Roman domination and Mycieleki's structure in graphs

Adel P. Kazemi
Department of Mathematics
University of Mohaghegh Ardabili
P.O.Box 5619911367, Ardabil, Iran
Email: adelpkazemi@yahoo.com

Abstract

For a graph $G = (V, E)$, a function $f : V \to \{0, 1, 2\}$ is called Roman dominating function (RDF) if for any vertex v with $f(v) = 0$, there is at least one vertex w in its neighborhood with $f(w) = 2$. The weight of an RDF f of G is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF of G is its Roman domination number and denoted by $\gamma_R(G)$. In this paper, we show that $\gamma_R(G) + 1 \leq \gamma_R(\mu(G)) \leq \gamma_R(G) + 2$, where $\mu(G)$ is the Mycielekian graph of G, and then characterize the graphs achieving equality in these bounds.

Keywords: Roman domination number, Mycieleskian graph.
2010 Mathematical Subject Classification: 05C69.

1 Introduction and primary results

The notation we use is as follows. Let G be a simple graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The *order* $| V |$ and *size* $| E |$ of G are respectively denoted by $n = n(G)$ and $m = m(G)$. If $E = \emptyset$, then G is called *empty graph.* The *open* and *closed neighborhoods* of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the *open* and *closed neighborhoods* of a subset $X \subseteq V(G)$ are $N_G(X) = \cup_{v \in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$, respectively. The *degree* of a vertex $v \in V$ is $deg(v) =| N(v) |$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree k, then G is said to be *k-regular.* We write K_n,

C_n and P_n for the *complete graph*, *cycle* and *path* of order n, respectively, while $K_{n_1,\dots,n_p}$ denotes a *complete p-partite graph*. For a subset $S \subseteq V$, the *induced subgraph* $G[S]$ is a subgraph of G with the vertex set S and for every vertices $u, v \in S$, $uv \in E(G[S])$ if and only if $uv \in E(G)$.

The research on domination in graphs has been an evergreen in the field of graph theory. Its basic concept is the dominating set and the domination number. The recent book *Fundamentals of Domination in Graphs* [4] lists, in an appendix, many varieties of dominating sets that have been studied. It appears that none of those listed are the same as Roman dominating sets. Thus, Roman domination appears to be a new variety of both historical and mathematical interest.

A subset $S \subseteq V(G)$ is a *dominating set*, briefly DS, in G, if every vertex in $V(G) - S$ has a neighbor in S. The minimum number of vertices of a DS in a graph G is called the *domination number* of G and denoted by $\gamma(G)$.

Let $f: V \to \{0,1,2\}$ be a function and let (V_0, V_1, V_2) be the ordered partition of V induced by f, where $V_i = \{v \in V \mid f(v) = i\}$ and $\mid V_i \mid = n_i$, for $i = 0,1,2$. We notice that there is an obvious one-to-one correspondence between f and the ordered partition (V_0, V_1, V_2) of V. Therefore, one can write $f = (V_0, V_1, V_2)$. Function $f = (V_0, V_1, V_2)$ is a *Roman dominating function*, abbreviated RDF, for G if $V_0 \subseteq N_G(V_2)$. If $W_2 \subseteq V_2$ and $W_1 \subseteq V_1$, then we say $W_1 \cup W_2$ *defends* $W_1 \cup N_G[W_2]$. For simplicity in notation, instead of saying that $\{v\}$ defends $\{w\}$, we say v *defends* w. The *weight* of f is the value $f(V) = \sum_{v \in V} f(v) = 2n_2 + n_1$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF of G, and we say a function $f = (V_0, V_1, V_2)$ is a γ_R- or $\gamma_R(G)$-*function* if it is an RDF for G and $f(V) = \gamma_R(G)$. More details about Roman domination number have given in many papers. For example reader can see [2, 3, 5, 6, 8, 9].

Let $G = (V^0, E^0)$ be a graph. The *Mycieleskian* $\mu(G)$ of G is the graph with vertex set $V^0 \cup V^1 \cup \{u\}$, where $V^1 = \{v_j^1 \mid v_j^0 \in V^0\}$, and edge set $E^0 \cup \{v_j^1 v_i^0 \mid v_j^0 v_i^0 \in E^0 \text{ and } v_j^1 \in V^1\} \cup \{v_j^1 u \mid v_j^1 \in V^1\}$. Interested readers may refer to [1, 7] to know more about the Mycieleskian graphs.

As stated in many references, for example in [4], the *Cartesian product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G)$.

Let $v \in S \subseteq V$. A vertex u is called a *private neighbor* of v with respect to S, or simply an *S-pn* of v, if $u \in N[v] - N[S - \{v\}]$. The set $pn(v; S) = N[v] - N[S - \{v\}]$ of all S-pn's of v is called the *private neighborhood set* of v with respect to S. Also an S-pn of v is an *external private neighbor* or *external* (denoted by *S-epn* of v) if it is a vertex of $V - S$. We also call the set $epn(v; S) = N(v) - N[S - \{v\}]$ of all S-epn's of v, the *external private neighborhood set* of v with respect to S. To see this definitions refer to [1, 4]. Obviously if $f = (V_0, V_1, V_2)$ is a γ_R-function, then for each $v \in V_2$, $epn(v; V_2) \neq \emptyset$ (we notice that for each vertex $v \in V_2$, $epn(v; V_2) \subseteq V_0$ and so $epn(v; V_2) \neq \emptyset$ if and only if $epn(v; V_2) \cap V_0 \neq \emptyset$).

Cockayne et al. in [2] have shown that for any graph G of order n and maximum degree Δ, $2n/(\Delta + 1) \leq \gamma_R(G)$, and for the classes of paths P_n and cycles C_n, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil 2n/3 \rceil$. Furthermore, they have shown that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$, where the lower bound is achieved only by $G = \overline{K_n}$, the empty graph with n vertices. A graph G is called a *Roman graph* if $\gamma_R(G) = 2\gamma(G)$ [2]. For example, the complete multipartite graph $K_{m_1,\ldots,m_n}$ is Roman if and only if $2 \notin \{m_1, \ldots, m_n\}$. As shown in [2], an equivalent condition for G to be a Roman graph is that G has a γ_R-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.

We now introduce two new concepts. A Roman graph G with γ_R-function $f = (V_0, \emptyset, V_2)$ we call a *special Roman graph* if the induced subgraph $G[V_2]$ has no isolated vertex, and its γ_R-function f we call a *special γ_R-function.*

In this paper, we first show that $\gamma_R(G) + 1 \leq \gamma_R(\mu(G)) \leq \gamma_R(G) + 2$ and then characterize the graphs achieving equality in these bounds.

In this entire paper, we assume that the induced subgraph by V_1 is an empty subgraph if $f = (V_0, V_1, V_2)$ is an RDF and $V_1 \neq \emptyset$.

We first present the Roman domination number of some known graphs.

Proposition 1.1. (Cockayne et al. [2] 2004) *If $m_1 \leq m_2 \leq \cdots \leq m_n$ are positive integers and G is the complete n-partite graph $K_{m_1,\ldots,m_n}$, then*

$$\gamma_R(G) = \begin{cases} m_1 + 1 & \text{if } 1 \leq m_1 \leq 2, \\ 4 & \text{otherwise.} \end{cases}$$

Proposition 1.2. *Let $t \geq 1$ and $n \geq 3$ be integers. If G is the cartesian product graph $P_t \times K_n$, then*

$$\gamma_R(G) = \begin{cases} 6\lfloor t/4 \rfloor + 1 & \text{if } n = 3,\ t \equiv 0 \pmod 4, \\ 6\lfloor t/4 \rfloor + 2r & \text{if } n = 3,\ t \equiv 1, 2 \pmod 4, \\ 6\lfloor t/4 \rfloor + 5 & \text{if } n = 3,\ t \equiv 3 \pmod 4, \\ 2t & \text{otherwise.} \end{cases}$$

Proof. Let $V(P_t \times K_n) = \{1, 2, \ldots, t\} \times \{1, 2, \ldots, n\}$. First let $n = 3$. Let $A = \{(4\ell + 1, 1), (4\ell + 3, 2) \mid 0 \leq \ell \leq \lfloor t/4 \rfloor - 1\}$ and $B = \{(4\ell + 2, 3), (4\ell +$

$4,3) \mid 0 \le \ell \le \lfloor t/4 \rfloor - 1\}$. One can easily verify that the following Roman dominating functions have minimum weight.

Case i. $t \equiv 0 \pmod 4$.
Let $f_0 = (W_0, W_1, W_2)$, where $W_2 = A$, $W_1 = B \cup \{(t,1)\}$ and $W_0 = V - (W_1 \cup W_2)$.

Case ii. $t \equiv 1 \pmod 4$.
Let $f_1 = (W_0', W_1', W_2')$, where $W_2' = A \cup \{(t,1)\}$, $W_1' = B$ and $W_0' = V - (W_1' \cup W_2')$.

Case iii. $t \equiv 2 \pmod 4$.
Let $f_2 = (W_0'', W_1'', W_2'')$, where $W_2'' = A \cup \{(t-1,1),(t,1)\}$, $W_1'' = B$ and $W_0'' = V - (W_1'' \cup W_2'')$.

Case iv. $t \equiv 3 \pmod 4$.
Let $f_3 = (W_0''', W_1''', W_2''')$, where $W_2''' = A \cup \{(t-2,1),(t,2)\}$, $W_1'' = B \cup \{(t-1,3)\}$ and $W_0''' = V - (W_1''' \cup W_2''')$.
Now let $n \ge 4$. Easily it can be seen that the weight of every RDF for $P_t \times K_n$ on the every copy of K_n is at least 2. Thus $\gamma_R(P_t \times K_n) \ge 2t$. Now since $f = (W_0, \emptyset, W_2)$ is an RDF with weight $2t$, when $W_2 = \{(\ell,1) \mid 1 \le \ell \le t\}$ and $W_0 = V - W_2$, we get $\gamma_R(P_t \times K_n) = 2t$. □

Proposition 1.3. *Let $t \ge 1$ and $n \ge 3$ be integers. If G is the cartesian product graph $C_t \times K_n$, then*

$$\gamma_R(G) = \begin{cases} 6\lfloor t/4 \rfloor + 2r & \text{if } n = 3,\ t \equiv r \pmod 4 \text{ and } 0 \le r \le 1, \\ 6\lfloor t/4 \rfloor + 2r - 1 & \text{if } n = 3,\ t \equiv r \pmod 4 \text{ and } 2 \le r \le 3, \\ 2t & \text{otherwise.} \end{cases}$$

Proof. Let $V(C_t \times K_n) = \{1,2,...,t\} \times \{1,2,...,n\}$. First let $n = 3$. Let $A = \{(4\ell+1,1),(4\ell+3,2) \mid 0 \le \ell \le \lfloor t/4 \rfloor - 1\}$ and $B = \{(4\ell+2,3),(4\ell+4,3) \mid 0 \le \ell \le \lfloor t/4 \rfloor - 1\}$. One can easily verify that the following Roman dominating functions have minimum weight.

Case i. $t \equiv 0 \pmod 4$.
Let $f_0 = (W_0, W_1, W_2)$, where $W_2 = A$, $W_1 = B$ and $W_0 = V - (W_1 \cup W_2)$.

Case ii. $t \equiv 1 \pmod 4$.
Let $f_1 = (W_0', W_1', W_2')$, where $W_2' = A \cup \{(t,1)\}$, $W_1' = B$ and $W_0' = V - (W_1' \cup W_2')$.

Case iii. $t \equiv 2 \pmod 4$.
Let $f_2 = (W_0'', W_1'', W_2'')$, where $W_2'' = A \cup \{(t-1,3)\}$, $W_1'' = (B - \{(t-2,3)\}) \cup \{(t-2,1),(t,2)\}$ and $W_0'' = V - (W_1'' \cup W_2'')$.

Case iv. $t \equiv 3 \pmod 4$.
Let $f_3 = (W_0''', W_1''', W_2''')$, where $W_2''' = A \cup \{(t-2,1),(t,2)\}$, $W_1'' = B \cup \{(t-1,3)\}$ and $W_0''' = V - (W_1''' \cup W_2''')$.
Similar to the proof of Proposition 1.2, we can prove $\gamma_R(C_t \times K_n) = 2t$, when $n \geq 4$. □

The following two propositions can be similarly proved and one can easily verify that these graphs are special Roman graphs.

Proposition 1.4. *If $t \geq 2$ and $4 \leq n_1 \leq n_2 \leq \ldots \leq n_p$ are integers, then $\gamma_R(P_t \times K_{n_1,\ldots,n_p}) = 4t$.*

Proposition 1.5. *If $t \geq 1$ and $n \geq 2$ are integers, then $\gamma_R(P_t \times K_{1,n}) = 2t$.*

2 Main Results

First we state our main theorem.

Theorem 2.1. *For each graph G, $\gamma_R(G) + 1 \leq \gamma_R(\mu(G)) \leq \gamma_R(G) + 2$.*

Proof. Let $V(G) = V^0$, and $V(\mu(G)) = V^0 \cup V^1 \cup \{u\}$. Let $f = (W_0, W_1, W_2)$ be a $\gamma_R(G)$-function. Since $g = (W_0 \cup V^1, W_1, W_2 \cup \{u\})$ is an RDF for $\mu(G)$, we have $\gamma_R(\mu(G)) \leq \gamma_R(G) + 2$. We now show that $\gamma_R(\mu(G)) \geq \gamma_R(G) + 1$. Let $g = (W_0, W_1, W_2)$ be an $\gamma_R(\mu(G))$-function. We continue our discussion in the following two cases.

Case 1. $u \in W_1 \cup W_2$. Let

$$\begin{aligned} W_2' &= (W_2 - (\{u\} \cup (W_2 \cap V^1))) \cup \{v_j^0 \mid v_j^1 \in W_2\}, \\ W_1' &= W_1 - \{v_j^0 \mid v_j^1 \in W_2\}, \\ W_0' &= V(G) - (W_1' \cup W_2'). \end{aligned}$$

Then the function $g' = (W_0', W_1', W_2')$ is an RDF for G, and

$$\gamma_R(G) \leq g'(V(G)) = 2|W_2'| + |W_1'| \leq 2|W_2| + |W_1| - 1 \leq \gamma_R(\mu(G)) - 1.$$

Hence $\gamma_R(\mu(G)) \geq \gamma_R(G) + 1$, as desired.

Case 2. $u \in W_0$.

Then $W_2 \cap V^1 \neq \emptyset$. If $W_1 \cap V^1 \neq \emptyset$, then it can be easily verify that the inequality is true. Now let $W_1 \subseteq V^0$. Consider $A = \{v_j^0 | v_j^1 \in W_2\}$. If

$(W_2 \cup W_1) \cap A \neq \emptyset$, then similar to Case 1, we can find an RDF g' for G such that $\gamma_R(G) \leq g'(V(G)) \leq \gamma_R(\mu(G)) - 1$, and hence $\gamma_R(\mu(G)) \geq \gamma_R(G) + 1$. Therefore, we assume that $(W_2 \cup W_1) \cap A = \emptyset$ and $W_2 = \{v^0_{s_i} | 1 \leq i \leq t\} \cup \{v^1_{j_\ell} | 1 \leq \ell \leq m\}$ such that $(W_1 \cup \{v^0_{s_i} | 1 \leq i \leq t\}) \cap \{v^0_{j_\ell} | 1 \leq \ell \leq m\} = \emptyset$.

Suppose that $epn(v^1_{j_k}; W_2) \cap V^0 = \emptyset$, for some $1 \leq k \leq m$. Then k is unique and $epn(v^1_{j_k}; W_2) = \{u\}$. Let

$$\begin{aligned} W_2' &= \{v^0_{s_i} | 1 \leq i \leq t\} \cup \{v^0_{j_\ell} | 1 \leq \ell \leq m, \text{ and } \ell \neq k\}, \\ W_1' &= W_1, \\ W_0' &= V(G) - (W_1' \cup W_2'). \end{aligned}$$

Then the function $g' = (W_0', W_1', W_2')$ is an RDF for G such that $\gamma_R(G) \leq \gamma_R(\mu(G)) - 2$, which implies $\gamma_R(\mu(G)) \geq \gamma_R(G) + 1$, as desired. Hence we may assume that for each $1 \leq \ell \leq m$, $epn(v^1_{j_\ell}; W_2) \cap V^0 \neq \emptyset$.

Let $\alpha^0_{j_\ell} \in epn(v^1_{j_\ell}; W_2) \cap V^0$, for each $1 \leq \ell \leq m$. Clearly $\{\alpha^1_{j_\ell} | 1 \leq \ell \leq m\} \cap (W_0 \cup W_1) = \emptyset$. Hence $\{\alpha^1_{j_\ell} | 1 \leq \ell \leq m\} \subseteq W_2$. Furthermore $m \geq 2$ and $\{\alpha^1_{j_\ell} | 1 \leq \ell \leq m\} = \{v^1_{j_\ell} | 1 \leq \ell \leq m\}$. Also for each $1 \leq \ell \leq m$, $| epn(v^1_{j_\ell}; W_2) \cap V^0 | = 1$. Let $\alpha^1_{j_1} = v^1_{j_2}$. We now add $v^0_{j_2}$ and $v^1_{j_2}$ to W_2 and W_1, respectively, and delete $v^1_{j_1}$ and $v^1_{j_2}$ of W_2. If necessary, we also add u to W_1. Then we obtain $g' = (W_0', W_1', W_2')$ as a new RDF for $\mu(G)$. If $m \geq 3$, then $g'(u) = 0$. Hence $g'(V(\mu(G)) \leq g(V(\mu(G))) - 1 = \gamma_R(\mu(G)) - 1$, a contradiction. Finally let $m = 2$ and choose $W_2'' = W_2'$, $W_1'' = W_1' - \{u, v^1_{j_2}\}$ and $W_0'' = V(G) - (W_1'' \cup W_2'')$. Since the function $g'' = (W_0'', W_1'', W_2'')$ is an RDF for G with weight $\gamma_R(\mu(G)) - 2$, we have $\gamma_R(G) \leq g''(V(G)) \leq \gamma_R(\mu(G)) - 2$, as desired. □

Our next aim is to characterize for which graphs G the Roman domination number of $\mu(G)$ is $\gamma_R(G) + 1$ or $\gamma_R(G) + 2$.

Theorem 2.2. *For every special Roman graph G, $\gamma_R(\mu(G)) = \gamma_R(G) + 1$.*

Proof. Theorem 2.1 implies $\gamma_R(\mu(G)) \geq \gamma_R(G) + 1$. Let $f = (V_0, \emptyset, V_2)$ be a special γ_R-function for G. By choosing $W_2 = V_2$, $W_1 = \{u\}$, and $W_0 = V_0 \cup V^1$, the function $g = (W_0, W_1, W_2)$ is an RDF for $\mu(G)$ with weight $\gamma_R(G) + 1$, which implies $\gamma_R(\mu(G)) = \gamma_R(G) + 1$. □

In the next theorem we show that the converse of Theorem 2.2 is also true.

Theorem 2.3. *If G is not a Roman graph, then $\gamma_R(\mu(G)) = \gamma_R(G) + 2$.*

Proof. In the contrary, let $g = (W_0, W_1, W_2)$ be a $\gamma_R(\mu(G))$-function with weight $\gamma_R(G)+1$, by Theorem 1. We also assume that if $|W_1| \geq 1$, then the induced subgraph $\mu(G)[W_1]$ is isomorphic to the empty graph $\overline{K_b}$, where $b = |W_1|$. In the next three cases, we show that $u \notin W_0 \cup W_1 \cup W_2$, and this completes our proof.

Case 1. $u \in W_2$.

Then $W_1 \subseteq V^0$. Let

$$\begin{aligned} V_2 &= (W_2 - \{u\}) \cup \{v_j^0 | v_j^1 \in W_2\}, \\ V_1 &= W_1 - \{v_j^0 | v_j^1 \in W_2\}, \\ V_0 &= V^0 - (V_1 \cup V_2). \end{aligned}$$

Then the function $f = (V_0, V_1, V_2)$ is an RDF for G with at most weight $\gamma_R(G) - 1$, a contradiction.

Case 2. $u \in W_1$.

Then $W_2 \subseteq V^0$. Since the induced subgraph $\mu(G)[W_1]$ is an empty graph, we have $W_1 - \{u\} \subseteq V^0$. Also since $v_j^0 \in W_1$ implies $v_j^1 \in W_1$, we have $W_1 = \{u\}$. Let $V_2 = W_2$, $V_1 = \emptyset$ and $V_0 = V^0 - V_2$. Then the function $f = (V_0, \emptyset, V_2)$ is a γ_R-function for G. Hence G is a Roman graph, a contradiction.

Case 3. $u \in W_0$.

Then $|W_2 \cap V^1| \geq 1$. Let $v_1^1 \in W_2 \cap V^1$. We may also assume that $v_1^0 \in W_0 \cup W_1$. Because if $v_1^0 \in W_2$, then with considering

$$\begin{aligned} V_2 &= (W_2 \cap V^0) \cup \{v_j^0 | v_j^1 \in W_2\}, \\ V_1 &= (W_1 \cap V^0) - \{v_j^0 | v_j^1 \in W_2\}, \\ V_0 &= V^0 - (V_1 \cup V_2), \end{aligned}$$

the function $f = (V_0, V_1, V_2)$ is an RDF for G with at most weight $\gamma_R(\mu(G)) - 2 = \gamma_R(G) - 1$, a contradiction. We now continue our discussion on the following two subcases.

Subcase 3.i. $v_1^0 \in W_1$.

Let $A = \{v_j^1 | v_j^0 \in W_1\}$. Since $\{i | v_i^0 \in W_1 \text{ and } v_i^1 \in W_0\} = \emptyset$, and g is a $\gamma_R(\mu(G))$-function, we have $|A \cap W_2| \leq 1$. More exactly $A \cap W_2 = \{v_1^1\}$. Easily we see that if $v_i^1 \in W_1$, then $v_i^0 \in W_1 \cup W_2$. Let $t = |\{i \mid v_i^0, v_i^1 \in W_1\}|$ and let $\ell = |\{i \mid v_i^0 \in W_2 \text{ and } v_i^1 \in W_1\}|$. If $t+\ell \geq 1$, then we can get an RDF with weight $\gamma_R(G) - 1$ for G, a contradiction. Now let $\ell = t = 0$.

Thus $W_1 = \{v_1^0\}$ and $epn(v_1^1; W_2) = \{u\}$. Hence with considering

$$\begin{aligned} V_2 &= (W_2 \cap V^0) \cup \{v_j^0 | v_j^1 \in W_2\} - \{v_1^0\}, \\ V_1 &= W_1 = \{v_1^0\}, \\ V_0 &= V^0 - (V_1 \cup V_2) \end{aligned}$$

the function $f = (V_0, V_1, V_2)$ is an RDF for G such that $f(V(G)) \leq \gamma_R(\mu(G)) - 2 = \gamma_R(G) - 1$, a contradiction.

Subcase 3.ii. $v_1^0 \in W_0$.

We recall $v_1^1 \in W_2$ and $u \in W_0$. By Subcase 3.i and the above discussion we may assume that if $v_i^1 \in W_2$, then $v_i^0 \in W_0$. We also know that if $v_i^0 \in W_2$, then $v_i^1 \notin W_2$. The assumption $v_1^0 \in W_0$ concludes that v_1^0 is defended by a vertex α of W_2. Suppose $\alpha \in V^0$ and let $\alpha = v_2^0$. If $epn(v_1^1; W_2) \cap V^0 = \emptyset$, then with deleting at least v_1^1 from W_2, we can find a function $f = (V_0, V_1, V_2)$ with weight at most $\gamma_R(\mu(G)) - 2 = \gamma_R(G) - 1$ such that $V_1 \cup V_2$ defends all vertices of G. Thus let $epn(v_1^1; W_2) \cap V^0 \neq \emptyset$ and let $epn(v_1^1; W_2) \cap V^0 = \{v_i^0 | 3 \leq i \leq t\}$, for some $t \geq 3$. Hence $\{v_i^1 | 3 \leq i \leq t\} \subseteq W_1 \cup W_2$. If $g(\cup_{i=3}^t v_i^1) \geq 2$, then with improving g we can makes an RDF g' for $\mu(G)$ with at most weight $\gamma_R(\mu(G)) - 1$, a contradiction. Now let $epn(v_1^1; W_2) \cap V^0 = \{v_3^0\}$ and $g(v_3^1) = 1$. In this case, we may find an RDF g' for $\mu(G)$ such that $g'(V(\mu(G))) \leq \gamma_R(\mu(G)) - 1$ (a contradiction) or $g'(V(\mu(G))) = \gamma_R(\mu(G))$ and $g'(u) = 1$, that is impossible by Case 2. Finally we assume that $W_2 \cap V^0$ does not defend v_1^0 and let $\alpha = v_2^1$. Then $epn(v_2^1; W_2) \cap V^0 \neq \emptyset$. Also we have $epn(v_1^1; W_2) \cap V^0 \neq \emptyset$. If $epn(v_2^1; W_2) \cap V^0 = \{v_1^0\}$, then with choosing

$$\begin{aligned} V_2' &= (W_2 \cap V^0) \cup \{v_j^0 | v_j^1 \in W_2\} - \{v_2^0\}, \\ V_1' &= (W_1 \cap V^0) - \{v_j^0 \in W_1 \mid v_j^1 \in W_2\}, \\ V_0' &= V(G) - (V_1' \cup V_2'), \end{aligned}$$

the function $g' = (V_0', V_1', V_2')$ is an RDF for G such that $g'(V(G)) \leq g(V(G)) - 2 = \gamma_R(G) - 1$, a contradiction. Now let $| epn(v_2^1; W_2) \cap V^0 | \geq 2$. Hence $v_j^1 \in W_1 \cup W_2$ if $v_j^0 \in epn(v_2^1; W_2) \cap V^0$. Now with choosing

$$\begin{aligned} W_2' &= (W_2 - \{v_1^1, v_2^1\}) \cup \{v_1^0, v_2^0\}, \\ W_1' &= W_1 - \{v_j^1 \mid v_j^0 \in epn(v_2^1; W_2)\}, \\ W_0' &= V(\mu(G)) - (W_1' \cup W_2'), \end{aligned}$$

the function $g' = (W_0', W_1', W_2')$ is an RDF for $\mu(G)$ such that

$$\begin{aligned} g'(V(\mu(G))) &= 2 \mid W_2' \mid + \mid W_1' \mid \leq 2 \mid W_2 \mid + \mid W_1 \mid -1 = g(V(\mu(G)))-1 \\ &= \gamma_R(\mu(G))-1, \end{aligned}$$

a contradiction. □

Theorem 2.4. *Let G be a graph. If for every $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ the induced subgraph $G[V_2]$ has an isolated vertex, then $\gamma_R(\mu(G)) = \gamma_R(G) + 2$.*

Proof. In the contrary, let $g = (W_0, W_1, W_2)$ be a γ_R-function for $\mu(G)$ with weight $\gamma_R(G) + 1$, by Theorem 1. We assume that if $| W_1 | \geq 1$, then the induced subgraph $\mu(G)[W_1]$ is isomorphic to the empty graph $\overline{K_b}$, where $b = | W_1 |$. In the next three cases, we show that $u \notin W_0 \cup W_1 \cup W_2$, and this completes our proof.

Case 1. $u \in W_2$.

Then $W_1 \subseteq V^0$. Let

$$\begin{aligned} V_2 &= (W_2 - \{u\}) \cup \{v_j^0 | v_j^1 \in W_2\}, \\ V_1 &= W_1 - \{v_j^0 | v_j^1 \in W_2\}, \\ V_0 &= V^0 - (V_1 \cup V_2). \end{aligned}$$

Then the function $f = (V_0, V_1, V_2)$ is an RDF for G with at most weight $\gamma_R(G) - 1$, a contradiction.

Case 2. $u \in W_1$.

Then $W_2 \subseteq V^0$. Since the induced subgraph $\mu(G)[W_1]$ is an empty graph, we have $W_1 - \{u\} \subseteq V^0$. Since $v_j^0 \in W_1$ implies $v_j^1 \in W_1$ and the induced subgraph $\mu(G)[W_1]$ is an empty graph, we have $W_1 = \{u\}$. Then the function $f = (V(G) - W_2, \emptyset, W_2)$ is a γ_R-function for G. Then $\delta(G[W_2]) = 0$. Hence there is a vertex $v_i^0 \in W_2$ such that it is adjacent to no vertex of $W_2 - \{v_i^0\}$. Therefore v_i^1 is adjacent to no vertex of W_2. Hence $v_i^1 \in W_1$, a contradiction.

Case 3. $u \in W_0$.

The proof of this case is exactly the proof of Case 3 of Theorem 2.3.

□

The next two theorems are immediate results of Theorems 2.2 and 2.4.

Theorem 2.5. *Let G be any graph. Then $\gamma_R(\mu(G)) = \gamma_R(G) + 1$ if and only if G is a special Roman graph.*

Theorem 2.6. *Let G be any graph. Then $\gamma_R(\mu(G)) = \gamma_R(G) + 2$ if and only if G is not a special Roman graph.*

There are many graphs that are not special Roman graph. For example, the complete graphs K_n, paths P_n, stars $K_{1,n}$, all for $n \geq 1$, cycles C_n for $n \geq 3$, and complete multipartite graphs $K_{2,m_2,\dots,m_n}$ for $2 \leq m_2 \leq \cdots \leq m_n$ are not special Roman graphs and their Roman domination number are 2, $\lceil 2n/3 \rceil$, 2, $\lceil 2n/3 \rceil$, and 3, respectively. The next proposition gives another non special Roman graph.

Proposition 2.7. *The Petersen graph $G(5)$ is not a special Roman graph, and $\gamma_R(G(5)) = 6$.*

Proof. Let $V(G(5)) = \{i | 1 \leq i \leq 10\}$ and

$$E(G(5)) = \{(i, i+1) | 1 \leq i \leq 9\} \cup \{(6,10), (1,5), (1,9), (2,7), (3,10), (4,8)\}.$$

Since $G(5)$ is 3-regular and $G(5)$ has 10 vertices, we have $\gamma_R(G(5)) \geq 6$. Let $V_2 = \{1, 8, 10\}$, $V_1 = \emptyset$, and $V_0 = V - V_2$. Since the function $f = (V_0, \emptyset, V_2)$ is an RDF with weight 6, we get $\gamma_R(G(5)) = 6$.

Finally we prove $G(5)$ is not a special Roman graph. By the given RDF in the previous paragraph, $G(5)$ is a Roman graph. Now let $f = (V_0, \emptyset, V_2)$ be an arbitrary γ_R-function for $G(5)$. We know $G(5)$ is a non-complete 3-partite graph with three parts $X = \{1, 4, 7, 10\}$, $Y = \{3, 6, 8\}$, and $Z = \{2, 5, 9\}$. Assume that a and b are two adjacent vertices of V_2. Since $| N(a) \cup N(b) | = 6$, and there is no other vertex c that dominates all of the four remained vertices, we get $f(V) \geq 7$, a contradiction. Hence $G(5)$ is not a special Roman graph. □

Corollary 2.8. *If $G \in \{K_{2,m_2,\dots,m_n} \mid 2 \leq m_2 \leq m_3 \leq \cdots \leq m_n\} \cup \{G(5)\} \cup \{C_n \mid n \geq 3\} \cup \{K_n, P_n, K_{1,n} \mid n \geq 1\}$, then $\gamma_R(\mu(G)) = \gamma_R(G) + 2$.*

3 Acknowledgements

The author thanks the referee for his/her helpful suggestions.

References

[1] R. Balakrishnan, S. Francis Raj, *Connectivity of the Mycielskian of a graph*, Discrete Mathematics 308 (2008) 2607-2610 .

[2] E. J. Cockayne, P. A. Dreyer Jr., S. M. Hedetniemi, S. T. Hedetniemi, *Roman domination in graphs*, Discrete Mathematics 278 (2004) 11-22.

[3] O. Favaron, H. Karami, R. Khoeilar, S. M. Sheikholeslami, *On the Roman domination number of a graph*, Discrete Mathematics 309 (2009) 3447-3451.

[4] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.

[5] M. A. Henning, *A Characterization of Roman trees*, Discussiones Mathematicae Graph Theory 22 (2002) 325-334.

[6] M. Liedloff, T. Kloks, J. Liu, S. L. Peng, *Efficient algorithms for Roman domination on some classes of graphs*, Discrete Appllied Mathematics 156 (2008) 3400-3415.

[7] W. Lin, J. Wu, P. C. B. Lam, G. Gu, *Several parameters of generalized Mycielskian's*, Discrete Appllied Mathematics 154 (2006) 1173-1182.

[8] C. S. ReVelle, K. E. Rosing, *Defendens imperium Romanum: a classical problem in military strategy*, Amer. Math. Monthly 107 (7) (2000) 585-594.

[9] I. Stewart, *Defend the Roman Empire!*, Sci. Amer. 281 (6) (1999) 136-139.