On Cordial Labelings of Fans with Other Graphs

Adel T. Diab and Sayed Anwer Elsaid Mohammed

Ain Shams University, Faculty of Science,
Department of Mathematics, Abbassia, Cairo, Egypt.

Abstract. A graph is said to be cordial if it has a 0 - 1 labeling that satisfies certain properties. A fan F_n is the graph obtained from the join of the path P_n and the null graph N_1. In this paper we investigate the cordiality of the join and the union of pairs of fans and graphs consisting of a fan with a path, and a cycle.

AMS Subject Classification: 05C78.

1 Introduction

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [10] in 1966 and Golomb[8] in 1972, while harmonious labelings were first studied by Graham and Sloane [9] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [1] in 1990. Whereas the label of an edge vw for graceful and harmonious labeling is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced edge label $(f(v) + f(w))$ (mod2), which of course equals $|f(v) - f(w)|$. Because arithmetic modulo2 is an integral part of computer science, cordial labelings have close connections with that field.An excellent reference on this subject is the survey by Gallian [7]. More precisely, cordial graphs are defined as follows.
Let $G = (V, E)$ be a graph, let $f : V \to \{0, 1\}$ be a labeling of its vertices, and let $f^* : E \to \{0, 1\}$ is the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w)$ (mod 2). (Thus, for any edge $e, f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels). Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edge. Such a labeling is called cordial if both $|v_0 - v_1| \le 1$ and $|e_0 - e_1| \le 1$. A graph is called cordial if it has a cordial labeling. A fan F_n is the graph obtained from the join of a path P_n and a null graph N_1. So the order of the fan F_n is n+1 and its size is 2n-1 for all n, in particular F_1=P_2 and F_2=C_3. Diab [2,3,5] has proved that the following: The join of a path P_n and a null graph N_m is cordial for all n and all m; the join $P_n + P_m$ of two paths P_n and P_m is cordial for all n and all m except for (n,m)=(2,2); the join $C_n + P_m$ of a cycle C_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (3,1), (3,2), or (3,3); the union $P_n \cup P_m$ of two paths P_n and P_m is cordial

for all n and all m except for the graph $2P_2$; the union $C_n \cup C_m$ of two cycles C_n and C_m is cordial for all n and all m if and only if n+m is not congruent to 2(mod4); the union $C_n \cup P_m$ of a cycle C_n and a path P_m is cordial for all n and all m if and only if it is not isomorphic to $C_n \cup P_1$ with $n \equiv 2(\text{mod}4)$. As stated in the above result we conclude that every fan $F_n = P_n + N_1$ is cordial for all n. In this paper we extend those results to investigate the cordiality of the join and the union of pairs of fans and graphs consisting of a fan and a path or a cycle. In section 3, we show that the join $F_n + F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1), (1,2), (1,3), (2,1), (2,2),(2,3), (3,1), (3,2) and (3,3). Also, we show that the union$F_n \cup F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1) and (2,2). In section 4, we show that the join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only (n,m) $\neq$ (1,2), (2,1), (2,2),(2,3) and (3,2). Also, we show that the union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,2). In section 5, we show that the join $F_n + C_m$ of a fan F_n and a cycle C_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,3), (2,3) and (3,3). Also, we prove that the union $F_n \cup C_m$ of a fan F_n and a cycle C_m is cordial for all n and all m if and only if (n,m) $\neq$ (2,3).

2 Terminology and notations

We introduce some notation and terminology for a graph with $4r$ vertices [2,3,4,5,6] , we let L_{4r} denote the labeling 00110011...0011. In most cases, we then modify this by adding symbols at one end or the other (or both). Thus $01L_{4r}$ denotes the labeling 0100110011...0011 of either F_{4r+2}, C_{4r+2} or P_{4r+2}(It should be to remark that for the labeling of the fan F_{4r+2}, we label the center of the fan by the first label which is 0 in $01L_{4r}$ and other labelings for the vertices of P_{4r+1} which are $1L_{4r}$). One exception to this is the labeling L'_{4r} obtained from L_{4r} by adding an initial 0 and deleting the last 1: that is, L'_{4r} is 000110011...11001 and $L"_{4r}$ obtained from L_{4r} by adding an initial 1 and deleting the last 1: that is, $L"_{4r}$ is 100110011...11001. For specific labeling L and M of G+H (or $G \cup H$), where G and H are paths or cycles or fans, we let $[L; M]$ denote the joint labeling. Additional notation that we use is the following.

For a given labeling of the join $G + H$ (or $G \cup H$), we let v_i and e_i (for i = 0,1) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0, v_1 = x_1 + y_1, e_0 = a_0 + b_0 + x_0y_0 + x_1y_1$ (or $e_0 = a_0+b_0$)and $e_1 = a_1+b_1+x_0y_1+x_1y_0$ (or $e_1 = a_1+b_1$) , thus, $v_0-v_1 = (x_0-x_1)+(y_0-y_1)$ and $e_0-e_1 = (a_0-a_1)+(b_0-b_1)+(x_0-x_1)(y_0-y_1)$ (or $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$).When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3 Joins and Union of Pairs of Fans

In [5], we determined that a join of a path P_n and a null graph N_m is cordial for all n and all m, and from this fact we conclude that every fan $F_n = P_n + N_1$ is cordial for all n. In this section, we extend this result to show that the join $F_n + F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2) and (3,3). Also, we prove that the union $F_n \cup F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1) and (2,2).

Lemma 3.1. The join $F_n + F_m$ of two fans F_n and F_m is cordial for all $n > 3$ and all $m > 3$.

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i for the fan F_n and B_j or B'_j or $B"_j$ for the fan F_m as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0,1,2,3$	Labeling of F_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r + 1$	$4r$	$4r - 1$
$i = 1$	$A_1 = 01L_{4r}$	$2r + 1$	$2r + 1$	$4r$	$4r + 1$
	$A'_1 = 10L_{4r}$	$2r + 1$	$2r + 1$	$4r + 1$	$4r$
$i = 2$	$A_2 = 001L_{4r}$	$2r + 2$	$2r + 1$	$4r + 1$	$4r + 2$
$i = 3$	$A_3 = 0011L_{4r}$	$2r + 2$	$2r + 2$	$4r + 2$	$4r + 3$

$m = 4s + j$, $j = 0,1,2,3$	Labeling of F_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = 0L_{4s}$	$2s + 1$	$2s$	$4s$	$4s - 1$
	$B'_0 = 1L_{4s}$	$2s$	$2s + 1$	$4s$	$4s - 1$
	$B"_0 = 1L'_{4s}$	$2s + 1$	$2s$	$4s - 1$	$4s$
$j = 1$	$B_1 = 01L_{4s}$	$2s + 1$	$2s + 1$	$4s$	$4s + 1$
	$B'_1 = 10L_{4r}$	$2r + 1$	$2r + 1$	$4r + 1$	$4r$
$j = 2$	$B_2 = 001L_{4s}$	$2s + 2$	$2s + 1$	$4s + 1$	$4s + 2$
	$B'_2 = 110L_{4s}$	$2s + 1$	$2s + 2$	$4s + 2$	$4s + 1$
$j = 3$	$B_3 = 0011L_{4s}$	$2s + 2$	$2s + 2$	$4s + 2$	$4s + 3$
	$B'_3 = 1100L_{4s}$	$2s + 2$	$2s + 2$	$4s + 3$	$4s + 2$

Table 3.1. Labelings of Wheels.

$n = 4r + i$, $i = 0,1,2,3$	$m = 4s + j$, $j = 0,1,2,3$	F_n	F_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	1
0	1	A_0	B_1	-1	0
0	2	A_0	B_2	0	-1
0	3	A_0	B_3	-1	0
1	0	A_1	B_0	1	0
1	1	A_1	B'_1	0	0
1	2	A_1	B'_2	-1	0
1	3	A_1	B'_3	0	0
2	0	A_2	B'_0	0	-1
2	1	A_2	B'_1	1	0
2	2	A_2	B'_2	0	-1
2	3	A_2	B'_3	1	0
3	0	A_3	B'_0	-1	0
3	1	A_3	B'_1	0	0
3	2	A_3	B'_2	-1	0
3	3	A_3	B'_3	0	0

Table 3.2. Combinations of labelings.

Lemma 3.2. The join $F_n + F_m$ of two fans F_n and F_m is cordial for all $n \leq 3$ and for all $m > 3$ (or vice versa).

Proof. Suppose m = 4s + j, where j = 1,2,3,4 and we consider the cases of n separately.

Case 1. n = 1. The following labelings suffice: $F_1 + F_{4s}$:[01;1L_{4s}], $F_1 + F_{4s+1}$: [01;10L_{4s}], $F_1 + F_{4s+2}$: [01;110L_{4s}] and $F_1 + F_{4s+3}$: [01;1100L_{4s}].

Case 2. n = 2. The following labelings suffice:$F_2 + F_{4s}$: [010;1L_{4s}], $F_2 + F_{4s+1}$: [010;10L_{4s}], $F_2 + F_{4s+2}$: [010;110L_{4s}] and $F_2 + F_{4s+3}$:[010;1100L_{4s}].

Case 3. n = 3. The following labelings suffice: $F_3 + F_{4s}$: [0011;1L_{4s}], $F_3 + F_{4s+1}$: [0011;10L_{4s}], $F_3 + F_{4s+2}$: [0011;110L_{4s}] and $F_3 + F_{4s+3}$: [0011;1100L_{4s}].This completes the proof.

Example 3.1. The graphs $F_1 + F_1$, $F_1 + F_2$, $F_1 + F_3$, $F_2 + F_1$, $F_2 + F_2$, $F_2 + F_3$, $F_3 + F_1$, $F_3 + F_2$ and $F_3 + F_3$ are not cordial.

Solution. It is easy to see that $F_1 + F_1 \equiv K_4$, $F_1 + F_2 \equiv F_2 + F_1 \equiv K_5$ and $F_2 + F_2 \equiv C_3 + C_3 \equiv K_6$ are not cordial from the fact that the complete graph K_n is cordial if and only if $n \leq 3$ (see [1]). By investigating all possible labelings, it is easy to see that $F_1 + F_3$, $F_2 + F_3$, $F_3 + F_1$, $F_3 + F_2$ and $F_3 + F_3$ does not have a cordial labeling.

Theorem 3.1. The join $F_n + F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1), (1,2), (1,3), (2,1), (2,2),(2,3), (3,1), (3,2) and (3,3).

Proof. The proof follows directly from lemma 3.1, lemma 3.2 and example 3.1, the theorem follows.

Lemma 3.3. The union $F_n \cup F_m$ of two fans F_n and F_m is cordial for all $n > 3$ and all $m > 3$.

Proof. For given values of i and j with $0 \le i \le 3$ and $0 \le j \le 3$, we use the labeling A_i or A'_i for the fan F_n, where $n > 3$ and B_j or B'_j or $B"_j$ for the fan F_m, where $m > 3$ as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 3.3. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	$m = 4s + j$, $j = 0, 1, 2, 3$	F_n	F_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	$B"_0$	0	0
0	1	A_0	B_1	-1	0
0	2	A_0	B_2	0	0
0	3	A_0	B_3	-1	0
1	0	A_1	B_0	1	0
1	1	A_1	B'_1	0	0
1	2	A_1	B'_2	-1	0
1	3	A_1	B'_3	0	0
2	0	A_2	B'_0	0	0
2	1	A_2	B'_1	1	0
2	2	A_2	B'_2	0	0
2	3	A_2	B'_3	1	0
3	0	A_3	B'_0	-1	0
3	1	A_3	B'_1	0	0
3	2	A_3	B'_2	-1	0
3	3	A_3	B'_3	0	0

Tabel 3.3. Combinations of labelings.

Lemma 3.4. The union $F_n \cup F_m$ of two fans F_n and F_m is cordial for all $n \le 3$ and all $m > 3$ (or vice versa).

Proof. Suppose m = 4s + j, where j = 1,2,3,4 and we consider the cases of n separately.

Case 1. n = 1. The following labelings suffice: $F_1 \cup F_{4s}$: [01;1L_{4s}], $F_1 \cup F_{4s+1}$: [01;10L_{4s}], $F_1 \cup F_{4s+2}$: [01;110L_{4s}] and $F_1 \cup F_{4s+3}$: [01;1100L_{4s}].

Case 2. n = 2. The following labelings suffice:$F_2 \cup F_{4s}$: [010;1L_{4s}], $F_2 \cup F_{4s+1}$: [010;10L_{4s}], $F_2 \cup F_{4s+2}$: [010;110L_{4s}] and $F_2 \cup F_{4s+3}$:[010;1100L_{4s}].

Case 3.n = 3. The following labelings suffice: $F_3 \cup F_{4s}$: [0011;1L_{4s}], $F_3 \cup F_{4s+1}$: [0011;10L_{4s}], $F_3 \cup F_{4s+2}$: [0011;110L_{4s}] and $F_3 \cup F_{4s+3}$: [0011;1100L_{4s}]. This completes the proof.

Example 3.2. The graphs $F_1 \cup F_1$ and $F_2 \cup F_2$ are not cordial.
Solution. Diab [2] has proved that the join $P_n + P_m$ of two paths P_n and P_m is cordial for all n and all m except for (n,m)=(2,2), and the union $C_n \cup C_m$ of two cycles C_n and C_m is cordial for all n and all m if and only if n+m is not congruent to 2(mod4), then the graphs $F_1 \cup F_1 = P_2 \cup P_2 = 2\ P_2$ and $F_2 \cup F_2 = C_3 \cup C_3$ are not cordial.
Lemma 3.5. The union $F_n \cup F_m$ of two fans F_n and F_m is cordial for all $n \leq 3$ and all $m \leq 3$ except for (n, m) = (1,1) and (2,2).
Proof. Appropriate labelings are the following: $F_1 \cup F_2$: [00;011], $F_1 \cup F_3$: [00;0111], $F_2 \cup F_1$: [011;00], $F_2 \cup F_3$: [011;0001], $F_3 \cup F_1$: [0111;00], $F_3 \cup F_2$: [0001;011] and $F_3 \cup F_3$: [0001;0111], the lemma follows.
Theorem 3.2. The union $F_n \cup F_m$ of two fans F_n and F_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,1) and (2,2).
Proof. The proof follows directly from lemma 3.3, lemma 3.4, lemma 3.5 and example 3.2, the theorem follows.

4 Joins and Unions of Fans and Paths

In this section we show that the join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only if (n,m)$\neq$ (1,2), (2,1), (2,2),(2,3) and (3,2). Also, we show that the union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,2).
Lemma 4.1. The join $F_n + P_m$ of a fan W_n and a path P_m is cordial for all $n > 3$ and all $m > 3$.
Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A_i' for the fan F_n, where $n > 3$ and B_j or B_j' for the path P_m, where $m > 3$ as given in Table4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4.2. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of F_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r+1$	$4r$	$4r-1$
$i = 1$	$A_1 = 01L_{4r}$	$2r+1$	$2r+1$	$4r$	$4r+1$
	$A_1' = 10L_{4r}$	$2r+1$	$2r+1$	$4r+1$	$4r$
$i = 2$	$A_2 = 110L_{4r}$	$2r+1$	$2r+2$	$4r+2$	$4r+1$
$i = 3$	$A_3 = 0011L_{4r}$	$2r+2$	$2r+2$	$4r+2$	$4r+3$

$m = 4s + j$, $j = 0,1,2,3$	Labeling of P_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s-1$
	$B'_0 = L"_{4s}$,	$2s$	$2s$	$2s-1$	$2s$
$j = 1$	$B_1 = L_{4s}0$	$2s+1$	$2s$	$2s$	$2s$
$j = 2$	$B_2 = L_{4s}01$	$2s+1$	$2s+1$	$2s$	$2s+1$
	$B'_2 = L_{4s}10$,	$2s+1$	$2s+1$	$2s+1$	$2s$
$j = 3$	$B_3 = L_{4s}001$	$2s+2$	$2s+1$	$2s+1$	$2s+1$

Table 4.1. Labelings of Fans F_n and paths P_m.

$n = 4r + i$, $i = 0,1,2,3$	$m = 4s + j$, $j = 0,1,2,3$	F_n	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B'_0	-1	0
0	1	A_0	B_1	0	0
0	2	A_0	B_2	-1	0
0	3	A_0	B_3	0	0
1	0	A_1	B_0	0	0
1	1	A'_1	B_1	1	1
1	2	A'_1	B_2	0	0
1	3	A'_1	B_3	1	1
2	0	A_2	B'_0	-1	0
2	1	A_2	B_1	0	0
2	2	A_2	B_2	-1	0
2	3	A_2	B_3	0	0
3	0	A_3	B'_0	0	0
3	1	A_3	B_1	1	-1
3	2	A_3	B'_2	0	0
3	3	A_3	B_3	1	-1

Table 4.2. Combinations of labelings.

Lemma 4.2. The join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all $n \leq 3$ and all $m > 3$.

Proof. We consider the cases of n separately.

Case 1. n = 1. The result follows from the fact that $F_1 = P_2$ and the following theorem, which states that the join $P_n + P_m$ of two paths P_n and P_m is cordial for all n and all m except for $P_2 + P_2$ (see [2]).

Case 2. n = 2. The result follows from the fact that $F_2 = C_3$ and the following theorem, which states that the join $C_n + P_m$ of a cycle C_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (3,1), (3,2), or (3,3)(see [3]).

Case 3. n = 3. Let m = 4r + j,where j = 1, 2, 3, 4, then the following labelings suffice. $F_3 + P_{4s}$: [0011;L_{4s}], $F_3 + P_{4s+1}$: [0011;L_{4s}0], $F_3 + P_{4s+2}$:[0011;L_{4s}10] and $F_3 + P_{4s+3}$:[0011;L_{4s}011]. This completes the proof.

Lemma 4.3. The join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all $n \leq 3$ and all $m \leq 3$ except for (n,m) = (1,2), (2,1), (2,2), (2,3) and (3,2).

Proof. Appropriate labelings are the following : $F_1 + P_1 \equiv P_2 + P_1$: [01,0], $F_1 + P_3 \equiv P_2 + P_3$:[01,011], $F_3 + P_1$:[0011,0] and $F_3 + P_1$:[0011,001], the lemma follows.

Example 4.1. The graphs $F_1 + P_2$, $F_2 + P_1$, $F_2 + P_2$, $F_2 + P_3$ and $F_3 + P_3$ are not cordial.

Solution. It is easy to see that $F_1 + P_2 \equiv P_2 + P_2 \equiv K_4$ and $F_2 + P_1 \equiv C_3 + P_1 \equiv K_4$, $F_2 + P_2 \equiv C_3 + P_2 \equiv K_5$ and $F_3 + P_3 \equiv C_3 + P_3$ are not cordial similar to example 3.1 and the fact that the join $C_n + P_m$ of a cycle C_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (3,1), (3,2), or (3,3). By investigating all possible labelings we see that $F_3 + P_3$ does not have a cordial labeling.

Lemma 4.4. The join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all $n > 3$ and all $m \leq 3$.

Proof. Let n = 4r + i, where i = 1, 2, 3, 4, then we consider the cases of m separately.

Case 1. m = 1. Appropriate labelings are the following: $F_{4r} + P_1$:[1L_{4r};0], $F_{4r+1} + P_1$: [10L_{4r};0], $F_{4r+2} + P_1$:[110L_{4r};0] and $F_{4r+3} + P_1$:[1100L_{4r};0].

Case 2. m = 2. Appropriate labelings are the following: $F_{4r} + P_2$:[1L_{4r};01], $F_{4r+1} + P_2$: [10L_{4r};01], $F_{4r+2} + P_2$:[110L_{4r};01] and $F_{4r+3} + P_2$:[1100L_{4r};01].

Case 3. m = 3. Appropriate labelings are the following: $F_{4r} + P_3$:[1L_{4r};001], $F_{4r+1} + P_3$: [10L_{4r};001], $F_{4r+2} + P_3$:[110L_{4r};001] and $F_{4r+3} + P_3$:[1100L_{4r};001], the lemma follows.

Theorem 4.1. The join $F_n + P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,2), (2,1), (2,2), (2,3) and (3,2).

Proof. The proof follows directly from lemma 4.1, lemma 4.2, lemma 4.3, lemma 4.4 and example 4.1, the theorem follows.

Lemma 4.5. The union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for all $n > 3$ and all $m > 3$.

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i for the fan F_n, where $n > 3$ and B_j or B'_j or for the path P_m , where $m > 3$ as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 4.3. Since these are all 0,1, or -1, the lemma follows.

$n=4r+i$, $i=0,1,2,3$	$m=4s+j$, $j=0,1,2,3$	F_n	P_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	0
0	1	A_0	B_1	0	1
0	2	A_0	B_2	-1	0
0	3	A_0	B_3	0	1
1	0	A_1	B_0	0	0
1	1	A_1'	B_1	1	1
1	2	A_1'	B_2	0	0
1	3	A_1'	B_3	1	1
2	0	A_2	B_0'	-1	0
2	1	A_2	B_1	0	1
2	2	A_2	B_2	-1	0
2	3	A_2	B_3	0	1
3	0	A_3	B_0'	0	0
3	1	A_3	B_1	1	-1
3	2	A_3	B_2'	0	0
3	3	A_3	B_3	1	-1

Table 4.3. Combinations of labelings.

Lemma 4.6. The union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for all $n \leq 3$ and all $m > 3$.

Proof. We consider the cases of n separately.

Case 1. n = 1. The result follows from the fact that $F_1 = P_2$ and the following theorem, which states that the union $P_n \cup P_m$ of two paths P_n and P_m is cordial for all n and all m except for $P_2 \cup P_2$ (see[2]).

Case 2. n = 2. The result follows from the fact that $F_2 = C_3$ and the following theorem, which states that the union $C_n \cup P_m$ of a cycle C_n and a path P_m is cordial for all n and all m if and only if it is not isomorphic to $C_n \cup P_1$ with $n \equiv 2(\text{mod}4)$(see[3]).

Case 3. n = 3. Let m = 4r + j,where j = 1, 2, 3, 4, then the following labelings suffice. $F_3 \cup P_{4s}$: [0011;L_{4s}], $F_3 \cup P_{4s+1}$: [0011;L_{4s}0], $F_3 \cup P_{4s+2}$:[0011;L_{4s}10] and $F_3 \cup P_{4s+3}$:[0011;L_{4s}011]. This completes the proof.

Lemma 4.7. The graphs $F_1 \cup P_1$, $F_1 \cup P_3$, $F_2 \cup P_1$, $F_2 \cup P_2$, $F_2 \cup P_3$, $F_3 \cup P_1$, $F_3 \cup P_2$ and $F_3 \cup P_3$ are cordial.

Proof. The following labelings suffice. $F_1 \cup P_1 \equiv P_2 \cup P_1$:[01;1], $F_1 \cup P_3 \equiv P_2 \cup P_3$:[01;001], $F_2 \cup P_1 \equiv C_3 \cup P_1$:[001;1], $F_2 \cup P_2 \equiv C_3 \cup P_2$:[001;11], $F_2 \cup P_3 \equiv C_3 \cup P_3$:[001;110], $F_3 \cup P_1$:[0011;0], $F_3 \cup P_2$:[0010;11] and $F_3 \cup P_3$:[0011;001], the lemma follows.

Lemma 4.8. The union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for

all $n > 3$ and all $m \leq 3$.
Proof. Let n = 4r + i, where i = 1, 2, 3, 4, then we consider the cases of m separately.
Case 1. m = 1. Appropriate labelings are the following: $F_{4r} \cup P_1$:[$1L_{4r}$;0], $F_{4r+1} \cup P_1$: [$10L_{4r}$;0], $F_{4r+2} \cup P_1$:[$110L_{4r}$;0] and $F_{4r+3} \cup P_1$:[$1100L_{4r}$;0].
Case 2. m = 2. Appropriate labelings are the following: $F_{4r} \cup P_2$:[$1L_{4r}$;01], $F_{4r+1} \cup P_2$: [$10L_{4r}$;01], $F_{4r+2} \cup P_2$:[$110L_{4r}$;01] and $F_{4r+3} \cup P_2$:[$1100L_{4r}$;01].
Case 3. m = 3. Appropriate labelings are the following: $F_{4r} \cup P_3$:[$1L_{4r}$;001], $F_{4r+1} \cup P_3$: [$10L_{4r}$;001], $F_{4r+2} \cup P_3$:[$110L_{4r}$;001] and $F_{4r+3} \cup P_3$:[$1100L_{4r}$;001], the lemma follows.
Example 4.2. The graph $F_1 \cup P_2$ is not cordial..
Solution. The solution follows directly from the fact that $F_1 \cup P_1 \equiv P_1 \cup P_1$, which is not cordial (see [2]).
Theorem 4.2. The union $F_n \cup P_m$ of a fan F_n and a path P_m is cordial for all n and all m if and only if (n,m) $\neq$ (1,2).
Proof. The proof follows directly from lemma 4.5, lemma 4.6, lemma 4.7, lemma 4.8 and example 4.2, the theorem follows.

5 Joins and Unions of Fans and Cycles

In this section, we show that the join $F_n + C_m$ of a fan F_n and a cycle C_m is cordial for all n and all m if and only if $(m, n) \neq$ (1,3), (2,3) and (3,3). Also, we prove that the union $W_n \cup C_m$ of a fan F_n a cycle C_m is cordial for all n and all m.
Lemma 5.1. The join $F_n + C_m$ of a fan F_n and cycles C_m is cordial for all $n > 3$ and all $m > 3$.
Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i for the fan F_n, where $n > 3$ and B_j for the cycle C_m, where $m > 3$ as given in Table 5.1. Using Table 5.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 5.2. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of F_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = 1L_{4r}$	$2r$	$2r + 1$	$4r$	$4r - 1$
$i = 1$	$A_1 = 01L_{4r}$	$2r + 1$	$2r + 1$	$4r$	$4r + 1$
	$A'_1 = 10L_{4r}$	$2r + 1$	$2r + 1$	$4r + 1$	$4r$
$i = 2$	$A_2 = 011L_{4r}$	$2r + 1$	$2r + 2$	$4r + 1$	$4r + 2$
	$A'_2 = 110L_{4r}$	$2r + 1$	$2r + 2$	$4r + 2$	$4r + 1$
$i = 3$	$A_3 = 0011L_{4r}$	$2r + 2$	$2r + 2$	$4r + 2$	$4r + 3$

$m = 4s + j$, $j = 0,1,2,3$	Labeling of C_m	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = L_{4s}$	$2s$	$2s$	$2s$	$2s$
$j = 1$	$B_1 = L_{4s}0$	$2s+1$	$2s$	$2s+1$	$2s$
$j = 2$	$B_2 = 01L_{4s}$	$2s+1$	$2s+1$	$2s$	$2s+2$
$j = 3$	$B_3 = L_{4s}001$,	$2s+2$	$2s+1$	$2s+1$	$2s+2$

Table 5.1. Labelings of a fan F_n and a cycle C_m.

$n = 4r + i$, $i = 0,1,2$	$m = 4s + j$, $j = 0,1,2,3$	F_n	C_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	-1	1
0	1	A_0	B_1	0	1
0	2	A_0	B_2	-1	-1
0	3	A_0	B_3	0	-1
1	0	A_1	B_0	0	-1
1	1	A_1	B_1	1	0
1	2	A'_1	B_2	0	-1
1	3	A'_1	B_3	1	0
2	0	A_2	B_0	-1	-1
2	1	A_2	B'_1	0	-1
2	2	A'_2	B_2	-1	-1
2	3	A'_2	B_3	0	-1
3	0	A_3	B_0	0	-1
3	1	A_3	B_1	1	0
3	2	A'_3	B_2	0	-1
3	3	A'_3	B_3	1	0

Table 5.2. Combinations of labelings.

Lemma 5.2. The join $F_n + C_3$ of a fan F_n and a cycle C_3 is cordial for all $n > 3$.
Proof. Let n = 4r+i, where i =1,2,3,4, then the following labelings suffice: $F_{4r} + C_3$:[1L_{4r};001], $F_{4r+1} + C_3$: [10L_{4r};001], $F_{4r+2} + P_1$:[110L_{4r};001] and $F_{4r+3} + C_3$: [1100L_{4r};001], the lemma follows.
Example 5.1. The graphs $F_1 + C_3$, $F_2 + C_3$ and $F_3 + C_3$ are not cordial.
Solution. It is easy to see that $F_1 + C_3 \equiv P_2 + C_3 \equiv K_5$ and $F_2 + C_3 \equiv C_3 + C_3 \equiv K_6$ are not cordial similar to example 3.1. By investigating all possible labelings we see that $F_3 + C_3$ does not have a cordial labeling.
Theorem 5.1. The join $F_n + C_m$ of a fan F_n and a cycle C_m is cordial for all n and all m if and only if $(n, m) \neq$ (1,3), (2,3) and (3,3).
Proof. The proof follows directly from lemma 5.1, lemma 5.2 and example 5.1, the theorem follows.
Lemma 5.4. The union $F_n \cup C_m$ of a fan F_n and a cycle C_m is cordial for

all $n > 3$ and $m > 3$.

Proof. For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 3$, we use the labeling A_i or A'_i for the fan F_n, where $n > 3$ and B_j for the cycle C_m, where $m > 3$ as given in Table 5.1. Using Table 5.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 5.3. Since these are all 0,1, or -1, the lemma follows.

$n = 4r + i$, $i = 0,1,2,3$	$m = 4s + j$, $j = 0,1,2,3$	F_n	P_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	-1	1
0	1	A_0	B_1	0	0
0	2	A_0	B_2	-1	-1
0	3	A_0	B_3	0	0
1	0	A_1	B_0	0	-1
1	1	A_1	B_1	1	0
1	2	A'_1	B_2	0	-1
1	3	A'_1	B_3	1	0
2	0	A_2	B_0	-1	-1
2	1	A_2	B'_1	0	0
2	2	A'_2	B_2	-1	-1
2	3	A'_2	B_3	0	0
3	0	A_3	B_0	0	-1
3	1	A_3	B_1	1	0
3	2	A'_3	B_2	0	-1
3	3	A'_3	B_3	1	0

Table 5.3. Combinations of labelings.

Lemma 5.4. The union $F_n \cup C_3$ of a fan F_n and a cycle C_3 is cordial for all $n > 3$.

Proof. Let n = 4r+i, where i =1,2,3,4, then the following labelings suffice: $F_{4r} \cup C_3$:[1L_{4r};001], $F_{4r+1} \cup C_3$: [10L_{4r};001], $F_{4r+2} \cup P_1$:[110L_{4r};001] and $F_{4r+3} \cup C_3$: [1100L_{4r};001], the lemma follows.

Example 5.2. The graphs $F_1 \cup C_3$ and $F_3 \cup C_3$ are cordial.

Solution. Appropriate labelings are the following : $F_1 \cup C_3 = P_2 \cup C_3$: [00;110] and $F_3 \cup C_3$: [0001;110].

Example 5.3. The graph $F_2 \cup C_3$ is not cordial.

Solution. The solution follows from the fact that $F_2 \cup C_3 \equiv C_3 \cup C_3$ and the following theorem, which states that the union $C_n \cup C_m$ of two cycles C_n and C_m is cordial for all n and all m if and only if n+m is not congruent to 2(mod4)(see[2]).

Theorem 5.2. The union $F_n \cup C_m$ of a fan F_n and a cycle C_m is cordial

for all n and all m if and only if $(n,m) \neq (2,3)$.
Proof. The proof follows directly from lemma 5.3, lemma 5.4, example 5.2 and example 5.3, the theorem follows.

References

[1] I. Cahit, On cordial and 3-equitable labelings of graphs, Utilities Math., 37 (1990), 189-198.

[2] A.T. Diab and E.A. Elsakhawi, Some Results on Cordial Graphs, Proc. Math. Phys. Soc. Egypt, No.7, pp. 67-87 (2002).

[3] A.T. Diab, Study of Some Problems of Cordial Graphs, Ars Combinatoria, to appear.

[4] A.T. Diab, On Cordial Labelings of the Second Power of Paths with Other Graphs, Ars Combinatoria, to appear.

[5] A.T. Diab, Generalization of Some Results on Cordial Graphs, Ars Combinatoria, to appear.

[6] A.T. Diab, On Cordial Labelings of Wheels with Other Graphs, preprint.

[7] J.A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics, DS6 (January 20, 2005).

[8] S.W. Golomb, How to number a graph in Graph Theory and Computing, R.C. Read, ed., Academic Press, New York (1972), 23-37.

[9] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Math., 1 (1980), 382-404.

[10] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internet. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967), 349-355.