A Cube–covering Problem*

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Abstract

We consider the problem of covering a unit cube with smaller cubes. The size of a cube is given by its side length and the size of a covering is the total size of the cubes used to cover the unit cube. We denote by $g_3(n)$ the smallest size of a minimal covering using n cubes. We present tight results for the upper and lower bounds of $g_3(n)$.

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1 Introduction

In 1932, Erdős defined a function $f(n)$ which denotes the maximum sum of n squares that can be packed into a unit square S. In [1], Erdős and Soifer gave some results of $f(n)$. Inspired by [1], Fan and Zhang [4] discussed the dual version, that is a square-covering problem. In this paper, we generalize this kind of covering problem to the case of cubes. That is, using smaller cubes to cover a unit cube, obtaining corresponding results.

First, we give the definition of a minimal cube-covering. The size of a given cube c is the side length of c and is denoted by $s(c)$. A covering C is given by a set of cubes S positioned inside a unit cube C in such a way that $0 < s(c) < 1$, for each $c \in S$, and any point of C is covered by at least one of the cubes in S.

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Given a covering $\mathcal{C}$ of the unit cube C using a set of cubes $S = \{c_1, \ldots, c_n\}$ of n smaller cubes, where $0 < s(c_i) < 1$, we denote by $s(\mathcal{C})$ the size of the covering $\mathcal{C}$, which is given by $\sum_{i=1}^{n} s(c_i)$.

A covering is said to be minimal if there is no other covering of C using a set of cubes S' where $S' = \{c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n\}$ or $S' = \{c_1, \ldots, c_{i-1}, c'_i, c_{i+1}, \ldots, c_n\}$, where $s(c'_i) < s(c_i)$. We denote by $g_3(n)$ the smallest size of a minimal covering using a set of n cubes. That is,

$$g_3(n) = \min\{s(\mathcal{C})\colon \mathcal{C} \text{ is a minimal covering of the unit cube with } n \text{ cubes}\}.$$

Let $\mathcal{C}$ be a covering of a unit cube C using a set of cubes $S = \{c_1, \ldots, c_n\}$. As each corner point of C has to be covered by a cube in S, and the size of a cube $c \in S$ is such that $0 < s(c) < 1$, we have that different corners of C must be covered by different cubes of S. Therefore, the following lemma is valid.

Lemma 1. *If $\mathcal{C}$ is a covering of the unit cube, then $\mathcal{C}$ has at least* 8 *cubes.*

For example of a covering, consider the case when $n = 8$. It is easy to show that $g_3(8) \leq 4$. To see this, we can use a set S with 8 cubes of size 1/2, each one positioned in a different corner of the unit cube. This covering is clearly minimal, as we cannot remove a cube from S or replace by a smaller cube to obtain a smaller covering. The next theorem shows that $g_3(n)$ must be at least 4.

Theorem 2. *For any $n \geq 8$, we have that $g_3(n) \geq 4$.*

Proof. Let $\mathcal{C}$ be a covering of the unit cube C with cubes in S. If a top face of C and a cube $c \in S$ have a common point, then c and the bottom face of C have no common points, because $0 < s(c) < 1$. Let $a_1, a_2, \cdots, a_s$ be the set of cubes in S which have common points with the top face of C and let $b_1, b_2, \cdots, b_t$ be the set of cubes in S which have common points with the bottom face of C, then $s + t \leq n$, and $\{a_1, \ldots, a_s\} \bigcap \{b_1, b_2, \ldots, b_t\} = \emptyset$.

For $i = 1, 2, \ldots, s$, the projection of a_i in the top face of C, is a square which has the same length with a_i. That is, the projections of $a_1, \ldots, a_s$ in the top face leads to a covering of the top face of C. By the known result for the square covering problem, the side length of these projections is no less than 2, so the total size of the cubes $a_1, \ldots, a_s$ is no less than 2. In the same way, the sum of the sizes of the cubes $b_1, b_2, \cdots, b_t$ is no less than 2. So,

$$g_3(n) \geq 2 + 2 = 4.$$

□

Obviously, the following corollary is valid.

Corollary 3. $g_3(8) = 4$.

Having a covering $\mathcal{C}$ with n cubes does not means that $g_3(n) \leq s(\mathcal{C})$. This may occur when we can still obtain a covering using smaller cubes or removing cubes in $\mathcal{C}$ and therefore, using less than n cubes. For example, consider a covering with $n \geq 8$ cubes having 8 cubes of size 1/2, each one in a different corner of the unit cube, and the remaining $n-8$ cubes of size ε, all in one of the corners. This leads to a covering $\mathcal{C}$ with size $4+(n-8)\varepsilon$, where ε can be made as close to zero as desired. Clearly this covering is not minimal, since the 8 cubes of size 1/2 already leads to a covering and we can remove the $n-8$ cubes of size ε. In this example, the smaller covering was easy to detect, but for a general covering, obtaining a smaller covering or proving that it is minimal may be a non-trivial task.

On the other hand, if we have a covering $\mathcal{C}$ that is minimal, each cube in $\mathcal{C}$ is necessary to obtain the covering of the unit cube. So, the following result is straightforward.

Lemma 4. *If $\mathcal{C}$ is a minimal covering of the unit cube and $\mathcal{C}$ has n cubes, then $g_3(n) \leq s(\mathcal{C})$.*

2 Main result

In this section, we present the upper and lower bounds for $g_3(n)$.

The next theorem shows that $g_3(n)$ cannot be greater than 4, for any $n \geq 9$.

Theorem 5. *For $n \geq 9$, we have $g_3(n) \leq 4+\delta$, where δ is a positive value that can be made as close to 0 as desired.*

Proof. We denote the unit cube by C. We present a minimal covering $\mathcal{C}$ with n cubes divided in two sets: a set L of large cubes and a set S of small cubes. The set L has three cubes with size $1-\varepsilon$ and one with size $1-\varepsilon'$, where $\varepsilon' = (n-7)\varepsilon$ and ε is a positive value such that $\varepsilon < \frac{1}{9n}$. The set S has two cubes of size ε' and $n-6$ cubes of size ε.

In the following, we show that these cubes leads to a minimal covering. The following facts are valid.

Fact 1. *The total length of the edges of C that can be covered by the cubes in S is smaller than 1.*

Proof. The result follows, as we have 2 cubes of size $\varepsilon' = (n-7)\varepsilon$ and $n-6$ cubes of size ε. Therefore, we have a total size of $2n\varepsilon - 14\varepsilon + n\varepsilon - 6\varepsilon = (3n-20)\varepsilon < 3n\varepsilon$. If a cube is positioned in a corner point of C, it partially cover 3 edges, and the total edge length covered by the cubes in S is less than $3 \cdot 3n\varepsilon = 9n\varepsilon$. The result follows, as we have $\varepsilon < 1/(9n)$. □

Fact 2. *If $L \cup S$ can cover C, then each edge of C must be intercepted by a cube in L.*

Proof. From Fact 1, the cubes in S can cover only a total edge length that is smaller than 1. As the length of an edge is 1, the edge must also be intercepted by a cube of L. □

Fact 3. *If $L \cup S$ can cover C, then each cube in L must cover a different corner point of C and each face of C has exactly two cubes of L covering opposite corners of the face.*

Proof. Let $c_i, i = 1, 2, 3, 4$ be the four cubes in L, and n_i is the number of edges of C which is intercepted by c_i. From Fact 1, $\sum_{i=1}^{4} n_i \geq 12$. But each cube cannot intercept more than 3 edges of C, which means that $n_i \leq 3$. So, $n_i = 3, i = 1, 2, 3, 4$. This happens only when c_i covers a corner of C.

Let there are n_{12} edges of C intercepted by c_1 and c_2 simultaneously. Then there are at least $12 - (n_1 + n_2 - n_{12}) = 6 + n_{12}$ edges intercepted by c_3 or c_4. So, $n_3 + n_4 \geq 6 + n_{12}$, and this cannot happen when $n_{12} > 0$ because $n_3 = n_4 = 3$. This means that each edge of C can by intercepted by only one cubes in L. This leads to a configuration where in each face of C, we have exactly two cubes of L covering opposite corners of the face.

□

So, consider the non-covered space after placing the large cubes of L. In Figure 1, we exemplify the placing of the large cubes: three cubes of size $(1 - \varepsilon)$ in positions $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$ and one cube of size $(1 - \varepsilon')$ in position $(1, 1, 1)$. The other possible coverings are symmetries of this one. To help visualize the covering in Figure 1, we only present the covering of the edges of C. The placing of the large cubes of L leads to 4 non-covered cuboid regions: one with dimensions $(\varepsilon, \varepsilon, \varepsilon)$ (at corner point $(0, 0, 0)$ in Figure 1) and three with dimensions $(\varepsilon, \varepsilon, \varepsilon')$ (at corner points $(0, 1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$, rotating if necessary).

We can consider these non-covered cuboid regions as one-dimensional bins, considering the largest edge of the cuboid as the size of a one-dimensional bin, that must be covered by one-dimensional items of size ε or ε' (all remaining cubes are the cubes in S, which has $n - 6$ cubes of size ε and 2 cubes of size ε'). In Figure 2, we present these bins with the size of a largest edge.

So, the total size of these bins is $\varepsilon + (n - 7)\varepsilon + 2\varepsilon' = \varepsilon + \varepsilon' + 2\varepsilon' = \varepsilon + 3\varepsilon'$. On the other hand, the total size of cubes in S is equal to $(n - 6)\varepsilon + 2\varepsilon'$ which is also $\varepsilon + 3\varepsilon'$. So, to have a covering of these bins (non-covered cuboids) with the cubes of S, we have to obtain a perfectly covering of the

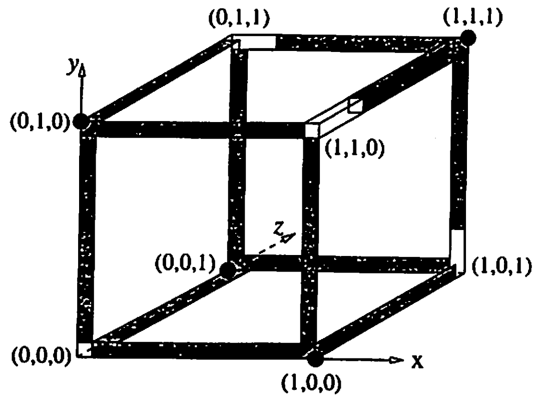

Figure 1: Edges of C covered after placing cubes in L.

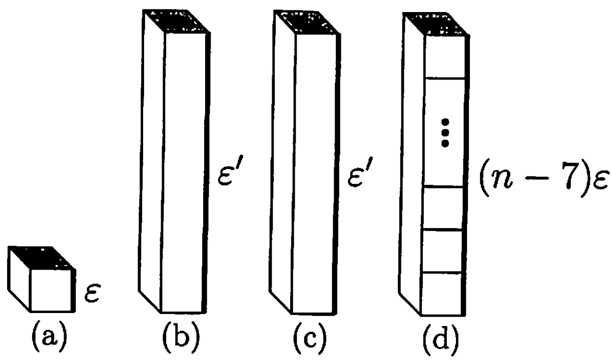

Figure 2: Non-covered cuboids after placing the large cubes of L.

bin size. In fact, the covering is easy to obtain, placing one cube of size ε in the cuboid of size ε (Figure 2 (a)), two cubes of size ε' covering the two cuboids with dimensions $(\varepsilon, \varepsilon, \varepsilon')$ (Figures 2 (b) and (c)) and the remaining $n-7$ cubes of size ε covering perfectly the remaining cuboid of dimension $(\varepsilon, \varepsilon, \varepsilon')$ (Figure 2 (d)).

To see that the above covering is minimal, note that we cannot replace one cube of S by a smaller cube, as the small cubes fit perfectly in the total length of the bins (cuboid largest edge). And we also cannot replace one large cube of L by a smaller cube, as there is no more small cubes to be used to cover the new larger cuboid regions.

Now, consider the size of the obtained covering. The cubes in L have total size $3(1-\varepsilon)+(1-\varepsilon') = 3-3\varepsilon+1-(n-7)\varepsilon = 4-(n-4)\varepsilon$. The cubes in S have total size $2\varepsilon'+(n-6)\varepsilon = 2(n-7)\varepsilon+(n-6)\varepsilon = (3n-20)\varepsilon$.

So, the size of the covering is $4 - (n-4)\varepsilon + (3n-20)\varepsilon = 4 + (2n-16)\varepsilon$. As ε can be made as close to 0 as desired, the size of the covering can also be made as close to 4 as desired. □

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