7,0_0,1_2,3_0,3_2,5_0),(0_0,5_2,5_6,6_2,7_6,3_4),(0_1,7_3,5_7,5_3,4_7,8_4),$
$(8_4,8_1,3_6,0_2,7_7,6_5),(0_1,6_3,6_6,6_4,8_6,1_6),(0_0,1_3,1_5,4_4,2_7,5_4),$
$(0_0,3_3,7_5,5_3,6_6,6_5),(0_0,0_1,8_2,2_3,1_4,5_5),(0_2,6_4,1_7,7_4,8_7,8_3),$
$(0_5,2_2,7_6,0_2,5_7,4_1),(8_7,4_2,6_5,2_2,5_5,4_5),(5_3,0_6,8_7,2_4,5_0,1_6),$
$(0_0,7_1,0_5,8_1,2_5,8_5).$ ∎

Lemma 3.3[13] *T-$HD(9^4)$, T-$HD(9^{2t+1})$ and T-$HD(18^{t+2})$ exist for $t \geq 1$.*

Lemma 3.4 *There exist H-$HD(9^t)$ and H-$HD(18^t)$ for $t \geq 3$.*

Proof. By [13], for $t \geq 3$, there exist H-$HD(9^t)$ ($t \neq 6, 8$) and H-$HD(18^t)$. We only need to construct:

<u>H-$HD(9^6)$</u> on $Z_9 \times Z_6$. Take the following base blocks module $(9, -)$.

$$(0_4, 0_0, 1_5, 2_5, 0_2, 0_5), (4_4, 0_1, 1_3, 4_5, 1_1, 2_5), (0_2, 4_5, 1_2, 7_1, 4_4, 8_1),$$
$$(0_0, 2_2, 1_0, 0_1, 2_4, 3_3), (0_2, 0_3, 2_1, 4_5, 1_4, 5_5), (4_0, 0_2, 2_4, 6_5, 8_3, 7_5),$$
$$(0_0, 6_3, 4_1, 0_5, 4_4, 3_1), (0_0, 8_3, 3_0, 5_4, 6_2, 5_5), (3_0, 0_1, 4_2, 6_3, 1_4, 7_5),$$
$$(0_0, 2_1, 1_0, 2_0, 2_3, 0_2), (0_0, 5_1, 2_2, 8_2, 7_3, 3_4), (0_5, 1_0, 5_1, 5_2, 0_3, 7_4),$$
$$(7_4, 8_5, 1_0, 0_1, 0_2, 7_3), (8_3, 1_4, 0_5, 2_0, 0_1, 5_2), (0_2, 6_3, 7_4, 8_5, 2_0, 1_1).$$

<u>H-$HD(9^8)$</u> on $Z_9 \times Z_8$:

$(0_1, 2_2, 4_6, 6_4, 3_3, 1_5), (6_1, 3_2, 7_3, 5_4, 1_6, 0_7), (0_2, 8_1, 2_5, 6_6, 2_7, 4_6)$ mod$(9, 8)$;
$(0_2, 0_1, 1_3, 1_7, 0_5, 0_6)$ mod$(9, 4)$. ■

4 Incomplete designs of particular orders

Among the following eight lemmas, the proofs of four Lemmas are given in this section, but the others (i.e., Lemmas 4.2, 4.4, 4.6 and 4.8) will be listed in Appendix B on our website: http://qdkang.hebtu.edu.cn (as electronic results, in Online).

Lemma 4.1 *There exist P-$ID(9+\omega, \omega)$ for $2 \leq \omega \leq 8$ and $\omega = 12$.*

Proof. A P-$ID(9+\omega, \omega)$ consists of $\omega + 4$ blocks.

<u>$\omega = 2$</u>: $(Z_3 \times Z_3) \bigcup \{a, b\}$,
$(0_1, 1_0, 2_0, a, 1_2, 1_1), (0_0, 2_2, 1_2, b, 0_1, 1_1)$ mod$(3, -)$.

<u>$\omega = 3$</u>: $Z_9 \bigcup \{a, b, c\}$, $(0, a, 1, b, 2, 6), (0, c, 3, 6, 4, 1), (6, a, 7, b, 8, 3),$
$(3, a, 4, b, 5, 0), (1, c, 2, 3, 7, 4), (2, 5, 4, 8, 7, 0), (5, c, 6, 1, 8, 0).$

<u>$\omega = 4$</u>: $Z_9 \bigcup \{a, b, c, d\}$, $(2, a, 1, b, 0, 4), (3, a, 4, b, 5, 6), (6, a, 7, b, 8, 4),$
$(0, 6, 1, 8, 3, 7), (3, c, 1, d, 2, 6), (7, c, 8, d, 0, 5), (4, c, 6, d, 5, 8), (7, 5, 1, 4, 2, 8).$

<u>$\omega = 5$</u>: $(Z_3 \times Z_3) \bigcup \{a, b, c, d, e\}$,
$(0_1, a, 0_0, b, 2_2, 1_2), (0_0, c, 1_2, d, 1_1, 2_2), (2_2, 0_0, 2_0, e, 2_1, 1_1)$ mod$(3, -)$.

<u>$\omega = 6$</u>: $Z_9 \bigcup \{a, b, c, d, e, f\}$, $(6, a, 3, b, 4, 8), (5, a, 7, b, 0, 8), (1, a, 8, b, 2, 5),$
$(5, c, 6, d, 8, 2), (0, e, 1, f, 2, 4), (2, c, 7, d, 3, 4),$
$(4, c, 0, d, 1, 5), (8, e, 7, f, 6, 2), (5, e, 3, f, 4, 7), (7, 0, 6, 1, 3, 8).$

<u>$\omega = 7$</u>: $Z_9 \bigcup \{a, b, c, d, e, f, g\}$, $(0, a, 6, b, 1, 5), (4, a, 5, b, 7, 6), (8, a, 3, b, 2, 6),$
$(g, 1, 4, 0, 3, 6), (2, c, 8, d, 6, g), (2, c, 3, d, 1, 6), (5, e, 6, f, 0, 8),$

$(4,e,2,f,3,5),(7,c,5,d,0,2),(1,e,7,f,8,6),(g,2,5,8,7,3)$.

$\underline{\omega=8}$: $(Z_3\times Z_3)\bigcup\{a,b,c,d,e,f,g,h\}$, $(0_0,a,0_2,b,0_1,1_0)$, $(1_1,c,0_0,d,1_2,2_2),(1_2,e,0_1,f,0_0,1_0),(2_2,g,0_0,h,0_1,1_1)$ mod$(3,-)$.

$\underline{\omega=12}$: $Z_9\bigcup\{a,b,c,d,e,f,g,h,m,n,p,q\}$,
$(0,p,1,q,2,8),(0,a,4,b,3,p),(7,e,0,f,4,q),(4,g,2,h,8,q)$,
$(1,a,2,b,5,p),(7,m,1,n,5,4),(7,a,8,b,6,p),(1,c,6,d,4,p)$,
$(2,c,3,d,7,p),(5,c,0,d,8,p),(3,g,1,h,7,q),(8,m,0,n,3,6)$,
$(6,g,0,h,5,q),(8,e,1,f,6,q),(5,e,2,f,3,q),(6,m,2,n,4,3)$. ■

Lemma 4.2 *There exist P-ID$(18+\omega,\omega)$ for $\omega=2,4,5,6,7$ and 8.*

Lemma 4.3 *There exists a Q-ID$(9+\omega,\omega)$ for $\omega=4,6,7,8,11$ and 14.*

Proof. A Q-ID$(9+\omega,\omega)$ consists of $\omega+4$ blocks.

$\underline{\omega=4}$: $Z_9\bigcup\{x_1,x_2,x_3,x_4\}$, $(0,x_1,1,x_2,2,8),(1,x_3,6,x_4,7,3)$,
$(8,x_3,2,x_4,5,1),(3,x_1,4,x_2,5,6),(7,2,3,8,4,0)$,
$(0,x_3,3,x_4,4,5),(6,x_1,7,x_2,8,0),(5,1,2,6,4,7)$.

$\underline{\omega=6}$: $Z_9\bigcup\{x_1,\cdots,x_6\}$, $(1,x_3,2,x_4,6,3),(4,7,5,2,8,6)$,
$(0,x_1,1,x_2,2,6),(3,x_1,4,x_2,5,6),(0,x_3,3,x_4,4,5),(6,x_1,7,x_2,8,5)$,
$(7,x_3,5,x_4,8,0),(2,x_5,3,x_6,6,7),(7,x_5,1,x_6,4,3),(8,x_5,0,x_6,5,3)$.

$\underline{\omega=7}$: $Z_9\bigcup\{x_1,\cdots,x_7\},(0,x_1,1,x_2,2,x_7),(x_7,2,6,4,7,5),(8,x_7,1,7,3,2)$,
$(3,x_1,4,x_2,5,6),(6,x_1,7,x_2,8,5),(0,x_5,6,x_6,7,8),(0,x_3,3,x_4,4,5)$,
$(1,x_3,2,x_4,6,3),(5,x_3,7,x_4,8,1),(2,x_5,3,x_6,5,4),(4,x_5,1,x_6,8,5)$.

$\underline{\omega=8}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_8\}$,
$(7,x_5,1,x_6,8,2),(8,x_7,1,x_8,2,4),(0,x_1,1,x_2,2,8),(3,x_1,4,x_2,5,8)$,
$(6,x_1,7,x_2,8,3),(0,x_3,3,x_4,4,5),(4,x_7,5,x_8,6,1),(1,x_3,2,x_4,6,3)$,
$(5,x_3,7,x_4,8,1),(2,x_5,3,x_6,4,5),(6,x_5,0,x_6,5,2),(7,x_7,0,x_8,3,4)$.

$\underline{\omega=11}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{11}\}$,
$(6,x_1,7,x_2,8,x_{11}),(5,x_3,7,x_4,8,x_{11}),(2,x_7,7,x_8,8,3)$,
$(4,x_3,0,x_4,6,x_{11}),(0,x_7,3,x_8,6,5),(2,x_5,5,x_6,6,x_{11})$,
$(8,x_5,1,x_6,4,x_{11}),(1,x_3,2,x_4,3,x_{11}),(1,x_7,4,x_8,5,7)$,
$(4,x_9,2,x_{10},5,7),(0,x_1,1,x_2,2,x_{11}),(6,x_9,1,x_{10},3,5)$,
$(8,x_9,0,x_{10},7,3),(7,x_5,0,x_6,3,x_{11}),(3,x_1,4,x_2,5,x_{11})$.

$\underline{\omega=14}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{14}\}$,
$(2,x_7,7,x_8,8,x_{14}),(4,x_9,2,x_{10},5,x_{14}),(0,x_7,3,x_8,6,x_{14})$,
$(3,x_1,4,x_2,5,x_{13}),(2,x_5,5,x_6,6,x_{13}),(5,x_{11},0,x_{12},6,x_{14})$,
$(8,x_9,0,x_{10},9,x_{14}),(7,x_{11},1,x_{12},4,x_{14}),(5,x_3,7,x_4,8,x_{13})$,
$(4,x_3,0,x_4,6,x_{13}),(1,x_7,4,x_8,5,x_{14}),(6,x_9,1,x_{10},3,x_{14})$,

$(3,x_{11},8,x_{12},2,x_{14}),(7,x_5,0,x_6,3,x_{13}),(1,x_3,2,x_4,3,x_{13}),$
$(6,x_1,7,x_2,8,x_{13}),(0,x_1,1,x_2,2,x_{13}),(8,x_5,1,x_6,4,x_{13}).$ ∎

Lemma 4.4 *There exists a Q-ID*$(18+\omega,\omega)$ *for* $\omega=3,11,12$ *and* 14.

Lemma 4.5 *T-ID*$(9+\omega,\omega)$ *exists for* $2\le\omega\le 8$ *and* $\omega=12,13,15,16$.

Proof. A *T-ID*$(9+\omega,\omega)$ consists of $\omega+4$ blocks.

$\underline{\omega=2}$: $(Z_3\times Z_3)\bigcup\{x_1,x_2\}$,
$(0_0,x_1,1_2,0_1,1_1,1_0),(x_2,2_1,0_0,1_1,1_2,2_2)$ mod$(3,-)$.

$\underline{\omega=3}$: $Z_9\bigcup\{x_1,x_2,x_3\}$, $(6,x_3,1,2,4,8),(1,5,4,0,3,7),(8,x_3,7,2,5,0),$
$(0,x_3,2,8,3,6),(0,x_1,1,3,2,x_2),(3,x_1,4,6,5,x_2),(6,x_1,7,5,8,x_2).$

$\underline{\omega=4}$: $Z_9\bigcup\{x_1,x_2,x_3,x_4\}$,
$(0,x_1,1,7,2,x_2),(3,x_1,4,0,5,x_2),(7,5,3,0,6,8),(3,x_3,6,1,5,x_4),$
$(7,x_3,4,2,8,x_4),(0,x_3,2,3,1,x_4),(6,x_1,7,0,8,x_2),(8,1,4,6,2,5).$

$\underline{\omega=5}$: $(Z_3\times Z_3)\bigcup\{x_1,x_2,\cdots,x_5\}$,
$(1_0,x_1,0_1,2_2,1_2,x_2),(2_1,x_3,1_0,0_0,1_2,x_4),$
$(0_2,x_5,0_0,2_2,1_1,0_1)$ mod$(3,-)$.

$\underline{\omega=6}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_6\}$, $(5,7,2,6,1,4),(3,x_5,4,7,8,x_6),$
$(0,x_1,1,8,4,x_2),(8,x_3,6,0,5,x_4),(1,x_3,2,0,3,x_4),(5,x_5,1,3,6,x_6),$
$(0,x_3,4,6,7,x_4),(2,x_1,3,5,6,x_2),(5,x_1,8,3,7,x_2),(7,x_5,0,8,2,x_6).$

$\underline{\omega=7}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_7\}$,
$(0,x_1,1,7,2,x_2),(3,x_1,4,2,5,x_2),(6,x_1,7,5,8,x_2),(1,x_5,2,x_7,6,x_6),$
$(3,x_3,5,1,4,x_4),(7,x_3,8,2,6,x_4),(0,x_5,4,x_7,5,x_6),(0,x_3,2,3,1,x_4),$
$(3,x_5,7,x_7,8,x_6),(6,5,4,7,0,8),(0,x_7,3,8,1,6).$

$\underline{\omega=8}$: $(Z_3\times Z_3)\bigcup\{x_1,x_2,\cdots,x_8\}$,
$(0_0,x_1,1_1,0_1,1_2,x_2),(2_1,x_3,0_0,1_2,0_2,x_4),$
$(0_2,x_7,0_1,2_2,0_0,x_8),(0_2,x_5,0_0,1_0,1_1,x_6)$ mod$(3,-)$.

$\underline{\omega=12}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{12}\}$,
$(0,x_7,4,x_{12},2,x_8),(3,x_3,1,x_{10},5,x_4),(1,x_{11},5,6,0,x_{12}),$
$(0,x_5,3,x_{11},6,x_6),(1,x_7,8,x_{12},3,x_8),(6,x_9,3,8,0,x_{10}),$
$(5,x_7,7,x_{12},6,x_8),(6,x_3,4,x_{10},7,x_4),(0,x_3,2,x_{10},8,x_4),$
$(5,x_5,4,x_{11},8,x_6),(1,x_5,2,x_{11},7,x_6),(6,x_1,7,x_9,8,x_2).$
$(2,3,5,0,7,8),(0,x_1,1,x_9,2,x_2),(7,1,4,8,6,2),(3,x_1,4,x_9,5,x_2).$

$\underline{\omega=13}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{13}\}$, $(4,x_7,5,2,1,x_8),(3,x_7,0,8,2,x_8),$
$(4,x_9,0,7,8,x_{10}),(6,x_1,8,x_{11},7,x_2),(3,x_1,5,x_{11},4,x_2),$
$(0,x_{11},6,4,3,x_{12}),(0,x_1,1,x_{11},2,x_2),(2,x_{13},4,1,5,7),$
$(0,x_3,2,x_{12},8,x_4),(6,x_3,5,x_{12},4,x_4),(7,x_7,6,3,8,x_8),$

$(2,x_5,6,x_{13},1,x_6),(2,x_9,3,1,7,x_{10}),(5,x_5,0,x_{13},7,x_6),$
$(3,x_3,7,x_{12},1,x_4),(6,x_9,1,8,5,x_{10}),(4,x_5,8,x_{13},3,x_6).$

$\underline{\omega=15}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{15}\}$, $(5,1,7,4,8,3),$
$(6,x_3,4,x_{12},8,x_4),(4,x_5,1,x_{13},6,x_6),(0,x_3,2,x_{12},5,x_4),$
$(4,x_{13},5,8,7,x_{14}),(3,x_1,4,x_{11},8,x_2),(6,x_1,5,x_{11},7,x_2),$
$(0,x_1,1,x_{11},2,x_2),(5,x_5,0,x_{13},3,x_6),(3,x_3,1,x_{12},7,x_4),$
$(6,x_9,3,x_{15},5,x_{10}),(0,x_{15},6,1,2,8),(2,x_9,4,x_{15},1,x_{10}),$
$(7,x_7,6,x_{14},8,x_8),(7,x_5,2,x_{13},8,x_6),(5,x_7,2,x_{14},1,x_8),$
$(4,x_7,0,x_{14},3,x_8),(0,x_9,7,x_{15},8,x_{10}),(0,x_{11},3,2,6,x_{12}).$

$\underline{\omega=16}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_{16}\}$, $(6,x_1,7,x_{13},8,x_2),(5,x_5,0,x_{15},3,x_6),$
$(0,x_1,1,x_{13},2,x_2),(3,x_1,4,x_{13},5,x_2),(6,x_9,8,1,7,x_{10}),$
$(3,x_3,1,x_{14},7,x_4),(6,x_3,4,x_{14},8,x_4),(0,x_9,3,2,1,x_{10}),$
$(2,x_9,4,7,5,x_{10}),(5,x_7,1,x_{16},2,x_8),(4,x_7,0,x_{16},6,x_8),$
$(4,x_{11},8,2,5,x_{12}),(7,x_{11},0,6,2,x_{12}),(7,x_7,3,x_{16},8,x_8),$
$(7,x_5,2,x_{15},8,x_6),(6,x_{13},3,8,0,x_{14}),(4,x_{15},5,8,7,x_{16}),$
$(1,x_{11},6,5,3,x_{12}),(0,x_3,2,x_{14},5,x_4),(4,x_5,1,x_{15},6,x_6).$ ■

Lemma 4.6 *There exists a T-$ID(18+\omega,\omega)$ for $\omega=2,5$ and 8.*

Lemma 4.7 *There exists a H-$ID(9+\omega,\omega)$ for $\omega=3,5$ and 7.*

Proof. There are $\omega+4$ blocks on the set $Z_9\bigcup\{x_1,x_2,\cdots,x_\omega\}$.

$\underline{\omega=3}$: $(0,1,2,3,4,x_1),(0,2,3,x_1,6,x_2),(2,5,0,x_2,7,6),(1,8,2,6,x_3,4),$
$(0,3,x_1,x_2,8,x_3),(5,1,6,x_2,4,7),(8,5,x_1,x_3,7,3).$

$\underline{\omega=5}$: $(x_3,0,x_4,x_5,1,2),(x_4,7,x_3,0,8,2),(7,5,0,2,6,8),$
$(x_2,7,x_1,x_5,8,4),(5,1,7,8,3,6),(3,x_3,5,6,x_4,4),$
$(0,x_1,2,3,x_2,1),(x_1,5,x_2,x_5,6,4),(x_5,3,0,1,4,2).$

$\underline{\omega=7}$: $(x_6,1,x_5,8,3,6,),(3,x_3,5,6,x_4,4),(x_5,0,x_6,5,2,6),(6,5,1,3,7,8),$
$(0,x_1,2,3,x_2,1),(x_4,7,x_3,0,8,2),(5,x_5,4,8,x_6,7),(x_7,3,0,1,4,2),$
$(x_1,5,x_2,x_7,6,4),(x_2,7,x_1,x_7,8,4),(x_3,0,x_4,x_7,1,2).$ ■

Lemma 4.8 H-$ID(18+\omega,\omega)$ *exists for* $\omega=2,4,6,8,11,13,15$ *and 17.*

5 Constructions for OPD and OCD

Among the following ten lemmas, the proofs of four Lemmas are given in this section, but the others (i.e., Lemmas 5.2, 5.4, 5.5, 5.7, 5.9 and 5.10) will be listed in Appendix C on http://qdkang.hebtu.edu.cn (as electronic results, in Online).

Lemma 5.1 *There exist P-OPD*$(9+\omega)$ *and P-OCD*$(9+\omega)$ *for* $2 \le \omega \le 8$.

Proof. The leave of the given P-$OPD(9+\omega)$, consisting of $\omega+4+\lfloor\frac{\omega(\omega-1)}{18}\rfloor$ blocks, are all subgraphs of P. Thus, the corresponding P-$OCD(9+\omega)$ can be obtained.

$\underline{\omega=2}$: $(Z_3 \times Z_3)\bigcup\{x_1,x_2\}$, $L=\{(x_1,x_2)\}$,
$(0_1,1_0,2_0,x_1,1_2,1_1),(0_0,2_2,1_2,x_2,0_1,1_1)$ mod $(3,-)$.

$\underline{\omega=3}$: $Z_9\bigcup\{x_1,x_2,x_3\}$, $L=\{(x_1,x_2),(x_2,x_3),(x_3,x_1)\}$,
$(0,x_1,1,x_2,2,6),(0,x_3,3,6,4,1),(6,x_1,7,x_2,8,3),(2,5,4,8,7,0)$,
$(3,x_1,4,x_2,5,0),(1,x_3,2,3,7,4),(5,x_3,6,1,8,0)$.

$\underline{\omega=4}$: Z_{13}, $L=\{(6,8),(10,3),(3,6),(6,1),(1,10),(1,11)\}$,
$(4,5,9,6,12,3),(11,3,7,4,8,9),(2,5,10,6,11,9),(0,1,2,3,4,10)$,
$(0,9,10,11,12,1),(12,8,10,7,2,4),(1,3,5,7,9,2),(0,5,6,7,8,1)$.

$\underline{\omega=5}$: Z_{14}, $L=\{(2,4)\}$, $(2,5,12,7,13,0),(1,6,10,13,11,2)$,
$(0,1,4,3,2,9),(0,9,10,11,12,1),(8,2,10,12,6,9),(1,3,5,7,9,13)$,
$(7,3,11,4,10,5),(9,4,5,11,8,13),(0,5,6,7,8,1),(13,6,4,12,3,8)$.

$\underline{\omega=6}$: $Z_9\bigcup\{x_1,x_2,\cdots,x_6\}$,
$(6,x_1,3,x_2,4,8),(5,x_1,7,x_2,0,8),(1,x_1,8,x_2,2,5),(5,x_3,6,x_4,8,2)$,
$(0,x_5,1,x_6,2,4),(2,x_3,7,x_4,3,4),(4,x_3,0,x_4,1,5),(8,x_5,7,x_6,6,2)$,
$(x_1,x_2,x_3,x_4,x_5,x_6),(5,x_5,3,x_6,4,7),(7,0,6,1,3,8)$,
$L=\{(x_6,x_1),(x_6,x_2),(x_6,x_3),(x_6,x_4),(x_2,x_4),(x_3,x_5)\}$.

$\underline{\omega=7}$: $Z_{13}\bigcup\{x_1,x_2,x_3\}$,
$(x_2,5,11,6,12,1),(10,x_3,4,12,8,5),(0,x_2,4,5,9,7),(3,x_1,4,6,5,1)$,
$(2,x_3,5,10,6,1),(9,x_3,1,11,3,12),(3,x_2,8,7,10,0),(6,x_1,7,0,8,1)$,
$(0,3,2,x_1,1,10),(x_1,9,10,11,12,2),(8,2,11,4,9,6),(1,x_2,2,4,7,5)$,
$(x_3,11,0,12,7,2)$, $L=\{(x_1,x_2),(x_2,x_3),(x_3,x_1)\}$.

$\underline{\omega=8}$: $(Z_5 \times Z_3)\bigcup\{x_1,x_2\}$, $L=\{(x_1,x_2)\}$, $(4_2,0_1,2_1,1_1,1_2,1_0)$,
$(0_0,x_1,1_2,x_2,1_1,1_0),(0_0,1_0,4_1,2_0,4_2,0_2)$ mod $(5,-)$. ■

Lemma 5.2 *P-OPD*$(18+\omega)$ *and P-OCD*$(18+\omega)$ *exist for* $2 \le \omega \le 8$.

Lemma 5.3 *Q-OPD*$(9+\omega)$ *and Q-OCD*$(9+\omega)$ *exist for* $\omega = 3,4,6,7,8$.

Proof. The leave of the given Q-$OPD(9+\omega)$, consisting of $\omega+4+\lfloor\frac{\omega(\omega-1)}{18}\rfloor$ blocks, are all subgraphs of Q. Thus, the corresponding Q-$OCD(9+\omega)$ can be obtained.

$\underline{\omega=3}$: Z_{12}, $L=\{(4,6),(4,7),(9,10)\}$,
$(1,3,5,6,8,7),(3,6,10,7,11,9),(2,4,5,7,9,6),(11,1,9,5,10,0)$,

$(0,1,2,3,4,5),(8,2,10,4,11,5),(0,6,7,8,9,10)$.

$\underline{\omega=4}$: Z_{13}, $L=\{(0,2),(0,4),(4,6),(4,7),(0,12),(7,12)\}$,
$(2,4,5,7,9,6),(5,8,10,9,11,0),(0,11,2,3,4,1),(12,1,9,3,11,5),$
$(1,3,5,6,8,7),(0,6,7,8,9,10),(10,3,6,11,7,1),(12,2,8,4,10,6)$.

$\underline{\omega=6}$: Z_{15}, $L=\{(0,10),(0,14),(2,14),(6,8),(6,14),(8,14)\}$,
$(0,2,3,4,5,7),(9,0,6,7,8,14),(11,0,13,10,12,1),(2,8,10,9,11,12),$
$(1,4,8,5,9,0),(1,2,6,3,7,10),(5,10,14,11,3,12),(3,8,12,9,13,14),$
$(4,6,10,9,11,2),(13,6,12,7,5,2),(14,13,1,12,4,7)$.

$\underline{\omega=7}$: Z_{16}, $L=\{(1,13),(2,13),(7,14)\}$, $(0,10,11,12,13,15),$
$(1,5,8,10,12,6),(3,5,6,10,14,11),(2,7,10,15,12,11),(1,2,4,7,9,3),$
$(12,8,14,9,3,4),(6,8,11,14,2,12),(15,1,11,13,14,9),(0,2,3,4,5,1),$
$(9,5,10,4,11,13),(0,6,7,8,9,14),(7,3,13,5,15,11),(4,6,13,8,15,14)$.

$\underline{\omega=8}$: $(Z_5\times Z_3)\bigcup\{x_1,x_2\}$, $L=\{(x_1,x_2)\}$,
$(0_0,x_1,1_2,x_2,1_1,0_2),(4_2,1_2,1_1,2_1,0_1,0_2),$
$(0_0,1_0,4_1,2_0,4_2,0_1)$ mod $(5,-)$. ■

Lemma 5.4 *There exist Q-OPD$(18{+}\omega)$ and Q-OCD$(18{+}\omega)$ for $2\le\omega\le 8$ and $\omega=11,12,14$.*

Lemma 5.5 *Q-OPD$(36+\omega)$ and Q-OCD$(36+\omega)$ exist for $\omega=3$ and 12.*

Lemma 5.6 *There exist T-OPD$(9{+}\omega)$ and T-OCD$(9{+}\omega)$ for $2\le\omega\le 8$.*

Proof. The leave of the given T-$OPD(9{+}\omega)$, consisting of $\omega{+}4{+}\lfloor\frac{\omega(\omega-1)}{18}\rfloor$ blocks, are all subgraphs of T. Thus, the corresponding T-$OCD(9+\omega)$ can be obtained.

$\underline{\omega=2}$: T-$OPD(9+2)$ is just T-$ID(9+2,2)$ in Lemma 4.5.

$\underline{\omega=3}$: Z_{12}, $L=\{(1,11),(10,11),(4,6)\}$,
$(0,1,2,3,4,5),(0,3,6,1,5,7),(0,4,8,1,7,9),(1,3,10,0,11,9),$
$(2,10,4,9,5,11),(6,8,2,7,3,9),(7,10,8,5,6,11)$.

$\underline{\omega=4}$: Z_{13}, $L=\{(9,10),(10,11),(11,12),(12,9),(10,12),(5,9)\}$,
$(0,3,6,1,5,7),(4,10,2,6,5,12),(0,4,11,5,8,9),(8,7,2,9,3,11),$
$(0,1,2,3,4,5),(8,0,10,1,12,6),(4,6,9,1,11,7),(8,1,3,10,7,12)$.

$\underline{\omega=5}$: Z_{14}, $L=\{(0,13)\}$, $(0,1,2,3,4,5),(1,12,8,3,10,13),$
$(0,7,9,1,3,12),(3,9,5,12,11,13),(0,8,10,1,5,11),(5,6,7,8,9,10),$
$(1,6,11,2,12,7),(6,10,2,8,4,13),(2,7,4,12,13,9),(0,3,6,8,11,4)$.

$\underline{\omega=6}$: $Z_9\bigcup\{x_1,\cdots,x_6\}$,
$(0,x_1,1,8,4,x_2),(2,x_1,3,5,6,x_2),(5,x_1,8,3,7,x_2),(1,x_3,2,0,3,x_4),$
$(0,x_3,4,6,7,x_4),(8,x_3,6,0,5,x_4),(3,x_5,4,7,8,x_6),(7,x_5,0,8,2,x_6),$

$(5,x_5,1,3,6,x_6),(5,7,2,6,1,4),(x_1,x_2,x_3,x_4,x_5,x_6),$

$L=\{(x_1,x_4),(x_1,x_5),(x_2,x_4),(x_2,x_6),(x_4,x_6),(x_3,x_5)\}.$

$\underline{\omega=7}$: Z_{16}, $L=\{(1,3),(5,9),(7,9)\}$, $(2,4,6,10,14,11)$, $(5,6,7,8,9,10),(0,3,6,1,5,8),(0,10,11,13,12,15),(10,3,8,11,7,15),$ $(3,11,9,14,1,15),(0,4,7,1,8,12),(13,2,10,1,9,4),(0,9,13,1,12,14),$ $(2,7,14,5,13,8),(0,1,2,3,4,5),(11,4,12,2,15,5),(15,13,6,12,3,14).$

$\underline{\omega=8}$: $(Z_5\times Z_3)\bigcup\{x_1,x_2\}$, $L=\{(x_1,x_2)\}$, $(3_0,0_0,0_1,1_1,4_2,0_2),$ $(0_1,x_1,1_0,1_2,3_2,x_2),(3_1,2_0,0_1,2_2,1_0,4_2)$ mod $(5,-)$. ■

Lemma 5.7 *T-$OPD(18+\omega)$ and T-$OCD(18+\omega)$ exist for $2\le\omega\le 8$.*

Lemma 5.8 *There exist H-$OPD(9+\omega)$ and H-$OCD(9+\omega)$ for $3\le\omega\le 8$.*

Proof. The leave of the given H-$OPD(9+\omega)$, consisting of $\omega+4+\lfloor\frac{\omega(\omega-1)}{18}\rfloor$ blocks, are all subgraphs of H. Thus, the corresponding H-$OCD(9+\omega)$ can be obtained.

$\underline{\omega=3}$: Z_{12}, $L=\{(5,7),(7,9),(3,8)\}$,
$(5,11,0,7,10,9),(0,3,5,7,8,9),(1,8,2,6,9,4),(11,1,6,7,4,10),$ $(0,1,2,3,4,5),(2,10,8,3,11,6),(0,2,3,5,6,7).$

$\underline{\omega=4}$: Z_{13}, $L=\{(1,6),(6,9),(1,12),(2,12),(2,9),(5,12)\}$, $(0,1,2,3,4,5),(0,10,1,5,11,12),(11,2,10,6,4,8),(0,2,3,5,6,7),$ $(10,3,12,11,6,8),(9,1,8,4,12,7),(11,7,5,4,9,10),(0,3,5,7,8,9).$

$\underline{\omega=5}$: Z_{14}, $L=\{(3,12)\}$, $(6,1,9,11,4,12),(11,0,10,12,7,13),$ $(5,7,1,4,13,9),(5,10,6,9,11,12),(0,3,5,7,8,9),(1,8,2,3,10,4),$ $(2,12,13,8,6,9),(0,2,3,5,6,7),(10,13,2,3,11,8),(0,1,2,3,4,5).$

$\underline{\omega=6}$: Z_{15}, $L=\{(0,11),(0,12),(8,11),(8,12),(7,12),(5,12)\}$, $(0,13,1,2,14,10),(0,5,1,7,6,4),(8,3,2,12,6,13),(0,8,1,2,9,7),$ $(0,2,4,5,3,1),(3,10,1,2,11,6),(1,7,2,11,12,4),(5,9,4,8,14,6),$ $(8,4,11,13,5,10),(14,12,10,9,11,13),(3,7,10,13,9,14).$

$\underline{\omega=7}$: Z_{16}, $L=\{(2,15),(5,15),(9,11)\}$, $(11,2,10,6,4,8),$ $(6,1,4,7,15,9),(14,8,10,12,9,13),(0,13,1,2,14,15),(0,3,5,7,8,9),$ $(5,13,12,3,14,7),(0,1,2,3,4,5),(1,8,3,6,12,15),(3,10,6,7,11,15),$ $(0,2,3,5,6,7),(4,12,2,5,9,7),(10,13,4,6,14,11),(0,10,1,5,11,12).$

$\underline{\omega=8}$: $(Z_5\times Z_3)\bigcup\{x_1,x_2\}$, $L=\{(x_1,x_2)\}$, $(x_1,2_1,1_0,4_1,3_2,0_0),$ $(x_2,0_1,1_1,2_2,4_2,0_0),(0_2,0_0,1_0,2_0,0_1,1_2)$, mod $(5,-)$. ■

Lemma 5.9 *H-$OPD(18+\omega)$ and H-$OCD(18+\omega)$ exist for $2\le\omega\le 8$ and $w=11,13,15,17$.*

Lemma 5.10 *There exist H-OPD*$(36+\omega)$ *and H-OCD*$(36+\omega)$ *for* $\omega = 2,4,6,8,11,13,15$ *and 17.*

6 Conclusion

Theorem 6.1 *There exist G-OPD*(v) *and G-OCD*(v) *for* $G \in \{P,Q,T,H\}$ *and* $v \geq 6$ *except*

(1). $p(7,P,1)=1, p(8,P,1)=2, p(9,P,1)=3, c(9,P,1)=5$;

(2). $p(7,Q,1)=1, p(8,Q,1)=2, p(9,Q,1)=3, c(9,Q,1)=5,$
$p(11,Q,1)=5, p(14,Q,1)=9$;

(3). $p(7,T,1)=1, p(8,T,1)=2, p(9,T,1)=3, c(9,T,1)=5$;

(4). $c(6,H,1)=3, p(7,H,1)=1, p(9,H,1)=3,$
$c(9,H,1)=5, p(11,H,1)=5.$

Proof. Let $v = 9t+\omega$. For $t=0$ and $6 \leq \omega \leq 9$, see Section 2. For $1 \leq t \leq 2$ and $2 \leq \omega \leq 8$, see Sections 2 and 5. Below, by the existence spectrum of G-$GD(v)$ in Lemma 1.1, consider only the cases $t \geq 3$ and $2 \leq \omega \leq 8$. By Lemma 1.2, we will use four recursive constructions as follows.

W1: $HD(9^t)$, $ID(9+\omega,\omega)$, $OPD(9+\omega) \Longrightarrow OPD(9t+\omega)$;

W2: $HD(9^{t-1})$, $ID(18+\omega, 9+\omega)$, $OPD(18+\omega) \Longrightarrow OPD(9t+\omega)$;

W3: $HD(18^{\frac{t}{2}})$, $ID(18+\omega,\omega)$, $OPD(18+\omega) \Longrightarrow OPD(9t+\omega)$;

W4: $HD(18^{\frac{t-1}{2}})$, $ID(27+\omega, 9+\omega)$, $OPD(27+\omega) \Longrightarrow OPD(9t+\omega)$.

Noting that the difference of the known G-$HD(m^t)$ for the graph G, the discussion is separated into two cases as follows.

Case $G \in \{P,T\}$: There exist $HD(9^4), HD(9^{2t+1})$ and $HD(18^{t+2})$ for $t \geq 1$ by Lemmas 3.1 and 3.3.

method	t	P-$OPD(9t+\omega)$	T-$OPD(9t+\omega)$
W1	$t=4$ or odd $t \geq 3$	$2 \leq \omega \leq 8$ (Lemmas 4.1, 5.1)	$2 \leq \omega \leq 8$ (Lemmas 4.5, 5.6)
W2	even $t \geq 6$	$\omega = 3$ (Lemmas 4.1, 5.2)	$\omega = 3,4,6,7$ (Lemmas 4.5, 5.7)
W3	even $t \geq 6$	$\omega = 2,4,5,6,7,8$ (Lemmas 4.2, 5.2)	$\omega = 2,5,8$ (Lemmas 4.6, 5.7)

Case $G \in \{Q,H\}$: There exist $HD(9^t)$ and $HD(18^t)$ for $t \geq 3$ by Lemmas 3.2 and 3.4.

method	t	$Q\text{-}OPD(9t+\omega)$	$H\text{-}OPD(9t+\omega)$
W1	$t \geq 3$	$\omega = 4,6,7,8$ (L4.3, 5.3)	$\omega = 3,5,7$ (L4.7, 5.8)
W2	$t \geq 4$	$\omega = 2,5$ (L4.3, 5.4)	
W3	even $t \geq 6$	$\omega = 3$ (L4.4, 5.4)	$\omega = 2,4,6,8$ (L 4.8, 5.9)
W4	odd $t \geq 7$	$\omega = 3$ (L4.4, 5.4)	$\omega = 2,4,6,8$ (L4.8, 5.9)
direct	$t = 3$ $t = 3,4,5$	$\omega = 2,5$ (L5.4) $\omega = 3$ (L5.4, 5.5)	$\omega = 2,4,6,8$ (L5.9,5.10)

This completes the proof. ■

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