Independence number and $[a, b]$-factors of graphs

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Abstract Let G be a graph. The cardinality of any largest independent set of vertices in G is called the independence number of G and is denoted by $\alpha(G)$. Let a and b be integers with $0 \le a \le b$. If $a = b$, it is assumed that G be a connected graph, furthermore, $a \ge \alpha(G)$, $a|V(G)| \equiv 0 \pmod 2$ if a is odd. We prove that every graph G has an $[a, b]$-factor if its minimum degree is at least $\left(\frac{b+\alpha(G)a-\alpha(G)}{b}\right)\left\lfloor\frac{b+\alpha(G)a}{2\alpha(G)}\right\rfloor - \frac{\alpha(G)}{b}\left(\left\lfloor\frac{b+\alpha(G)a}{2\alpha(G)}\right\rfloor\right)^2 + \theta\frac{\alpha(G)^2}{b} + \frac{a}{b}\alpha(G)$, where $\theta = 0$ if $a < b$, and $\theta = 1$ if $a = b$. This degree condition is sharp.

Keywords: Graph, Factor, $[a, b]$-Factor, Independence number.

1 Introduction

The graphs considered in this paper will be finite, undirected, without loops and multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Notation and definition not given in this note can found in [1].

For $S \subseteq V(G)$ the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) \backslash S]$. For any vertex x of G, the degree of x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$. Furthermore, $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The cardinality of any largest independent set of vertices in G is called the independence number of G and is denoted by $\alpha(G)$. Let a and b be integers with $a \le b$. An $[a, b]$-factor of G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for each $x \in V(G)$. If $r = a = b$, then an $[a, b]$-factor of G is called an r-factor. A graph is called $K_{1,t}$-free if it contains no $K_{1,t}$ as an induced subgraph.

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The following results on factors are known.

Theorem A (Y.Li and M.Cai [2]) Let G be a graph of order $|G|$, and let a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if $\delta(G) \geq a$, $m \geq 2a + b + (a^2 - a)/b$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{a|G|}{a+b}$$

for any two non-adjacent vertices x and y of G.

Theorem B (H.Matsuda [5]) Let G be a graph of order $|G|$, and let a and b be integers such that $1 \leq a < b$. Then G has an $[a, b]$-factor if $\delta(G) \geq a$, $|G| \geq 2(a + b)(a + b - 1)/b$ and

$$|N_G(x) \cup N_G(y)| \geq \frac{a|G|}{a+b}$$

for any two non-adjacent vertices x and y of G.

Theorem C (K.Ota and T.Tokuda [6]) Let t and r be positive integers and $t \geq 3$. If r is odd, we assume that $r \geq t - 1$. Let G be a connected graph with $r|G|$ even. If G is a $K_{1,t}$-free graph and the minimum degree of G is at least

$$\left(\frac{t(r+1)-1}{r}\right)\left\lceil\frac{rt}{2(t-1)}\right\rceil - \frac{t-1}{r}\left(\left\lceil\frac{rt}{2(t-1)}\right\rceil\right)^2 + t - 3, \qquad (1)$$

then G has an r-factor.

Theorem D(J.Li [3]) Let G be a graph, and let t, a and b be integers such that $0 \leq a < b$ and $t \geq 3$. If G is a $K_{1,t}$-free graph and its minimum degree is at least

$$\left(\frac{(t-1)(a+1)+b}{b}\right)\left\lceil\frac{b+a(t-1)}{2(t-1)}\right\rceil - \frac{t-1}{b}\left(\left\lceil\frac{b+a(t-1)}{2(t-1)}\right\rceil\right)^2 - 1, \qquad (2)$$

then G has an $[a, b]$-factor.

In this paper we shall prove the following theorem, which is a new degree condition for graphs to have $[a, b]$-factors in term of independence number of G.

Theorem 1 Let G be a graph, a and b be integers with $0 \leq a \leq b$. If $a = b$, it is assumed that G be a connected graph, furthermore, $a \geq \alpha(G)$, $a|V(G)| \equiv 0 \pmod 2$ if a is odd. We prove that every graph G has an $[a, b]$-factor if its minimum degree is at least

$$\left(\frac{b+\alpha(G)a-\alpha(G)}{b}\right)\left\lfloor\frac{b+\alpha(G)a}{2\alpha(G)}\right\rfloor - \frac{\alpha(G)}{b}\left(\left\lfloor\frac{b+\alpha(G)a}{2\alpha(G)}\right\rfloor\right)^2 + \theta\frac{\alpha(G)^2}{b} + \frac{a}{b}\alpha(G), \qquad (3)$$

where $\theta = 0$ if $a < b$, and $\theta = 1$ if $a = b$.

In Section 3, we will show that the condition (3) in Theorem 1 is sharp.

2 Proof of Theorem 1

Let S and T be disjoint subsets of $V(G)$. We write $e_G(S,T) = |\{xy \in E(G) : x \in S,\ y \in T\}|$ and $d_{G-S}(T) = \sum_{x\in T} d_{G-S}(x)$. In particular, $e_G(x,T)$ means $e_G(\{x\},T)$. Let

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| - h_G(S,T),$$

where $h_G(S,T)$ is the number of components C of $G-(S\cup T)$ for $a=b$ and $b|V(C)| + e_G(V(C),T) \equiv 1 \pmod 2$. Such a component C is called an odd component. Clearly, if $a<b$, then $h_G(S,T)=0$ by the definition of odd component.

We use the following Lemma.

Lemma 1 (L.Lovász [4]) Let G be a graph. Let a and b be nonnegative integers with $a \le b$. Then G contains an $[a,b]$-factor if and only if

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| - h_G(S,T) \ge 0$$

for all disjoint subsets S and T of $V(G)$.

We now prove Theorem 1.

By Lemma 1, to prove the theorem we need only to show that for all disjoint subsets S and T of $V(G)$

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| - h_G(S,T) \ge 0,$$

where $0 \le a \le b$.

At first, we prove the following claim.

Claim 1 $\delta(G) \ge y + \frac{\alpha(G)}{b}(y-\theta x+1)(a-y) + \theta\frac{\alpha(G)^2}{b}$ for any integer y if $a<b$ or any integer $y \le a$ and any integer $x \in [0,\alpha(G)]$ if $a=b$, where $\theta = 0$ if $a<b$, and $\theta=1$ if $a=b$.

Proof. We fix $\alpha(G), a$ and b, and define $f(x,y)$ to be the right-hand side of the above inequality. Note that

$$\frac{df}{dy} = -\frac{2\alpha(G)}{b}y + \frac{\alpha(G)}{b}(a+\theta x - 1) + 1, \quad \frac{df}{dx} = \frac{\theta\alpha(G)}{b}(y-a) \le 0.$$

If $\theta = 0$, then $f(x,y) = f(0,y)$. Hence, among all integers y, $f(x,y) = f(0,y)$ is maximum when y is the nearest integer to $\frac{b+a\alpha(G)}{2\alpha(G)} - \frac{1}{2}$(this can be got by $\frac{df}{dy} = 0$), i.e., when $y = \left\lfloor \frac{b+a\alpha(G)}{2\alpha(G)} \right\rfloor$.

If $\theta = 1$, we fix y. Note that $\frac{df}{dx} \le 0$, among all integers $x \in [0,\alpha(G)]$ and fixed $y \le a$, $f(x,y) = f(0,y)$ is maximum when $x=0$. So, among all integers $y \le a$ and all integers $x \in [0,\alpha(G)]$, $f(x,y) = f(0,y)$ is maximum when $x = 0$ and y is the nearest integer to $\frac{b+a\alpha(G)}{2\alpha(G)} - \frac{1}{2}$(this can be got by $x = 0$ and $\frac{df}{dy} = 0$), i.e., when $x = 0$ and $y = \left\lfloor \frac{b+a\alpha(G)}{2\alpha(G)} \right\rfloor (\leq a)$.

It is easy to check that $f\left(0, \left\lfloor \frac{b+a\alpha(G)}{2\alpha(G)} \right\rfloor\right)$ is identical to the expression (3). Hence, $f(x,y) \leq f\left(0, \left\lfloor \frac{b+a\alpha(G)}{2\alpha(G)} \right\rfloor\right) \leq \delta(G)$ for any integer y if $a < b$ or any integer $y \leq \alpha(G)$, and any integer $x \in [0, \alpha(G)]$ if $a = b$. ∎

We define x_i and $N_i (i \geq 1)$ as follows: If $T \neq \emptyset$, let $x_1 \in T$ be a vertex such that $d_{G-S}(x_1) - a$ is minimum, and $N_1 = (N_G(x_1) \cup \{x_1\}) \cap T$. For $i \geq 2$, if $T - \bigcup_{j<i} N_j \neq \emptyset$, let $x_i \in T - \bigcup_{j<i} N_j$ be a vertex such that $d_{G-S}(x_i) - a$ is as small as possible, and $N_i = (N_G(x_i) \cup \{x_i\}) \cap (T - \bigcup_{j<i} N_j)$.

We suppose $x_1, \ldots, x_m$ are defined and x_{m+1} cannot be defined. By definition, $\{x_1, \ldots, x_m\}$ is an independent set of G, and T is the disjoint union of $N_1, \ldots, N_m$.

For $A \subset T$, we define $\lambda(A)$ to be the number of odd components C such that $e(C, A) > 0$.

Claim 2 $|S| \geq \frac{1}{\alpha(G)} \sum_{i=1}^{m} e(x_i, S) + \frac{1}{b}(h_G(S,T) - \lambda(T))$.

Proof. Let $l = h_G(S,T) - \lambda(T)$. Then there exist l odd components $C_1, C_2, \cdots, C_l$ such that $e(C_i, T) = 0$ for $1 \leq i \leq l$.

Case 1 $S = \emptyset$.

If $a < b$, then $l = 0$ (by the definition of odd components).

If $r = a = b$, since G is connected, we can see that $l \leq 1$. Moreover, if $l = 1$, then $T = \emptyset$, and hence $G = C_1$ is an odd component, $r|G| = r|C_1|$ is odd. A contradiction to the assumption that $r|G|$ is even. Hence $l = 0$.

In this case, the claim becomes $0 \geq 0$, which clearly holds.

Case 2 $S \neq \emptyset$.

If $a < b$, then $l = 0$. Note that $\{x_1, \ldots, x_m\}$ is an independent set of G and $|\{x_1, \ldots, x_m\}| \leq \alpha(G)$, every vertex $v \in S$ is adjacent to at most $\alpha(G)$ vertices of $\{x_1, \ldots, x_m\}$. Therefore

$$\alpha(G)|S| \geq e(\{x_1, \ldots, x_m\}, S) = \sum_{i=1}^{m} e(x_i, S),$$

or

$$|S| \geq \frac{1}{\alpha(G)} \sum_{i=1}^{m} e(x_i, S) = \frac{1}{\alpha(G)} \sum_{i=1}^{m} e(x_i, S) + \frac{1}{b} l.$$

If $r = a = b$, since G is connected, $e(C_j, S) > 0$ for $0 \leq j \leq l$. Here we choose a vertex $z_j \in V(C_j)$ such that $e(z_j, S) > 0$ for each j. Let $X = \{x_1, \ldots, x_m, z_1, \ldots, z_l\}$. Since X is an independent set of G and $|X| \leq \alpha(G)$, every vertex $v \in S$ is adjacent to at most $\alpha(G)$ vertices of X.

Therefore

$$\alpha(G)|S| \geq e(X,S) = \sum_{i=1}^{m} e(x_i, S) + \sum_{j=1}^{l} e(z_j, S) \geq \sum_{i=1}^{m} e(x_i, S) + l.$$

Here if r is even, then $l = 0$ (by the definition of odd components); and if r is odd, then $l \geq \frac{\alpha(G)}{r} l = \frac{\alpha(G)}{b} l$ (by the assumption $r \geq \alpha(G)$). Hence the claim holds. ∎

By claim 2,

$$\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| - h_G(S,T) \\
&\geq \frac{b}{\alpha(G)} \sum_{i=1}^{m} e_G(x_i, S) + d_{G-S}(T) - a|T| - \lambda(T) \\
&\geq \sum_{i=1}^{m} \left(\frac{b}{\alpha(G)} e_G(x_i, S) + (d_{G-S}(x_i) - a)|N_i| - \lambda(N_i) \right).
\end{aligned}$$

We show the following inequality that implies $\delta_G(S,T) \geq 0$:

$$\frac{b}{\alpha(G)} e_G(x_i, S) + (d_{G-S}(x_i) - a)|N_i| - \lambda(N_i) \geq 0 \qquad (4)$$

for each i $(1 \leq i \leq m)$. Here we fix i $(1 \leq i \leq m)$ and define

$$d = d_{G-S}(x_i), \quad \lambda = \lambda(N_i).$$

It is clearly that $0 \leq \lambda \leq \alpha(G)$, and

$$1 \leq |N_i| \leq d - e_G(x_i, V(G) - (S \cup T)) + 1 \leq d - \lambda + 1,$$

and

$$e(x_i, S) = d_G(x_i) - d_{G-S}(x_i) \geq \delta(G) - d.$$

We divide the proof into two cases.

Case 1 $a < b$.

$\lambda(N_i) = 0$ by the definition of odd components.

If $d - a \geq 0$, then

$$\frac{b}{\alpha(G)} e_G(x_i, S) + (d-a)|N_i| \geq 0$$

and thus (4) holds. Hence we may assume $d - a < 0$. Claim 1 with $y = d$ yields $\delta(G) \geq d + \frac{\alpha(G)}{b}(a-d)(d+1)$. Hence,

$$\begin{aligned}
& \tfrac{b}{\alpha(G)} e_G(x_i, S) + (d-a)|N_i| \\
\geq\ & \tfrac{b}{\alpha(G)} (\delta(G) - d) + (d+1)(d-a) \\
\geq\ & \tfrac{b}{\alpha(G)} \left(d + \tfrac{\alpha(G)}{b}(d+1)(a-d) - d \right) + (d+1)(d-a) \\
=\ & 0.
\end{aligned}$$

Thus, (4) holds.

Case 2 $r = a = b$.

Since $\lambda(N_i) \le \alpha(G)$, we have

$$\begin{aligned}&\frac{b}{\alpha(G)}e_G(x_i,S) + (d_{G-S}(x_i) - a)|N_i| - \lambda(N_i)\\ &\qquad \ge \frac{r}{\alpha(G)}e_G(x_i,S) + (d-r)|N_i| - \alpha(G).\end{aligned}$$

We divide the proof into two subcases.

Subcase 2.1 $d - r \ge 0$.

If $d - r - \alpha(G) \ge 0$, then

$$\frac{r}{\alpha(G)}e_G(x_i,S) + (d-r)|N_i| - \alpha(G) \ge \frac{r}{\alpha(G)}e_G(x_i,S) + d - r - \alpha(G) \ge 0.$$

therefore (4) holds. hence we may assume $d - r - \alpha(G) < 0$.

Subcase 2.1.1 $\alpha(G) \ge r$.

Claim 1 with $y = r, x \in [0, \alpha(G)]$ yields $e_G(x_i, S) \ge \delta(G) - d \ge r + \frac{\alpha(G)^2}{r} - d$. Hence,

$$\begin{aligned}&\tfrac{r}{\alpha(G)}e_G(x_i,S) + (d-r)|N_i| - \alpha(G)\\ \ge\ & \tfrac{r}{\alpha(G)}(r + \tfrac{\alpha(G)^2}{r} - d) + d - r - \alpha(G)\\ =\ & (d-r)(1 - \tfrac{r}{\alpha(G)}) \ge 0.\end{aligned}$$

Thus, (4) holds.

Subcase 2.1.2 $r > \alpha(G)$.

Claim 1 with $y = d - \alpha(G)(< r), x = 0$ yields

$$\begin{aligned}&e_G(x_i,S) \ge \delta(G) - d\\ \ge\ & d - \alpha(G) + \tfrac{\alpha(G)}{r}(r - d + \alpha(G))(d - \alpha(G) + 1) + \tfrac{\alpha(G)^2}{r} - d\\ =\ & \tfrac{\alpha(G)}{r}(r - d + \alpha(G))(d - \alpha(G) + 1) + \tfrac{\alpha(G)^2}{r} - \alpha(G).\end{aligned}$$

Note that $d - r - \alpha(G) < 0$ implies $r + \alpha(G) - d \ge 1$, hence,

$$\begin{aligned}&\frac{r}{\alpha(G)}e_G(x_i,S) + (d-r)|N_i| - \alpha(G)\\ \ge\ & \frac{r}{\alpha(G)}\left(\frac{\alpha(G)}{r}(r - d + \alpha(G))(d - \alpha(G) + 1) + \frac{\alpha(G)^2}{r} - \alpha(G)\right)\\ & + d - r - \alpha(G)\\ =\ & (d - \alpha(G))(r + \alpha(G) - d) - (r - \alpha(G))\\ \ge\ & (r - \alpha(G))(r + \alpha(G) - d) - (r - \alpha(G))\\ =\ & (r - \alpha(G))(r + \alpha(G) - d - 1) \ge 0.\end{aligned}$$

Thus, (4) holds.
Case 2.2 $d-r<0$.

From Claim 1 with $y=d(<r), x=\lambda$, we obtain

$$\begin{aligned} e_G(x_i,S)\geq\delta(G)-d &\geq d+\frac{\alpha(G)}{r}(r-d)(d-\lambda+1)+\frac{\alpha(G)^2}{r}-d \\ &= \frac{\alpha(G)}{r}(r-d)(d-\lambda+1)+\frac{\alpha(G)^2}{r}. \end{aligned}$$

Note that $(d-r)|N_i|\geq(d-r)(d-\lambda+1)$, hence,

$$\begin{aligned} &\frac{r}{\alpha(G)}e_G(x_i,S)+(d-r)|N_i|-\alpha(G) \\ \geq\ & \frac{r}{\alpha(G)}\left(\frac{\alpha(G)}{r}(r-d)(d-\lambda+1)+\frac{\alpha(G)^2}{r}\right) \\ &+(d-r)(d-\lambda+1)-\alpha(G) \\ =\ & 0. \end{aligned}$$

Thus, (4) holds. The proof is complete.

3 Remarks

Remark 1 In Theorem 1, the degree condition is sharp if $r=a=b$.

To demonstrate this remark with an example, let n be an integer with $n\geq1$, $\alpha(G)=2n$, $r=2\alpha(G)^2$,

$$y=\left\lfloor\frac{r(1+\alpha(G))}{2\alpha(G)}\right\rfloor=\frac{r(1+\alpha(G))}{2\alpha(G)},$$

$$x=\frac{\alpha(G)}{r}(r-y)(y+1)+\frac{\alpha(G)^2}{r}-1.$$

Let L be the complete graph K_x, and let M be $\alpha(G)$ disjoint copies of K_{y+1}. Let $G=L+M$, where $L+M$ denotes the join of L and M. Then G is a connected graph with

$$\begin{aligned} \delta(G) &= d_G(v)=y+x \\ &= \left(1+\alpha(G)-\frac{\alpha(G)}{r}\right)\left\lfloor\frac{r(1+\alpha(G))}{2\alpha(G)}\right\rfloor \\ &\quad -\frac{\alpha(G)}{r}\left(\left\lfloor\frac{r(1+\alpha(G))}{2\alpha(G)}\right\rfloor\right)^2+\frac{\alpha(G)^2}{r}+\alpha(G)-1, \end{aligned}$$

where $v\in V(M)$. Application of Lemma 1 in Section 2 with $S=V(L)$ and $T=V(M)$ proves that G have not an $[a,b]$-factor.

Remark 2 In Theorem 1, the degree condition is sharp if $a<b$.

Let $\alpha(G) \equiv 0 \pmod{b}$, and

$$y = \left\lfloor \frac{b+\alpha(G)a}{2\alpha(G)} \right\rfloor,$$

$$x = \frac{\alpha(G)}{b}(a-y)(y+1) - 1.$$

Let L be the complete graph K_x, and let M be $\alpha(G)$ disjoint copies of K_{y+1}. Let $G = L + M$, where $L + M$ denotes the join of L and M. Then G is a graph with

$$\begin{aligned}\delta(G) &= d_G(v) = y + x \\ &= \lceil \tfrac{b+\alpha(G)a-\alpha(G)}{b} \lfloor \tfrac{b+\alpha(G)a}{2\alpha(G)} \rfloor - \tfrac{\alpha(G)}{b}(\lfloor \tfrac{b+\alpha(G)a}{2\alpha(G)} \rfloor)^2 + \tfrac{a}{b}\alpha(G) - 1 \rceil,\end{aligned}$$

where $v \in V(M)$. Application of Lemma 1 in Section 2 with $S = V(L)$ and $T = V(M)$ proves that G have not an $[a, b]$-factor.

Remark 3 Expression (3) is identical to expression (2) for $a < b$ and $\alpha(G) = t - 1$.

It is easy to prove that

$$\left(\frac{b+\alpha(G)a+\alpha(G)}{b}\right)\left\lceil \frac{b+\alpha(G)a}{2\alpha(G)} \right\rceil - \frac{\alpha(G)}{b}\left(\left\lceil \frac{b+\alpha(G)a}{2\alpha(G)} \right\rceil\right)^2 + \theta\frac{\alpha(G)^2}{b} - 1 \qquad (5)$$

is identical to the expression (3).

Remark 4 Let t be an integer with $t \geq 4$, and
$\Phi = \{G : G$ is a graph satisfying conditions of Theorem 1 with $\alpha(G) = t - 1, a = b\}$,

$\Psi = \{G : G$ is a $K_{1,t}$–free graph satisfying conditions of Theorem $C\}$.

It is easy to see that $\Phi \subset \Psi$. So, Theorem C holds for Ψ, and Theorem 1 holds for Φ but not for Ψ.

In fact, expression (1)> expression (5) for $\alpha(G) = t - 1$, and $r = a = b \geq \alpha(G) + 7, \theta = 1$.

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