# On the Constructions of New Families of Graceful Graphs

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**Abstract**

Suppose that graphs $H$ and $G$ are graceful, and that at least one of $H$ and $G$ has an α-labeling. Four graph operations on $H$ and $G$ are provided. By utilizing repeatedly or in turn the four graph operations, we can construct a large number of graceful graphs. In particular, if both $H$ and $G$ have α-labelings, then each of the graphs obtained by the four graph operations on $H$ and $G$ has an $\alpha$-labeling.

## 1. Introduction

All graphs considered here will be finite, undirected, and without loops and multiple edges. For any graph $G$ with $n$ edges, the symbols $V(G)$ and $E(G)$ will denote its vertex set and edge set, respectively. A *graceful labeling* of $G$ is an injection $f$ of $V(G)$ into the set $\{0, 1, \ldots, n\}$ with the property: if, for each edge $e \in E(G)$ with the end vertices $u, v \in V(G)$, the value $f^*(e)$ of the edge $e$ is defined by $f^*(e) = |f(u) - f(v)|$, then $f^*$ is a bijection of $E(G)$ onto the set $\{1, 2, \ldots, n\}$. A graceful labeling $f$ is an *α-labeling* if there is an integer $\lambda$ ($0 \le \lambda \le n - 1$) such that for each edge $(u, v)$,

$$min\{f(u), f(v)\} \le \lambda < max\{f(u), f(v)\}.$$

Clearly, a graph admitting an $\alpha$-labeling is necessarily bipartite. For the sake of convenience, we shall call an $\alpha$-labeling a $\lambda$-graceful labeling. A graph with a graceful labeling or a λ-graceful labeling is said to be *graceful* or *λ-graceful*, respectively.

Let $G$ be a graceful graph with $n$ edges and let $f$ be a graceful labeling of $G$. The graceful labeling $f^c$ of $G$ given for each vertex $u$ by

$$f^c(u) = n - f(u)$$

is called *complementary labeling* [6] of $f$.

Suppose that $f$ is a $\lambda$-graceful labeling of $G$, and that $(A, B)$ is a bipartition of $G$, that is, a partition of $V(G)$ into two independent subsets $A$ and $B$. Throughout this paper, we will assume $A$ to be the part of the bipartition of the vertex set of $G$ for which $f(u) \le \lambda$, and $B$ the part of the bipartition of the vertex set of $G$ for which $f(u) > \lambda$.

The *inverse labeling* $f^i$ [6] of $f$ is given by

$$f^i(u) = \begin{cases} \lambda - f(u) & \text{if } u \in A, \\ n + \lambda + 1 - f(u) & \text{if } u \in B. \end{cases}$$

Note that if $f$ is a $\lambda$-graceful labeling of $G$ then $f^c$ and $f^i$ are $(n - \lambda - 1)$- and $\lambda$-graceful labelings of $G$, respectively. Let $f^{c,i}$ and $f^{i,c}$ be inverse and complementary labelings of $f^c$ and $f^i$ of $G$ defined as

$$f^{c,i}(u) = \begin{cases} 2n - \lambda - f^c(u) & \text{if } u \in A, \\ n - \lambda - 1 - f^c(u) & \text{if } u \in B; \end{cases}$$

and

$$f^{i,c}(u) = n - f^i(u), \text{ for each vertex } u \in V(G).$$

It should be noted that both $f^{c,i}$ and $f^{i,c}$ are also $(n - \lambda - 1)$-graceful labelings of $G$. In fact,

$$f^{c,i} = f^{i,c} = \begin{cases} n - \lambda + f(u) & \text{if } u \in A, \\ f(u) - \lambda - 1 & \text{if } u \in B. \end{cases}$$

If $f(u) = i$, then $f^c(u) = n - i$. Moreover, $f^i(u) = \lambda - i$, $f^{c,i}(u) = n - \lambda + i$, if $u \in A$ and $f^i(u) = n + \lambda + 1 - i$, $f^{c,i}(u) = i - \lambda - 1$, if $u \in B$. Consequently, if $f(u_1) = \lambda$, $f(u_2) = \lambda + 1$ and $f(u_3) = n$ then we have $f^i(u_1) = f^{c,i}(u_2) = f^c(u_3) = 0$.

Snevily [8] proved that if two graphs $G_1$ and $G_2$ have α-labelings then their weak tensor product $G_1 \overline{\otimes} G_2$ has an α-labeling. Koh, Rogers, and Tan [4, 5] provided methods for combining graceful trees to yield larger graceful trees. Wu [11, 12] gave a number of methods for constructing larger graceful graphs from graceful graphs. Further results on graceful labelings can refer to a dynamic survey [2].

We also find graceful labelings and λ-graceful labelings attractive because of the following theorems.

**Theorem 1.1.** [7] *Let $G$ be a graph with $n$ edges having an $\alpha$-labeling. Then the complete graph $K_{2pn+1}$ can be decomposed into the isomorphic copies of $G$, where $p$ is any positive integer.*

**Theorem 1.2.** [10] *Suppose that $G$ is a graph with $n$ edges, and let $\Theta_k G$ be the class of graphs obtained from $G$ by adding $k$ ($\geq 1$) distinct pendent edges to the vertices of $G$. If $G$ is graceful, then the complete graph $K_{2(n+k)+1}$ can be decomposed into the isomorphic copies of $H$ for each positive integer $k$ and every $H \in \Theta_k G$.*

## 2. A necessary condition

The necessary condition for an Eulerian graph to have a graceful labeling was presented by Rosa [7].

**Theorem 2.1.** [7] *If an Eulerian graph G with n edges has a graceful labeling, then* $n \equiv 0$ *or* 3 (*mod* 4).

In [9] Sheppard proved that there are exactly $n!$ graceful graphs with $n$ edges. Thus, we first investigate the number of $\lambda$-graceful graphs with $n$ edges. By $|G(n,\lambda)|$ we mean the number of $\lambda$-graceful graphs with $n$ edges (including isomorphic graphs). Since for any graph $G$ with $n$ edges a $\lambda$-graceful labeling is also a ($n - \lambda - 1$)-graceful labeling, it suffices to consider the $\lambda$-graceful labeling with $0 \le \lambda \le \left\lfloor \frac{n-1}{2} \right\rfloor$.

**Theorem 2.2.**

(1) *If n is even, then* $|G(n,\lambda)| = 1^2 2^2 \cdots \lambda^2 (\lambda+1)^{n-2\lambda},\ 0 \le \lambda \le \frac{n-2}{2}$.

(2) *If n is odd, then* $|G(n,\lambda)| = \begin{cases} 1^2 2^2 \cdots \lambda^2 (\lambda+1)^{n-2\lambda}, & 0 \le \lambda \le \frac{n-3}{2}, \\ 1^2 2^2 \cdots (\frac{n-1}{2})^2 \frac{n+1}{2}, & \lambda = \frac{n-1}{2}. \end{cases}$

*Proof.*

(1) Suppose that $G$ is a graph with $n$ edges. For each $j$, where $1 \le j \le n$, let $S_\lambda(j)$ denote the set of edges $(u, v)$ such that $|f(u) - f(v)| = j$ for some $\lambda$-graceful labeling $f$, and let $|S_\lambda(j)|$ be the number of distinct edges in $S_\lambda(j)$. For brevity, if $f$ is a $\lambda$-graceful labeling, we describe an edge $(u, v)$ by its vertex-labels $(f(u), f(v))$. Observing the value of each edge in the $\lambda$-graceful graph $G$, we have

$S_0(j) = \{(j, 0)\}$, $1 \le j \le n$, and

$$S_i(j) = \begin{cases} \{(i+1, i-j+1), (i+2, i-j+2), \cdots, (i+j, i)\}, & 1 \le j \le i, \\ \{(j, 0), (j+1, 1), \cdots, (j+i, i)\}, & i+1 \le j \le \frac{n}{2}. \end{cases}$$

$$(1 \le i \le \frac{n-2}{2})$$

It is easy to see that

$|S_0(j)| = 1$, $1 \le j \le n$, and

$$|S_i(j)| = \begin{cases} j, & 1 \le j \le i, \\ i+1, & i+1 \le j \le \frac{n}{2}. \end{cases} \quad (1 \le i \le \frac{n-2}{2})$$

The proof then follows from the fact that

$$|G(n, \lambda)| = |S_\lambda(1)| \cdot |S_\lambda(2)| \cdots |S_\lambda(n)|.$$

(2) The proof is similar to that of (1) and omitted. □

*Remark.* The $\lambda$-graceful graph considered in Theorem 2.2 could be disconnected. As an example consider the $\lambda$-graceful graph $G$ with $E(G) = \{(7, 0), (6, 0), (7, 2), (5, 1), (6, 3), (4, 2), (4, 3)\}$.

For the following Theorem the reader is referred to [3, ch.2, §6, Th.2].

**Theorem 2.3.** *Equation* $\sum_{i=1}^{p} d_i x_i \equiv \binom{n}{2}$ *(mod n) has a solution* $(x_1, x_2, \ldots, x_p)$ *of integers if and only if g.c.d.* $(d_1, d_2, \ldots, d_p, n) \mid \binom{n}{2}$.

Assume $V(G) = \{u_1, u_2, \ldots, u_p\}$ to be the vertex set of $G$, and $d(u_i) = d_i$ to be the degree of vertex $u_i$ in $G$, $1 \le i \le p$. Consider, now, the necessary condition for $G$ to have a $\lambda$-graceful labeling.

**Theorem 2.4.** *Let* $(d_1, d_2, \ldots, d_p)$ *be the degree sequence of G. If a graph G with n edges is* $\lambda$*-graceful, then g.c.d.* $(d_1, d_2, \ldots, d_p, n) \mid \binom{n}{2}$.

*Proof.* Suppose that $f$ is any $\lambda$-graceful labeling of $G$, and let $f(u_i) = r_i$, where $u_i \in V(G)$ and $1 \le i \le p$. Let $(f(v_i), f(w_i))$ denote the edge of $G$ satisfying $|f(v_i) - f(w_i)| = i$. If $r_i > \lambda$, then set $x_i = r_i$; if $r_i \le \lambda$, then set $x_i = -r_i$. Consider the following equation

$$\begin{aligned} \sum_{i=1}^{p} d_i x_i &\equiv \sum_{i=1}^{n} |f(v_i) - f(w_i)| \\ &\equiv 1 + 2 + \ldots + n \\ &\equiv \binom{n}{2} + n \\ &\equiv \binom{n}{2} \pmod{n}. \end{aligned}$$

Clearly, it has a solution of integers. By Theorem 2.3, we have therefore *g.c.d.*

$(d_1, d_2, \ldots, d_p, n) \mid \binom{n}{2}$. □

As an immediate consequence of Theorem 2.4, we have the following.

**Corollary 2.5.** *Let H be a k-regular bipartite graph with* $|V(H)| = v$. *If one of the following conditions holds, then H is not λ-graceful.*

(1) $v \equiv 1 \ (mod\ 4)$ *and* $k \equiv 0 \ (mod\ 4)$.

(2) $v \equiv 2 \ (mod\ 4)$ *and* $k \equiv 0 \ (mod\ 2)$.

(3) $v \equiv 3 \ (mod\ 4)$ *and* $k \equiv 0 \ (mod\ 4)$.

## 3. The constructions

We start with introducing the definitions of the following four graph operations on graphs $H$ and $G$. Suppose that $H$ and $G$ are vertex-disjoint graphs with distinguished vertices $v$ and $u$ and distinguished edges $(v_1, v_2)$ and $(u_1, u_2)$, respectively.

(1) The *vertex-amalgamated* operation $H \odot G$ is the graph obtained from $H$ and $G$ by amalgamating $H$ and $G$ at vertices $v$ and $u$, that is, by identifying $v$ with $u$.

(2) The *edge-amalgamated* operation $H \ominus G$ is the graph obtained from $H$ and $G$ by amalgamating $H$ and $G$ at edges $(v_1, v_2)$ and $(u_1, u_2)$, that is, by identifying $(v_1, v_2)$ with $(u_1, u_2)$.

(3) The *vertex-edge-attached* operation $H \oplus G$ is the graph obtained by adjoining to the graphs $H$ and $G$ a new vertex $w$ accompanied two edges $(w, v)$ and $(w, u)$.

(4) The *edge-attached* operation $H \circleddash G$ is the graph obtained from $H$ and $G$ by attaching one edge to vertices $v$ and $u$ of graphs $H$ and $G$.

Although the vertices $v$ and $u$ and the edges $(v_1, v_2)$ and $(u_1, u_2)$ do no explicitly appear in each notation, it will be always clear from the context which vertices or edges are identified or adjoined.

In what follows we will assume that the graphs $H$ and $G$ with $m$ and $n$ edges have respectively $\lambda_1$- and $\lambda_2$-graceful labelings $h$ and $g$, let $(A, B)$ be the bipartition of $G$, and let $E_1$ and $E_2$ denote the sets of values of edges of graphs $H$ and $G$, respectively.

**Theorem 3.1.** *If* $h(v) = 0$ *and* $g(u) = 0$, *then the graph* $H \odot G$ *is* $(\lambda_1 + \lambda_2)$*-graceful.*

*Proof.* Let $f$ be a labeling of $H \odot G$ defined as

$$f(x)=\begin{cases} g^{i}(x) & \text{if } x \in A, \\ \lambda_2 + h(x) & \text{if } x \in V(H), \\ m+g^{i}(x) & \text{if } x \in B. \end{cases}$$

Clearly, the values of vertices of the graph $H \odot G$ are all distinct. Moreover, $E_1 = \{|f(x)-f(y)| :$ all edges $(x, y) \in E(H)\} = \{1, 2, \ldots, m\}$ and $E_2 = \{|f(x)-f(y)| :$ all edges $(x, y) \in E(G)\} = \{m+1, m+2, \ldots, m+n\}$. Thus $f$ is a graceful labeling of the graph $H \odot G$.

Let $(C, D)$ be the bipartition of $H$ satisfying that $h(v) \leq \lambda_1$, if $v \in C$ and $h(v) > \lambda_1$, if $v \in D$. In order to prove that the labeling $f$ is a $\lambda$-graceful labeling of $H \odot G$ with $\lambda = \lambda_1 + \lambda_2$, it is enough to show that for any edge $(x, y)$ in $H \odot G$ with $x \in A \cup C$ and $y \in B \cup D$, $f(x) \leq \lambda_1 + \lambda_2 < f(y)$.

Suppose that $x_1 \in A$, $x_2 \in C$ and $y_1 \in B$, $y_2 \in D$. It is obvious that $f(x_1) \leq \lambda_2$, $f(x_2) \leq \lambda_1 + \lambda_2$ and $f(y_1) \geq m + \lambda_2 + 1$, $f(y_2) \geq \lambda_1 + \lambda_2 + 1$. Consequently, for all vertices $x \in A \cup C$ and all vertices $y \in B \cup D$, we have $f(x) \leq \lambda_1 + \lambda_2 < f(y)$ and the desired result follows. □

**Corollary 3.2.** *If $h(v) = 0, \lambda_1, \lambda_1 + 1$, or $m$ and $g(u) = 0, \lambda_2, \lambda_2 + 1$, or $n$, then the graph $H \odot G$ is $\lambda$-graceful for some $\lambda$ satisfying $0 \leq \lambda \leq m + n$.*

*Proof.* We may assume that $h(v) = 0$, for it is not, we could redefine $h$ as

$$\tilde{h}=\begin{cases} h^{i} & \text{if } f(v)=\lambda_1, \\ h^{c,i} & \text{if } f(v)=\lambda_1+1, \\ h^{c} & \text{if } f(v)=m. \end{cases}$$

It is clear that $\tilde{h}$ is $\lambda'$-graceful for some $\lambda'$, where $0 \leq \lambda' \leq m-1$ and $\tilde{h}(v) = 0$. Likewise, we may assume $g(u) = 0$. The result follows immediately from Theorem 3.1. □

**Theorem 3.3.** *Let $(h(v_1), h(v_2))$ and $(g(u_1), g(u_2))$ be the distinguished edges of $H$ and $G$, respectively. If $(h(v_1), h(v_2)) = (0, m)$ or $(\lambda_1, \lambda_1 + 1)$ and $(g(u_1), g(u_2)) = (0, n)$ or $(\lambda_2, \lambda_2 + 1)$, then $H \ominus G$ is $\lambda$-graceful.*

*Proof.* Since if $(h(v_1), h(v_2)) = (\lambda_1, \lambda_1 + 1)$ and $(g(u_1), g(u_2)) = (\lambda_2, \lambda_2 + 1)$, then $(h^{i}(v_1), h^{i}(v_2)) = (0, m)$ and $(g^{i}(u_1), g^{i}(u_2)) = (0, n)$. Thus we also assume that $(h(v_1), h(v_2)) = (0, m)$ and $(g(u_1), g(u_2)) = (0, n)$. Let $f$ be a labeling of the graph $H \ominus G$ given as

$$f(x)=\begin{cases} g^i(x) & \text{if } x\in A,\\ \lambda_2+h(x) & \text{if } x\in V(H),\\ m-1+g^i(x) & \text{if } x\in B.\end{cases}$$

By easy calculation, it can be verified that $f$ is a $\lambda$-graceful labeling of $H \ominus G$. □

**Theorem 3.4.** *If $h(v) = 0$, $\lambda_1$, $\lambda_1 + 1$, or $m$ and $g(u) = 0$, $\lambda_2$, $\lambda_2 + 1$, or $n$, then the graph $H \oplus G$ is $\lambda$-graceful.*

*Proof.* As in Theorem 3.1, we may assume that $h(v) = g(u) = 0$. Let us introduce a labeling $f$ of $H \oplus G$ as

$$f(x)=\begin{cases} g^i(x) & \text{if } x\in A,\\ \lambda_2+1+h(x) & \text{if } x\in V(H),\\ m+\lambda_2+2 & \text{if } x=w,\\ m+2+g^i(x) & \text{if } x\in B.\end{cases}$$

A routine verification shows that the labeling $f$ is indeed a $\lambda$-graceful labeling of $H \oplus G$. □

**Theorem 3.5.** *If either $g(u) = i$ and $h(v) = i$, or $\lambda_1 - i$ for $0 \le i \le min\{\lambda_1, \lambda_2\}$, or $g(u) = i$ and $h(v) = \lambda_1 + 1 + i$, or $m - i$ for $0 \le i \le min\{m - \lambda_1 - 1, \lambda_2\}$, then the graph $H \Theta G$ is $\lambda$-graceful.*

*Proof.* Suppose that $g(u) = i$ and $h(v) = i$, or $\lambda_1 - i$ for $0 \le i \le min\{\lambda_1, \lambda_2\}$, or $g(u) = i$ and $h(v) = \lambda_1 + 1 + i$, or $m - i$ for $0 \le i \le min\{m - \lambda_1 - 1, \lambda_2\}$.

Let $f$ be a labeling of $H \Theta G$ given by

$$f(x)=\begin{cases} g^i(x) & \text{if } x\in A,\\ m+1+g^i(x) & \text{if } x\in B,\\ \lambda_2+1+h^c(x) & \text{if } x\in V(H), g(u)=i, \text{and } h(v)=i,\\ \lambda_2+1+h^{c,i}(x) & \text{if } x\in V(H), g(u)=i, \text{and } h(v)=\lambda_1-i,\\ \lambda_2+1+h^i(x) & \text{if } x\in V(H), g(u)=i, \text{and } h(v)=\lambda_1+1+i,\\ \lambda_2+1+h(x) & \text{if } x\in V(H), g(u)=i, \text{and } h(v)=m-i.\end{cases}$$

Evidently, the values of vertices of the graph $H \Theta G$ are all distinct. An easy computation shows that $E_1 = \{1, 2, \dots, m\}$ and $E_2 = \{m + 2, m + 3, \dots, m + n + 1\}$. To prove that $f$ is a graceful labeling of $H \Theta G$, it suffices to show that $|f(v)-f(u)| = m + 1$. This can be done by the following consequences.

If $g(u) = i$ and $h(v) = i$, $0 \le i \le min\{\lambda_1, \lambda_2\}$, then

$$|f(v) - f(u)| = |(\lambda_2 + 1 + h^c(v)) - g^i(u)| = m + 1.$$

If $g(u) = i$ and $h(v) = \lambda_1 - i$, $0 \le i \le min\{\lambda_1, \lambda_2\}$, then

$$|f(v) - f(u)| = |(\lambda_2 + 1 + h^{c,i}(v)) - g^i(u)| = m + 1.$$

If $g(u) = i$ and $h(v) = \lambda_1 + 1 + i$, $0 \le i \le min\{m - \lambda_1 - 1, \lambda_2\}$, then

$$|f(v) - f(u)| = |(\lambda_2 + 1 + h^i(v)) - g^i(u)| = m + 1.$$

If $g(u) = i$ and $h(v) = m - i$, $0 \le i \le min\{m - \lambda_1 - 1, \lambda_2\}$, then

$$|f(v) - f(u)| = |(\lambda_2 + 1 + h(v)) - g^i(u)| = m + 1.$$

The remainder of the proof is similar to that in Theorem 3.1 and the details are omitted. □

By analogous argument, it follows that if $h$ is only a graceful labeling of $H$, then the graphs $H \odot G$, $H \ominus G$, $H \oplus G$, and $H \Theta G$ are graceful.

Combining Theorems 3.1, 3.3, 3.4, and 3.5, we have

**Theorem 3.6.** *Let $G_i$ ($1 \le i \le k$) be $\lambda_i$-graceful and let the symbol $\otimes$ be one of the operations $\odot$, $\ominus$, $\oplus$, and $\Theta$ with appropriately chosen distinguished vertices or edges. Then the graph*

$$G_1 \otimes G_2 \otimes \ldots \otimes G_k \quad (k \ge 3)$$

*is $\lambda$-graceful.*

*Remark.* In Theorem 3.6, if $G_i$ ($1 \le i \le k$) are trees, Chen, Lü, and Yeh [1] have obtained an analogous result.

It is natural to ask whether there exist graphs $G$ and $H$ such that for any vertex $u$ in $G$ and any vertex $v$ in $H$, the operations on $G$ and $H$ mentioned above can be applied. A graph $G$ is called a 0-*moveable graceful* (resp. 0-*moveable $\lambda$-graceful*) *graph* if for each vertex $z$ in $G$ there exists a graceful (resp. $\lambda$-graceful) labeling $g$ satisfying $g(z) = 0$. By virtue of Theorems 3.1, 3.4, and 3.5, we have the following.

**Theorem 3.7.** *If $H$ and $G$ are 0-moveable $\lambda$-graceful graphs, then the graphs $H \odot G$, $H \oplus G$, and $H \Theta G$ are $\lambda$-graceful, where $0 \le h(v) \le m$ and $0 \le g(u) \le n$. In particular, if $H$ is just a 0-moveable graceful graph, then the graphs $H \odot G$, $H \oplus G$, and $H \Theta G$ are graceful.*

Finally we shall extend the edge-attached operation on graphs $G_1$ and $G_2$ to that on graphs $G_1, G_2, \ldots, G_k$ ($k \ge 3$). To avoid cumbersome notation, if $G_i \cong G$ for $1 \le i \le k$, then we simply write $\Theta(G_1, G_2, \ldots, G_k)$ as $\Theta\, G^k$.

**Theorem 3.8.** *Suppose that graphs $G_i$ ($1 \le i \le k$) with $n_i$ edges are all $\lambda^*$-graceful having labeling $f_i$ and that $f_1(u_1) = f_2(u_2) = \ldots = f_k(u_k) = j$ where $0 \le j$*

$\leq \lambda^*$. *Then the graph* $\Theta(G_1, G_2, \ldots, G_k)$ *is* $\lambda$*-graceful. Consequently, if* $G$ *is* $\lambda$*-graceful then the graph* $\Theta\, G^k$ *is also* $\lambda$*-graceful.*

*Proof.* Let $(A_i, B_i)$ be the bipartition of $G_i$ such that $f_i(x_i) < f_i(y_i)$ for $x_i \in A_i$ and $y_i \in B_i$ and let $a_i = \lambda^* + 1$ and $b_i = n_i - \lambda^*$, $1 \leq i \leq k$.

*Case* 1*:* $k$ is odd.

Set $S_i = a_i + b_{i-1} + a_{i+2} + b_{i+3} + \ldots + a_k$, if $i$ is odd and set $S_i = b_i + a_{i+1} + b_{i+2} + a_{i+3} + \ldots + a_k$, if $i$ is even. Let $f$ be a labeling of the graph $\Theta(G_1, G_2, \ldots, G_k)$ given as

$$f(u) = \begin{cases} S_{i+1} + f_i(u) & \text{if } u \in A_1 \cup B_1 \text{ or } u \in A_i, \\ \sum_{t=1}^{i-1} n_t + S_{i+1} + i - 1 + f_i(u) & \text{if } u \in B_i, \end{cases} \qquad (i = 3, 5, \ldots, k)$$

and

$$f(u) = \begin{cases} \sum_{t=1}^{i} n_t + S_{i+1}' + i - \lambda^* - 1 + f_i(u) & \text{if } u \in A_i, \\ S_{i+1} - \lambda^* - 1 + f_i(u) & \text{if } u \in B_i. \end{cases} \qquad (i = 2, 4, \ldots, k-1)$$

It can be checked that the labels of vertices of the graph $\Theta(G_1, G_2, \ldots, G_k)$ are all distinct.

Next we shall show that $f$ is a graceful labeling of $\Theta(G_1, G_2, \ldots, G_k)$. Let $W_i$ ($1 \leq i \leq k$) denote the set of values of edges $(x_i, y_i)$ of subgraph $G_i$ in the graph $\Theta(G_1, G_2, \ldots, G_k)$, where $x_i \in A_i$ and $y_i \in B_i$. Observing the construction of the graph $\Theta(G_1, G_2, \ldots, G_k)$, we have

$$W_1 = \{ |f_1(x_1) - f_1(y_1)| : \text{all edges } (x_1, y_1) \in E(G_1)\} = \{1, 2, \ldots, n_1\};$$

If $i$ ($\geq 3$) is odd, then

$$W_i = \{ \sum_{t=1}^{i-1} n_t + i - 1 + f_i(y_i) - f_i(x_i) : \text{all edges } (x_i, y_i) \in E(G_i)\}$$
$$= \{ \sum_{t=1}^{i-1} n_t + i,\ \sum_{t=1}^{i-1} n_t + i + 1, \ldots, \sum_{t=1}^{i} n_t + i - 1\}; \text{ and}$$

If $i$ is even, then

$$W_i = \{ \sum_{t=1}^{i} n_t + i - f_i(y_i) + f_i(x_i)) : \text{all edges } (x_i, y_i) \in E(G_i)\}$$
$$= \{ \sum_{t=1}^{i-1} n_t + i,\ \sum_{t=1}^{i-1} n_t + i + 1, \ldots, \sum_{t=1}^{i} n_t + i - 1\}.$$

Let $T_i$ ($1 \leq i \leq k-1$) be the value of the edge $(f(u_i), f(u_{i+1}))$ in the graph $\Theta(G_1, G_2, \ldots, G_k)$. It is clear that $T_i = \sum_{t=1}^{i} n_t + i$. By routine computation, it follows that $\{T_1, T_2, \ldots, T_{k-1}\} \cup W_1 \cup \ldots \cup W_k = \{1, 2, \ldots, \sum_{t=1}^{k} n_t + k - 1\}$ and so $f$ is a graceful labeling of $\Theta(G_1, G_2, \ldots, G_k)$.

It remains to show that $f$ is also a $\lambda$-graceful labeling of $\Theta(G_1, G_2, \ldots, G_k)$ with

$\lambda = S_2 + \lambda^*$. Let $A = A_1 \cup B_2 \cup A_3 \cup B_4 \cup \ldots \cup A_k$ and $B = B_1 \cup A_2 \cup B_3 \cup A_4 \cup \ldots \cup B_k$. It is sufficient to prove that for any edge $(x, y)$ in $\Theta(G_1, G_2, \ldots, G_k)$ with $x \in A$ and $y \in B$, $f(x) \le S_2 + \lambda^* < f(y)$. This can be done as follows.

If $x \in A_1$ and $y \in B_1$, then $f(x) = S_2 + f_1(x)$, $f(y) = S_2 + f_1(y)$ and we have $S_2 + f_1(x) \le S_2 + \lambda^* < S_2 + f_1(y)$.

If $i$ is even and $x \in B_i$, $y \in A_i$, then $f(x) = S_{i+1} - \lambda^* - 1 + f_i(x)$ and $f(y) = \sum_{t=1}^{i} n_t + S_{i+1} + i - \lambda^* - 1 + f_i(y)$. Since $S_{i+1} - \lambda^* - 1 + f_i(x) < S_{i+1} < S_2 + \lambda^*$ and $S_2 + \lambda^* = b_2 + a_3 + \ldots + b_i + S_{i+1} + \lambda^* = \sum_{t=2}^{i} n_t + S_{i+1} < \sum_{t=1}^{i} n_t + S_{i+1} + i - \lambda^* - 1 + f_i(y)$, it follows that $f(x) \le S_2 + \lambda^* < f(y)$.

If $i$ is odd ($\ge 3$) and $x \in A_i$, $y \in B_i$, then $f(x) = S_{i+1} + f_i(x)$ and $f(y) = \sum_{t=1}^{i-1} n_t + S_{i+1} + i - 1 + f_i(y)$. Similarly, we obtain $f(x) \le S_2 + \lambda^* < f(y)$.

Finally, we need to determine the labels on the edges between the subgraphs $G_i$ and $G_{i+1}$ $(1 \le i \le k-1)$ in $\Theta(G_1, G_2, \ldots, G_k)$.

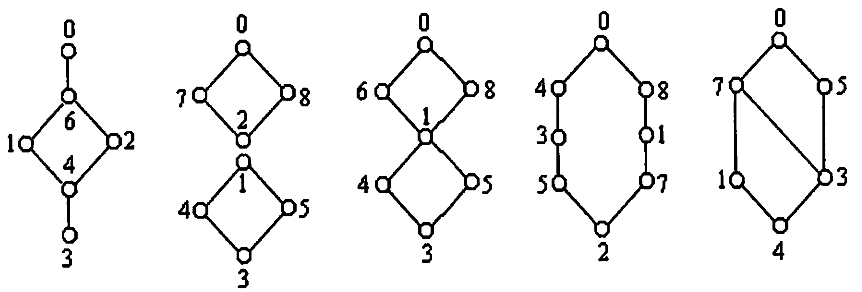
If either $x \in A_i$, $y \in A_{i+1}$, or $x \in A_{i+1}$, $y \in A_{i+2}$ for $i = 1, 3, \ldots, k-2$, say the former, then $f(x) = S_{i+1} + f_i(x)$ and $f(y) = \sum_{t=1}^{i+1} n_t + S_{i+2} + i - \lambda^* + f_{i+1}(y)$, and so $f(x) \le S_2 + \lambda^* < f(y)$.

*Case* 2*:* $k$ is even.

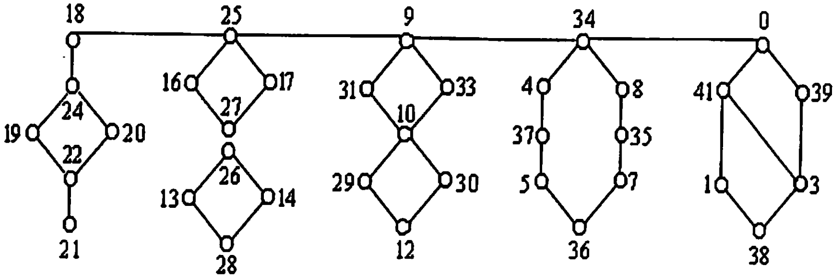
Similar to that of Case 1 and omitted. □

*Remark.* Suppose that graphs $G_i$ $(1 \le i \le k)$ are connected with $n$ edges each. It is proved in [12] that if graphs $G_i$ $(1 \le i \le k)$ are $\lambda_i$-graceful with $\lambda_i = \lambda_{k-i+1}$ for $1 \le i \le \lfloor k/2 \rfloor$, then the graph $\Theta(G_1, G_k, G_2, G_{k-1}, \ldots, G_{\lfloor (k+2)/2 \rfloor})$ is graceful.

We demonstrate the construction above with an example. Consider the 3-graceful graphs $G_i$ with 3-graceful labeling $f_i$ $(1 \le i \le 5)$, depicted in Figures 5-(1)~(5). Choosing $f_1(u_1) = f_2(u_2) = \ldots = f_5(u_5) = 0$ and utilizing Theorem 3.8, the graph $\Theta(G_1, G_2, G_3, G_4, G_5)$ of Figure 5-(6) then follows.



(1) $G_1$ (2) $G_2$ (3) $G_3$ (4) $G_4$ (5) $G_5$




(6) $\Theta(G_1, G_2, G_3, G_4, G_5)$
Figure 5.

In [6] Rosa proved that the cycle $C_{4k}$ ($k \geq 1$) is $\lambda$-graceful with $\lambda = 2k - 1$. Combining the result and Theorem 3.8, we have

**Corollary 3.9.** *Let $r_1, r_2, \ldots, r_s$ ($s \geq 2$) be positive integers with $1 \leq r_1 \leq r_2 \leq \ldots \leq r_s$. Then the graph $\Theta(C_{4r_1}, C_{4r_2}, \ldots, C_{4r_s})$ is $\lambda$-graceful.*

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