Bijective proofs of Gould-Mohanty's and Raney-Mohanty's identities

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Abstract. Using the model of words, we give bijective proofs of Gould-Mohanty's and Raney-Mohanty's identities, which are respectively multivariable generalizations of Gould's identity

$$\sum_{k=0}^{n}\binom{x-kz}{k}\binom{y+kz}{n-k}=\sum_{k=0}^{n}\binom{x+\epsilon-kz}{k}\binom{y-\epsilon+kz}{n-k}$$

and Rothe's identity

$$\sum_{k=0}^{n}\frac{x}{x-kz}\binom{x-kz}{k}\binom{y+kz}{n-k}=\binom{x+y}{n}.$$

1. Introduction

A famous generalization of the binomial theorem is Abel's identity [1]:

$$\sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}=(x+y)^{n}, \tag{1}$$

which also has a company identity as follows:

$$\sum_{k=0}^{n}\binom{n}{k}xy(x-kz)^{k-1}(y+kz)^{n-k-1}=(x+y+nz)(x+y)^{n-1}. \tag{2}$$

It is not difficult to see that (1) and (2) are respectively limiting cases of the following convolution formulas due to Rothe [17]:

$$\sum_{k=0}^{n}\frac{x}{x-kz}\binom{x-kz}{k}\binom{y+kz}{n-k}=\binom{x+y}{n}, \tag{3}$$

$$\sum_{k=0}^{n}\frac{xy}{(x-kz)(y-(n-k)z)}\binom{x-kz}{k}\binom{y-(n-k)z}{n-k}$$
$$=\frac{x+y}{x+y-nz}\binom{x+y-nz}{n}. \tag{4}$$

Gould [5,6] reproved (3) and (4) and also obtained the following identity

$$\sum_{k=0}^{n}\binom{x-kz}{k}\binom{y+kz}{n-k}=\sum_{k=0}^{n}\binom{x+\epsilon-kz}{k}\binom{y-\epsilon+kz}{n-k}. \qquad (5)$$

Another proof of (3) and (4) was given by Sprugnoli [19]. It is not difficult to see that (4) can be deduced from (3). Blackwell and Dubins [2] gave a combinatorial proof of Rothe's identity (4), which can also be proved in the model of lattice paths (using [13, p. 9] or [10, (1.1)]). Recently, the author [8] gives simple bijective proofs of Gould's identity (5) and Rothe's identity (3) in the model of binary words.

Hurwitz [9] established a multivariable generalization of Abel's identities (1) and (2) (see also [20]). For a curious q-analogue of Rothe's identity (3), we refer the reader to [18] and references therein.

In order to state a multivariable generalization of Rothe's identities in the literature, we need first to introduce some notation. Let m be a fixed natural number throughout the paper. For $\mathbf{a}=(a_1,\ldots,a_m)\in\mathbb{N}^m$ and $\mathbf{b}=(b_1,\ldots,b_m)\in\mathbb{C}^m$, set $|\mathbf{a}|=a_1+\cdots+a_m$, $\mathbf{a}!=a_1!\cdots a_m!$, $\mathbf{a}+\mathbf{b}=(a_1+b_1,\ldots,a_m+b_m)$, $\mathbf{a}\cdot\mathbf{b}=a_1b_1+\cdots+a_mb_m$, and $\mathbf{b}^{\mathbf{a}}=b_1^{a_1}\cdots b_m^{a_m}$. For any complex parameter x and $\mathbf{n}=(n_1,\ldots,n_m)\in\mathbb{Z}^m$, we define the *multinomial coefficient* $\binom{x}{\mathbf{n}}$ by

$$\binom{x}{\mathbf{n}}=\begin{cases}x(x-1)\cdots(x-|\mathbf{n}|+1)/\mathbf{n}!, & \text{if } \mathbf{n}=(n_1,\ldots,n_m)\in\mathbb{N}^m,\\ 0, & \text{otherwise.}\end{cases}$$

Using generating functions, Mohanty [12] proved the following multivariable generalization of Rothe's identities (3) and (4):

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\frac{x}{x-\mathbf{k}\cdot\mathbf{z}}\binom{x-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}=\binom{x+y}{\mathbf{n}}, \qquad (6)$$

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}}\frac{xy}{(x-\mathbf{k}\cdot\mathbf{z})(y-(\mathbf{n}-\mathbf{k})\cdot\mathbf{z})}\binom{x-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y-(\mathbf{n}-\mathbf{k})\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}$$
$$=\frac{x+y}{x+y-\mathbf{n}\cdot\mathbf{z}}\binom{x+y-\mathbf{n}\cdot\mathbf{z}}{\mathbf{n}}. \qquad (7)$$

However, an important special case of (7) (where $z_i=i$) was already contained in the earlier work of Raney [16] on a combinatorial approach to the Lagrange inversion. Hence we would call both (6) and (7) *Raney-Mohanty's identities.* Unaware of Mohanty's work, in 1988 Louck [11] proposed a "conjecture" equivalent to (7), which caught the interests of three different people independently and was solved by them by three different methods:

Paule [15] proved (7) by the Lagrange inversion approach, Strehl [20] gave a completely combinatorial approach, while Zeng [21] used mathematical induction.

Moreover, Mohanty and Handa [14] established the following identity

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+y-|\mathbf{k}|}{\mathbf{n}-\mathbf{k}}\binom{|\mathbf{k}|}{\mathbf{k}}\mathbf{z}^{\mathbf{k}}, \tag{8}$$

which is a multivariable generalization of Jensen's identity [7]:

$$\sum_{k=0}^{n} \binom{x+kz}{k}\binom{y-kz}{n-k} = \sum_{k=0}^{n} \binom{x+y-k}{n-k} z^k.$$

It follows immediately from Mohanty-Handa's identity (8) that

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+\epsilon-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{y-\epsilon+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}. \tag{9}$$

Since (9) is obviously a multivariable generalization of Gould's identity (5) and it also follows from one of the generating functions established by Mohanty in [12], we call (9) *Gould-Mohanty's identity*.

To the knowledge of the author, there are no combinatorial proofs of Mohanty-Handa's identity (8) and Gould-Mohanty's identity (9). In this paper, continuing the work of [8], we shall give bijective proofs of Gould-Mohanty's identity and Raney-Mohanty's identity (6) in the model of words.

2. Proof of Gould-Mohanty's identity

It suffices to prove Gould-Mohanty's identity (9) for the special case:

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{p-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{q+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{p+1-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}\binom{q-1+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}, \tag{10}$$

where $p, q \in \mathbb{N}$ and $\mathbf{n}, \mathbf{z} \in \mathbb{N}^m$. Furthermore, we need only to prove that (10) holds for all integers $p \geq \mathbf{n}\cdot\mathbf{z}$ and $q \geq 1$. In this case, each multinomial coefficient in (10) is nonnegative and therefore has a combinatorial interpretation.

Let $\Gamma = \{a, b_1, \ldots, b_m\}$ denote an alphabet with a grading $||a|| = 1$ and $||b_i|| = z_i + 1$ $(1 \leq i \leq m)$. For a word $w = w_1 \cdots w_n \in \Gamma^*$, its *length* n is denoted by $|w|$ and its *weight* by $||w|| = ||w_1|| + \cdots + ||w_n||$, and we call the word $w_n w_{n-1} \cdots w_1$ the *reverse* of w. Let $|w|_{b_i}$ be the number of b_i's appearing in w, and let

$$\Gamma_{p,\mathbf{k}} := \{w \in \Gamma^* : ||w|| = p \text{ and } |w|_{b_i} = k_i,\ i = 1, \ldots, m\},$$

where $\mathbf{k} = (k_1, \ldots, k_m)$. It is easy to see that $\Gamma_{p,\mathbf{k}} \subseteq \Gamma^{p-\mathbf{k}\cdot\mathbf{z}}$ and

$$\#\Gamma_{p,\mathbf{k}} = \binom{p - \mathbf{k}\cdot\mathbf{z}}{\mathbf{k}}, \tag{11}$$

where $\mathbf{z} = (z_1, \ldots, z_m)$.

Furthermore, let

$$\Gamma_{p,\mathbf{k}}^{(r)} := \{w \in \Gamma_{p,\mathbf{k}} : w \text{ has a prefix of weight } r\}.$$

For $p, q \geq \mathbf{n}\cdot\mathbf{z}$, an obvious bijection

$$\Gamma_{p+q,\mathbf{n}}^{(p)} \longleftrightarrow \biguplus_{\mathbf{k}} \Gamma_{p,\mathbf{k}} \times \Gamma_{q,\mathbf{n}-\mathbf{k}}$$

leads to

$$\#\Gamma_{p+q,\mathbf{n}}^{(p)} = \sum_{\mathbf{k}} \binom{p - \mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{q - (\mathbf{n}-\mathbf{k})\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}. \tag{12}$$

Thus, the identity (10) is equivalent to

$$\#\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p)} = \#\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p+1)}. \tag{13}$$

We need the following simple fact.

Lemma 1. *Let $u, v \in \Gamma^*$ with $||u||, ||v|| \geq \mathbf{n}\cdot\mathbf{z} + 1$, where $n_i = |u \cdot v|_{b_i}$ $(1 \leq i \leq m)$. Then there exist nonempty prefixes x of u and y of v such that $||x|| = ||y||$.*

Proof. Since the proof is easy and very similar to the proof of [8, Lemma 1], we omit it here. □

Now we can prove (13) by the following theorem.

Theorem 2. *For all $p \geq \mathbf{n}\cdot\mathbf{z}$ and $q \geq 1$, there is a bijection between $\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p)}$ and $\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p+1)}$.*

Proof. Suppose that $w = u \cdot v \in \Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p)}$, where $||u|| = p$ and $||v|| = q + \mathbf{n}\cdot\mathbf{z}$. Applying Lemma 1 to v and the reverse of $u \cdot a$, one sees that u has a suffix x (perhaps empty), i.e., $u = u' \cdot x$, and v has a prefix y, i.e., $v = y \cdot v'$, such that $||x|| = ||y|| - 1$. Choosing such x and y with minimal length, then $w' = u' \cdot \overline{y} \cdot \overline{x} \cdot v' \in \Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p+1)}$ and $w \mapsto w'$ is a bijection. Here $\overline{x}$ and $\overline{y}$ are respectively the reverses of x and y. □

In the same manner, we may also give a direct bijection from $\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p)}$ to $\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}^{(p+r)}$ for all $p \geq \mathbf{n}\cdot\mathbf{z}$ and $q \geq r \geq 1$.

3. Proof of Raney-Mohanty's identity

We again assume that $p \geq \mathbf{n} \cdot \mathbf{z}$ and $q \geq 1$. Moreover, let $z_i \geq 1$ for all i. For each $w \in \Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}$, let $w = u \cdot v$ denote the unique factorization with $||u|| \geq p$ but as small as possible. Then we have the following possibilities:

- If $||u|| = p$, then $w \in \Gamma^{(p)}_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}}$ and all these words have been counted in Section 2.
- If $||u|| = p + j$ for some $1 \leq j \leq \max\{z_1, \ldots, z_m\}$, then the last letter of u must a b_i for some $1 \leq i \leq m$. Namely, $u = u' \cdot b_i$ for some $u' \in \Gamma_{p+j-z_i-1,\mathbf{k}-\mathbf{e}_i}$, where $\mathbf{e}_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}^m$ with the 1 being in the i-th position. The corresponding v belongs to $\Gamma_{q+\mathbf{n}\cdot\mathbf{z}-j,\mathbf{n}-\mathbf{k}}$. It is clear that the mapping $w \mapsto (u', v)$ may be inverted.

Hence there is a bijection

$$\Gamma_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}} \longleftrightarrow \Gamma^{(p)}_{p+q+\mathbf{n}\cdot\mathbf{z},\mathbf{n}} \biguplus_{i=1}^{m} \biguplus_{j=1}^{z_i} \biguplus_{\mathbf{k}=0}^{\mathbf{n}} \Gamma_{p+j-z_i-1,\mathbf{k}-\mathbf{e}_i} \times \Gamma_{q+\mathbf{n}\cdot\mathbf{z}-j,\mathbf{n}-\mathbf{k}},$$

which, together with (11) and (12), gives the identity

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \left(\binom{p-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{q+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} + \sum_{i=1}^{m} \sum_{j=1}^{z_i} \binom{p-\mathbf{k}\cdot\mathbf{z}+j-1}{\mathbf{k}-\mathbf{e}_i} \binom{q+\mathbf{k}\cdot\mathbf{z}-j}{\mathbf{n}-\mathbf{k}} \right) = \binom{p+q}{\mathbf{n}}. \tag{14}$$

However, by (9), for all $1 \leq i \leq m$ and $1 \leq j \leq z_i$, we have

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p-\mathbf{k}\cdot\mathbf{z}+j-1}{\mathbf{k}-\mathbf{e}_i} \binom{q+\mathbf{k}\cdot\mathbf{z}-j}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=0}^{\mathbf{n}} \binom{p-\mathbf{k}\cdot\mathbf{z}-1}{\mathbf{k}-\mathbf{e}_i} \binom{q+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}. \tag{15}$$

Substituting (15) into (14), we obtain

$$\sum_{\mathbf{k}=0}^{\mathbf{n}} \left(\binom{p-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} + \sum_{i=1}^{m} z_i \binom{p-\mathbf{k}\cdot\mathbf{z}-1}{\mathbf{k}-\mathbf{e}_i} \right) \binom{q+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \binom{p+q}{\mathbf{n}}. \tag{16}$$

Noticing that

$$\binom{p-\mathbf{k}\cdot\mathbf{z}-1}{\mathbf{k}-\mathbf{e}_i} = \frac{k_i}{p-\mathbf{k}\cdot\mathbf{z}} \binom{p-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}},$$

the identity (16) may be simplified as

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{p}{p-\mathbf{k}\cdot\mathbf{z}} \binom{p-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{q+\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \binom{p+q}{\mathbf{n}},$$

which is Raney-Mohanty's identity (6).

For the $m = 1$ case, the above bijection also leads to a double sum extension of the q-Chu-Vandermonde formula (see [8]). It is also possible to give a similar q-analogue of (14). However we omit it here and leave it to the interested reader.

4. Some remarks

We point out that (7) is a consequence of (6), since the left-hand side of the former may be written as

$$\frac{1}{x+y-\mathbf{n}\cdot\mathbf{z}} \left(\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{xy}{x-\mathbf{k}\cdot\mathbf{z}} \binom{x-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-(\mathbf{n}-\mathbf{k})\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} \right.$$
$$\left. + \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{xy}{y-(\mathbf{n}-\mathbf{k})\cdot\mathbf{z}} \binom{x-\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-(\mathbf{n}-\mathbf{k})\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} \right).$$

It is also worth mentioning that Mohanty-Handa's identity (8) can be deduced from Raney-Mohanty's identity (6). Indeed, note that

$$\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{x}{x+\mathbf{k}\cdot\mathbf{z}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}$$
$$+ \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} \frac{\mathbf{k}\cdot\mathbf{z}}{x+\mathbf{k}\cdot\mathbf{z}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} \binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}$$
$$= \binom{x+y}{\mathbf{n}} + \sum_{i=1}^{m} \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{n}} z_i \binom{x-1+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}-\mathbf{e}_i} \binom{y-\mathbf{k}\cdot\mathbf{z}}{\mathbf{n}-\mathbf{k}}.$$

Then (8) follows from (6) by induction on $|\mathbf{n}|$. However, I am unable to give a combinatorial proof of Mohanty-Handa's identity.

Finally, we remark that a further generalization of (8) was given by Chu [3] by using the following generating functions due to Mohanty [12]:

$$\sum_{\mathbf{k}\geq\mathbf{0}} \frac{x}{x+\mathbf{k}\cdot\mathbf{z}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} u_1^{k_1} \cdots u_m^{k_m} = v^x,$$
$$\sum_{\mathbf{k}\geq\mathbf{0}} \binom{x+\mathbf{k}\cdot\mathbf{z}}{\mathbf{k}} u_1^{k_1} \cdots u_m^{k_m} = \frac{v^x}{1-\sum_{i=1}^{m} u_i z_i v^{z_i-1}},$$

where v satisfies the functional equation $\sum_{i=1}^{m} u_i v^{z_i} = v - 1$.

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