On connection between α-labelings and edge-antimagic labelings of disconnected graphs

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Abstract

A *labeling* of a graph is any map that carries some set of graph elements to numbers (usually to the positive integers). An (a, d)-*edge-antimagic total labeling* on a graph with p vertices and q edges is defined as a one-to-one map taking the vertices and edges onto the integers $1, 2, \ldots, p+q$ with the property that the sums of the labels on the edges and the labels of their endpoints form an arithmetic sequence starting from a and having a common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices.

We use the connection between α-labelings and edge-antimagic labelings for determining a super (a, d)-edge-antimagic total labelings of disconnected graphs.

Keywords: (a, d)-edge-antimagic total labeling, super (a, d)-edge-antimagic total labeling, (a, d)-edge-antimagic vertex labeling, α-labeling

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1 Introduction

For a (p,q) graph G, a bijective function $f : V(G) \to \{1,2,\ldots,p\}$ is a *vertex labeling* of G and the associated edge-weight $w_f(uv)$ of an edge $uv \in E(G)$ is $w_f(uv) = f(u) + f(v)$. A bijective function $g : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ is a *total labeling* of G and the associated edge-weight $w_g(uv) = g(u) + g(uv) + g(v)$ for $uv \in E(G)$.

A vertex labeling f of G is *(a,d)-edge-antimagic vertex* if the set of all the edge-weights is $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$. We use the notation (a,d)-EAV to refer to these labelings. An *(a,d)-edge-antimagic total labeling* of G is the total labeling with the property that the edge-weights form an arithmetic sequence starting from a and having common difference d, where $a > 0$ and $d \geq 0$ are two fixed integers. For this labeling we use the notation (a,d)-EAT. Definition of (a,d)-EAT labeling was introduced by Simanjuntak *et al.* in [14] as a natural extension of a notion of *"magic valuation"* ($(a,0)$-EAT labeling) defined by Kotzig and Rosa in [10]. Kotzig and Rosa [10] showed that all caterpillars have "magic valuations" and conjectured that all trees have "magic valuations".

An (a,d)-EAT labeling is called *super* if the smallest possible labels appear on the vertices. A super (a,d)-EAT labeling is a natural extension of a notion of *"super edge-magic labeling"* defined by Enomoto *et al.* in [5]. For more information about "magic valuations" and "super edge-magic labelings" the reader is referred to [8] and [18].

A graph that admits an (a,d)-EAV labeling or a super (a,d)-EAT labeling is called an (a,d)-EAV graph or super (a,d)-EAT graph, respectively.

A *graceful labeling* of a (p,q) graph G is an injection $h : V(G) \to \{1,2,\ldots, q+1\}$ such that, when each edge uv is assigned the label $|h(u) - h(v)|$, the resulting edge labels (or weights) are distinct. A graph that admits a graceful labeling is said to be *graceful*. When the graceful labeling h has the property that there exists an integer λ such that for each edge uv either $h(u) \leq \lambda < h(v)$ or $h(v) \leq \lambda < h(u)$, h is called an *α-labeling*. The number λ is called the *boundary value* of h. A graph with an α-labeling is necessarily bipartite and the boundary value must be the smaller of the two vertex labels that yield the edge label 1. A graph that admits an α-labeling is called an α-graph. Graceful labelings and α-labelings are probably the most popular kind among the several classes of the graph labelings. They were introduced by Rosa in [12]. The Ringel-Kotzig conjecture that all trees are graceful is a very popular open problem. Some methods for constructing the graceful labelings and α-labelings for certain families of trees can be found in [1, 4, 13, 15].

We will use the connection between α-labelings and (a,d)-EAV labelings for determining super (a,d)-EAT labelings of disconnected graphs.

There are known certain results for the super edge-antimagicness of forests. Namely, Ivančo and Lučkaničová [9] described some constructions of super edge-magic (super $(a,0)$-edge-antimagic total) labelings for $K_{1,m} \cup K_{1,n}$. The super (a,d)-EAT labelings for $P_n \cup P_{n+1}$, $nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been described by Sudarsana *et al.* in [16], and $(a,0)$-EAT labelings for nP_3 can be found in [3].

2 Arithmetic sequences

This section contains the tools that allow us to determine the type of a sequence after combining two different sequences. It will be useful later.

Lemma 1. *Let $\mathcal{M}$ be an arithmetic sequence $\mathcal{M} = \{a + d(i-1) : 1 \le i \le k+1\}$, for the positive integers a, d and k, k even. Then there exists a permutation $\mathcal{P}(\mathcal{M})$ of the elements of $\mathcal{M}$ such that $\mathcal{M} + \mathcal{P}(\mathcal{M})$ is an arithmetic sequence with first term $2a + \frac{kd}{2}$ and a common difference d.*

Proof. Suppose that $\mathcal{M} = \{p_i : p_i = a + d(i-1), 1 \le i \le k+1\}$ for k even and $a, d > 0$. Consider the following permutation $\mathcal{P}(\mathcal{M}) = \{q_i : 1 \le i \le k+1\}$ of the elements of $\mathcal{M}$ where

$$q_i = \begin{cases} a + \frac{(k-i+1)d}{2} & \text{if } i \text{ is odd, } 1 \le i \le k+1 \\ a + \frac{(2k-i+2)d}{2} & \text{if } i \text{ is even, } 2 \le i \le k. \end{cases}$$

We claim that $\mathcal{M} + \mathcal{P}(\mathcal{M})$ is an arithmetic sequence. In fact,

$$p_i + q_i = \begin{cases} 2a + \frac{(k+i-1)d}{2} & \text{if } i \text{ is odd, } 1 \le i \le k+1 \\ 2a + \frac{(2k+i)d}{2} & \text{if } i \text{ is even, } 2 \le i \le k. \end{cases}$$

Thus, $\mathcal{M} + \mathcal{P}(\mathcal{M})$ is the arithmetic sequence with first term $2a + \frac{kd}{2}$ and common difference d. □

Lemma 2. *Let $\mathcal{N}$ be a sequence $\mathcal{N} = \{c + d(i-1) : 1 \le i \le \frac{k+1}{2}\} \cup \{c + di : \frac{k+3}{2} \le i \le k+1\}$, for positive integers c, d and k, k odd. Then there exists a permutation of the elements of an arithmetic sequence $\mathcal{S} = \{r + d(i-1) : 1 \le i \le k+1\}$ such that $\mathcal{N} + \mathcal{P}(\mathcal{S})$ is an arithmetic sequence with first term $c + r + \frac{(k+1)d}{2}$ and common difference d.*

Proof. Let $\mathcal{N} = \{n_i : n_i = c + d(i-1), 1 \le i \le \frac{k+1}{2}\} \cup \{n_i : n_i = c + di, \frac{k+3}{2} \le i \le k+1\}$ be a sequence for k odd and $c, d > 0$. Let $\mathcal{S} = \{r + d(i-1) : 1 \le i \le k+1\}$ be an arithmetic sequence. There are three cases to describe a requested permutation $\mathcal{P}(\mathcal{S}) = \{h_i : 1 \le i \le k+1\}$.

Case 1. For $k \equiv 5 \pmod 6$, where $k \ge 11$, we define

$$h_i = \begin{cases} r + (k-1)d & \text{if } i = 1 \\ r + (k-3)d & \text{if } i = 2 \\ r + (k-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } 3 \le i < \frac{k-1}{2} \\ r + (k-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } 4 \le i < \frac{k-1}{2} \\ r + (k+3-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } 5 \le i \le \frac{k-1}{2} \\ r + kd & \text{if } i = \frac{k+1}{2} \\ r + (k-4)d & \text{if } i = \frac{k+3}{2} \\ r + (k-2)d & \text{if } i = \frac{k+5}{2} \\ r + (k-5)d & \text{if } i = \frac{k+7}{2} \\ r + (2k-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } \frac{k+9}{2} \le i \le k-1 \\ r + (2k-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } \frac{k+11}{2} \le i \le k \\ r + (2k+3-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } \frac{k+13}{2} \le i \le k+1. \end{cases}$$

For $k = 5$ the permutation is

$$h_i = \begin{cases} r + 4d & \text{if } i = 1 \\ r + 2d & \text{if } i = 2 \\ r + 5d & \text{if } i = 3 \\ r + d & \text{if } i = 4 \\ r + 3d & \text{if } i = 5 \\ r & \text{if } i = 6. \end{cases}$$

Case 2. For $k \equiv 1 \pmod 6$, where $k \ge 7$, we construct

$$h_i = \begin{cases} r + (k-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } 1 \le i < \frac{k-1}{2} \\ r + (k-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } 2 \le i < \frac{k-1}{2} \\ r + (k+3-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } 3 \le i \le \frac{k-1}{2} \\ r + kd & \text{if } i = \frac{k+1}{2} \\ r + (k-1)d & \text{if } i = \frac{k+3}{2} \\ r + (2k-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } \frac{k+5}{2} \le i \le k-1 \\ r + (2k-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } \frac{k+7}{2} \le i \le k \\ r + (2k+3-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } \frac{k+9}{2} \le i \le k+1, \end{cases}$$

and for $k = 1$

$$h_i = \begin{cases} r + d & \text{if } i = 1 \\ r & \text{if } i = 2. \end{cases}$$

Case 3. For $k \equiv 3 \pmod 6$, where $k \geq 9$, we define

$$h_i = \begin{cases} r+(k-1)d & \text{if } i=1 \\ r+(k-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } 2 \leq i < \frac{k-1}{2} \\ r+(k-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } 3 \leq i < \frac{k-1}{2} \\ r+(k+3-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } 4 \leq i \leq \frac{k-1}{2} \\ r+kd & \text{if } i = \frac{k+1}{2} \\ r+(2k-2i)d & \text{if } i \equiv 0 \pmod 3 \text{ and } \frac{k+3}{2} \leq i \leq k \\ r+(2k+3-2i)d & \text{if } i \equiv 1 \pmod 3 \text{ and } \frac{k+5}{2} \leq i \leq k+1 \\ r+(2k-2i)d & \text{if } i \equiv 2 \pmod 3 \text{ and } \frac{k+7}{2} \leq i \leq k-1. \end{cases}$$

For $k=3$ we define the permutation in the following way

$$h_i = \begin{cases} r+2d & \text{if } i=1 \\ r+3d & \text{if } i=2 \\ r & \text{if } i=3 \\ r+d & \text{if } i=4. \end{cases}$$

There is no problem in seeing that, in all the consider cases, each integer h_i, $1 \leq i \leq k+1$, belongs to S and $\{n_i + h_i : 1 \leq i \leq k+1\} = \{c+r+\frac{(k+1)d}{2}, c+r+\frac{(k+3)d}{2}, c+r+\frac{(k+5)d}{2}, \ldots, c+r+\frac{(3k-1)d}{2}, c+r+\frac{(3k+1)d}{2}\}$. This produces the desired result. □

3 Disjoint union of graphs

Let G be a graph of order n and size $n-1$. We denote by mG a disjoint union of m copies of G. Our main goal in this section is to show that if G admits an α-labeling then mG admits a super (a,d)-EAT labeling.

We start by basic counting to determine an upper bound of difference d for a super (a,d)-EAT labeling. Let (p,q) graph be a super (a,d)-EAT. It is easy to see that the minimum possible edge-weight is at least $p+4$ and the maximum possible edge-weight is no more than $3p+q-1$. Thus $a+(q-1)d \leq 3p+q-1$ and $d \leq \frac{2p+q-5}{q-1}$. For $p = mn$, $q = m(n-1)$ and $m \geq 1$, $n \geq 3$, we have that $d < 4$.

Next lemma presents a connection between α-labeling and $(a,1)$-EAV labeling.

Lemma 3. *Let G be a graph of order n and size $n-1$, $n \geq 3$. If G admits an α-labeling, and m is odd, $m \geq 1$, then mG admits an $(a,1)$-EAV labeling.*

Proof. Suppose that G is an α-graph. It is known (see [11] or [2]) that if graph G of order n and size $n-1$ admits an α-labeling, then G also admits an $(a,1)$-EAV labeling. Hence, for $m=1$ we have the desired result.

Figueroa-Centeno *et al.* [6] showed that a (p,q) graph H is super edge-magic if and only if there exists a bijective function $f: V(H) \to \{1,2,\ldots,p\}$ such that the set $\{f(u)+f(v) : uv \in E(H)\}$ consists of q consecutive integers. In our terminology it means that a (p,q) graph H is super $(b,0)$-EAT if and only if there exists its $(b-p-q,1)$-EAV labeling. With respect to the previous result it follows that if a graph G of order n and size $n-1$ admits an α-labeling then G also admits a super edge-magic labeling.

It was proved by Figueroa-Centeno *et al.* (see [7], Theorem 2.1) that if H is a super edge-magic bipartite or tripartite graph, and m is odd, then mH is super edge-magic. Evidently, if G admits an α-labeling, and m is odd, then mG admits an $(a,1)$-EAV labeling. □

Lemma 4. *Let G be a graph of order n and size $n-1$, $n \geq 3$. If G admits an α-labeling, and m is odd, $m \geq 1$, then mG admits a super $(a+2mn-m,0)$-EAT and a super $(a+mn+1,2)$-EAT labeling.*

Proof. In light of Lemma 3 we propose that f is an $(a,1)$-EAV labeling of mG, where the set of the edge-weights gives the sequence $\{a, a+1, a+2,\ldots,a+mn-m-1\}$.

Case 1. The difference is $d=0$.

We extend the vertex labeling f into a labeling g such that

$g(u) = f(u)$ for every vertex $u \in V(mG)$

$g(uv) = 2mn - m + a - (g(u)+g(v))$ for every edge $uv \in E(mG)$.

Since $a \leq g(u)+g(v) \leq a+mn-m-1$, we have that $mn+1 \leq g(uv) \leq 2mn-m$ and thus g is a total labeling. Every edge $uv \in E(mG)$ has edge-weight $g(u)+g(uv)+g(v) = a+2mn-m$. This implies that mG is super $(a+2mn-m,0)$-EAT.

Case 2. The difference is $d=2$.

We consider a labeling h defined in the following way

$h(u) = f(u)$ for every vertex $u \in V(mG)$

$h(uv) = mn+1-a+(h(u)+h(v))$ for every edge $uv \in E(mG)$.

Evidently, h is a total labeling and as $a \leq h(u)+h(v) \leq a+mn-m-1$ and $mn+1 \leq h(uv) \leq 2mn-m$ the set of the edge-weights is $\{a+mn+1, a+mn+3,\ldots,a+3mn-2m-1\}$. Thus, mG is super $(a+mn+1,2)$-EAT. □

Lemma 5. *Let G be a graph of order n and size $n-1$, $n \geq 4$ even. If G admits an α-labeling, then mG admits a super $(b,1)$-EAT labeling for every $m \geq 1$.*

Proof. Let us distinguish two cases:

Case 1. m is odd

As G is an α-graph of order n and size $n-1$, according to Lemma 3 there exists an $(a,1)$-EAV labeling f of mG. Thus the set of the edge-weights gives the sequence $\mathcal{M} = \{a+(i-1) : 1 \leq i \leq k+1\}$, where $k = m(n-1)-1$. As n is even and if m odd then k is even. With respect to Lemma 1, for $d = 1$, there exists a permutation $\mathcal{P}(\mathcal{M})$ of the elements of $\mathcal{M}$ such that $\mathcal{M} + [\mathcal{P}(\mathcal{M}) - a + mn + 1]$ is an arithmetic sequence with the first term $a + \frac{m(3n-1)+1}{2}$ and the common difference $d = 1$.

If $[\mathcal{P}(\mathcal{M}) - a + mn + 1]$ is an edge labeling of mG with the labels $mn + 1, mn+2, \dots, 2mn - m$, then $\mathcal{M} + [\mathcal{P}(\mathcal{M}) - a + mn + 1]$ determines the set of the edge-weights under the resulting total labeling. Hence, mG is super $(b,1)$-EAT for $b = a + \frac{m(3n-1)+1}{2}$.

Case 2. m is even

Assume that f is an α-labeling of a graph G with n vertices and $n-1$ edges, and V_1, V_2 are its bipartite sets. Without loss of generality, we may assume that the vertex labeled by the boundary value λ belongs to V_1. So, $f(u) < f(v)$ for any $u \in V_1$ and $v \in V_2$.

We denote by $V(mG) = \bigcup_{j=1}^{m}\{u_j, v_j : u_j \in V_1^j, v_j \in V_2^j\}$ the vertex set of a disjoint union of m copies of G. i.e. $\bigcup_{j=1}^{m}\{V_1^j \cup V_2^j\} = V(mG)$.

Consider the vertex labeling g of mG such that for every $u_j \in V_1^j$, $1 \leq j \leq m$, we put

$g(u_j) = m[f(u) - 1] + j \quad \text{if } u \in V_1$

and for every $v_j \in V_2^j$, $1 \leq j \leq m$, we put

$$g(v_j) = \begin{cases} m[n + \lambda - f(v)] + \frac{m+1-j}{2} & \text{if } v \in V_2 \text{ and } j \text{ is odd} \\ m[n + \lambda + 1 - f(v)] + \frac{2-j}{2} & \text{if } v \in V_2 \text{ and } j \text{ is even.} \end{cases}$$

Since $1 \leq f(u) \leq \lambda$ and $\lambda + 1 \leq f(v) \leq n$, thus the function g assigns the labels $1, 2, 3, \dots, m\lambda - 1, m\lambda$ to all vertices $u_j \in V_1^j$, $1 \leq j \leq m$, and the labels $m\lambda + 1, m\lambda + 2, \dots, mn - 1, mn$ to all vertices $v_j \in V_2^j$, $1 \leq j \leq m$. Therefore g is an injective function from $\bigcup_{j=1}^{m}\{V_1^j \cup V_2^j\}$ into $\{1, 2, \dots, mn\}$.

If uv is an edge in G, $u \in V_1$, $v \in V_2$, then u_jv_j is the edge in mG, where $u_j \in V_1^j$, $v_j \in V_2^j$, for $1 \le j \le m$. For the edge-weight of u_jv_j we have

$$g(u_j) + g(v_j) = \begin{cases} m[n + \lambda - (f(v) - f(u))] + \frac{1+j-m}{2} & \text{if } j \text{ is odd} \\ m[n + \lambda - (f(v) - f(u))] + \frac{2+j}{2} & \text{if } j \text{ is even.} \end{cases}$$

We can see that, for each edge $uv \in E(G)$, the edge-weights of the corresponding edges in mG produce a sequence $\mathcal{N} = \{c + d(i-1) : 1 \le i \le \frac{k+1}{2}\} \cup \{c + di : \frac{k+3}{2} \le i \le k+1\}$ for $c = m[n + \lambda - \frac{1}{2} - (f(v) - f(u))] + 1$, $d = 1$ and $k = m - 1$. For $f(v) - f(u) = l$, we have $n-1$ sequences $\mathcal{N}_l$, $1 \le l \le n-1$.

Now, we define an arithmetic sequence $\mathcal{S}_l = \{r_l + d(i-1) : 1 \le i \le k+1\}$ for $d = 1$, $k = m - 1$ and

$$r_l = \begin{cases} \frac{m}{2}[2n - 1 + l] + 1 & \text{if } l \text{ is odd} \\ \frac{m}{2}[3n - 2 + l] + 1 & \text{if } l \text{ is even.} \end{cases}$$

We can see that $\bigcup_{l=1}^{n-1} \mathcal{S}_l = \{mn+1, mn+2, \ldots, 2mn - m - 1, 2mn - m\}$. From Lemma 2, it follows that for each sequence $\mathcal{N}_l$, $1 \le l \le n-1$, there exists a permutation of the elements of the arithmetic sequence $\mathcal{S}_l$ such that $\mathcal{N}_l + \mathcal{P}(\mathcal{S}_l)$, $1 \le l \le n-1$, is an arithmetic sequence with a first term

$$\begin{cases} \frac{m}{2}[4n + 2\lambda - l - 1] + 2 & \text{if } l \text{ is odd} \\ \frac{m}{2}[5n + 2\lambda - l - 2] + 2 & \text{if } l \text{ is even,} \end{cases}$$

and a common difference $d = 1$. It is a matter for routine checking to see that $\bigcup_{l=1}^{n-1}\{\mathcal{N}_l + \mathcal{P}(\mathcal{S}_l)\} = \{\frac{m}{2}[3n+2\lambda]+2, \frac{m}{2}[3n+2\lambda]+3, \ldots, \frac{m}{2}[5n+2\lambda-2]+1\}$.

If the arithmetic sequence $\bigcup_{l=1}^{n-1} \mathcal{S}_l$ is a set of edge labels of mG then $\bigcup_{l=1}^{n-1}\{\mathcal{N}_l + \mathcal{P}(\mathcal{S}_l)\}$ describes the set of the corresponding edge-weights of mG. It implies that mG has a super $(\frac{m}{2}[3n + 2\lambda] + 2, 1)$-EAT labeling. □

Using three previous lemmas the following theorem can be proved.

Theorem 1. *Let G be an α-graph of order n and size $n-1$, $n \ge 3$. The graph mG is super (a, d)-EAT if either*

(i) $d \in \{0, 2\}$ and m is odd, $m \ge 1$, or

(ii) $d = 1$ and n is even, $m \ge 1$.

The next result gives a connection between the α-labelings and the $(a, 2)$-EAV labelings.

Lemma 6. *Let G be an α-graph of order n and size $n-1$ and $\{V_1, V_2\}$ be* *the bipartition of its vertex set. If* $||V_1| - |V_2|| \leq 1$, *then* mG *is* $(m+2,2)$-*EAV, for every* $m \geq 1$.

Proof. It is proved in [2] that if G is an α-graph of order n and size $n-1$ and $||V_1| - |V_2|| \leq 1$ then G is $(3,2)$-EAV. Hence the desired result holds for $m = 1$.

Let f be an α-labeling of graph G of order n and size $n-1$ and V_1, V_2 be the bipartite sets of G. We may assume that $0 \leq |V_1| - |V_2| \leq 1$ and the vertex labeled by the boundary value λ belongs to V_1. In the case that the vertex labeled by the boundary value λ does not belong to V_1 under the α-labeling f then a new labeling

$$f^*(x) = n + 1 - f(x), \quad \text{for } x \in V(G)$$

is an α-labeling as well and its boundary value $n-\lambda$ is appeared on a vertex of V_1.

Now, we consider the vertex labeling g of mG such that for every $u_j \in V_1^j$, $1 \leq j \leq m$, we define

$$g(u_j) = m[2f(u) - 2] + j \quad \text{if } u \in V_1$$

and for every $v_j \in V_2^j$, $1 \leq j \leq m$, we define

$$g(v_j) = m[2n + 1 - 2f(v)] + j \quad \text{if } v \in V_2.$$

Since $1 \leq f(u) \leq \lambda$ and $\lambda + 1 \leq f(v) \leq n$, thus the function g assigns the labels $\{1,2,\ldots,m\} \cup \{2m+1, 2m+2, \ldots, 3m\} \cup \cdots \cup \{m(2\lambda-4)+1, m(2\lambda-4)+2, \ldots, m(2\lambda-3)\} \cup \{m(2\lambda-2)+1, m(2\lambda-2)+2, \ldots, m(2\lambda-1)\}$ to all vertices $u_j \in V_1^j$, $1 \leq j \leq m$, and the labels $\{m+1, m+2, \ldots, 2m\} \cup \{3m+1, 3m+2, \ldots, 4m\} \cup \cdots \cup \{m(2n-2\lambda-3)+1, m(2n-2\lambda-3)+2, \ldots, m(2n-2\lambda-2)\} \cup \{m(2n-2\lambda-1)+1, m(2n-2\lambda-1)+2, \ldots, m(2n-2\lambda)\}$ to all vertices $v_j \in V_2^j$, $1 \leq j \leq m$. If $0 \leq |V_1| - |V_2| \leq 1$ then $\lambda = \lceil \frac{n}{2} \rceil$ and evidently g is an injective function with the labels $1, 2, 3, \ldots, mn-1, mn$.

Moreover, if uv is an edge in G, $u \in V_1$, $v \in V_2$, then u_jv_j is the edge in mG, where $u_j \in V_1^j$, $v_j \in V_2^j$, for $1 \leq j \leq m$. For the edge-weight of u_jv_j, $1 \leq j \leq m$, we have

$$g(u_j) + g(v_j) = m(2n-1) + 2j - 2m[f(v) - f(u)].$$

Since f is an α-labeling, thus $1 \leq f(v) - f(u) \leq n-1$ for $uv \in E(G)$ and the edge-weights of mG form an arithmetic sequence $\{m+2, m+4, \ldots, 2mn-m-2, 2mn-m\}$. Thus, g is an $(m+2,2)$-EAV labeling of mG. □

Theorem 2. *Let* G *be an* α-*graph of order* n *and size* $n-1$ *and* $\{V_1, V_2\}$

be the bipartition of its vertex set. If $||V_1| - |V_2|| \leq 1$, *then* mG *is super* (a, d)*-EAT. for* $d \in \{1, 3\}$ *and every* $m \geq 1$.

Proof. It follows from Lemma 6 that if a graph G satisfies the assumptions of the theorem then mG is $(m+2, 2)$-EAV for every $m \geq 1$. Let g be an $(m+2, 2)$-EAV labeling of mG with the set of edge-weights $\{g(u) + g(v) : uv \in E(mG)\} = \{m+2, m+4, \ldots, 2mn - m - 2, 2mn - m\}$.

We extend the vertex labeling g into a total labeling h_1 and a total labeling h_2 by adding the edge labels from a set $\{mn+1, mn+2, \ldots, 2mn - m - 1, 2mn - m\}$ where

$h_1(u) = h_2(u) = g(u)$ for every vertex $u \in V(mG)$,

$h_1(uv) = 2mn - m + 1 + \frac{m - [h_1(u) + h_1(v)]}{2}$ and $h_2(uv) = mn + \frac{[h_2(u) + h_2(v)] - m}{2}$ for every edge $uv \in E(mG)$.

It easily follows that if $\{h_1(u) + h_1(v) : uv \in E(mG)\} = \{m+2, m+4, \ldots, 2mn-m-2, 2mn-m\}$ then the set of edge-weights is $\{h_1(u)+h_1(v)+h_1(uv) : uv \in E(mG)\} = \{2mn+2, 2mn+3, \ldots, 3mn-m, 3mn-m+1\}$. The reader can also easily verify that $\{h_2(u) + h_2(v) + h_2(uv) : uv \in E(mG)\} = \{mn+m+3, mn+m+6, \ldots, 4mn-2m-3, 4mn-2m\}$. This implies the desired result. □

4 Disjoint union of caterpillars

In this section we study a super edge-antimagicness of forests in which every component is a caterpillar. The *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path. Sugeng *et al.* in [17] described some constructions of the super (a, d)-EAT labelings of the caterpillars for $d \in \{0, 1, 2, 3\}$.

Let T be a caterpillar of order n and mT be a disjoint union of m copies of T. Rosa [12] showed that all caterpillars have an α-labeling. Therefore all results from previous section hold for T and mT. Moreover we complete one case when $d = 1$ and n odd.

Lemma 7. *There is a super* $(a, 1)$*-EAT labeling for a caterpillar of order* n, $n \geq 3$ *odd.*

Proof. We consider a caterpillar T of order n, $n \geq 3$ odd. Any caterpillar is bipartite. We denote by $\{V_1, V_2\}$ the bipartition of the vertex set of the caterpillar T, i.e. $V(T) = V_1(T) \cup V_2(T)$. We can draw the vertices of T in two rows, such that each row is containing only the vertices from one partite set. Clearly, it is possible to make the drawing of T such that there are no edge crossings. Let $e_1^*, e_2^*, \dots, e_{n-1}^*$ be the edges of T ordered from left to right. If one of the endpoints of the edge $e_{\frac{n+1}{2}}^*$ is of degree 1 then we denote it by v_1. If both endpoints of $e_{\frac{n+1}{2}}^*$ have the degrees greater then 1, we denote by v_1 the vertex which is common vertex of the edges $e_{\frac{n+1}{2}}^*$ and $e_{\frac{n+3}{2}}^*$. The next vertices ordered from v_1 to right in the same partition we denote by $v_2, v_3, \dots, v_t$. We continue in the same partition at the beginning and we denote the vertices ordered from left to v_1 by $v_{t+1}, v_{t+2}, \dots, v_{t+s}$. So, $v_{t+1}, v_{t+2}, \dots, v_{t+s}, v_1, v_2, \dots, v_t$ are ordered vertices in the first partition, say $V_1(T)$. Let $u_1, u_2, \dots, u_{n-t-s}$ be the vertices in the second partition, say $V_2(T)$, ordered from left to right.

Consider the labeling $f : V(T) \to \{1, 2, \dots, n\}$ defined by

$$f(v_l) = \begin{cases} l & \text{if } 1 \le l \le t \\ n - t - s + l & \text{if } t + 1 \le l \le t + s \end{cases}$$

$$f(u_l) = t + l \quad \text{if } 1 \le l \le n - t - s.$$

Now, we rename the edges of T such that

$$e_i = \begin{cases} e^*_{\frac{n+1}{2}-1+i} & \text{if } 1 \le i \le \frac{n-1}{2} \\ e^*_{i+1-\frac{n-1}{2}} & \text{if } \frac{n+1}{2} \le i \le n-1. \end{cases}$$

We can see that the set of the edge-weights gives a sequence $\mathcal{N} = \{w(e_i) : w(e_i) = c + (i-1), 1 \le i \le \frac{k+1}{2}\} \cup \{w(e_i) : w(e_i) = c + i, \frac{k+3}{2} \le i \le k+1\}$ for $k = n-2$, where c is an edge-weight of the edge $e^*_{\frac{n+1}{2}} = e_1$. With respect to Lemma 2, for $d = 1$, there exists a permutation of the elements of an arithmetic sequence $\mathcal{S} = \{r + d(i-1) : 1 \le i \le k+1\}$ for $d = 1$, $k = n-2$, $r = n+1$, such that $\mathcal{N} + \mathcal{P}(\mathcal{S})$ is an arithmetic sequence with the first term $c + \frac{3n+1}{2}$ and a common difference $d = 1$. If $\mathcal{S}$ is a set of edge labels of T then $\mathcal{N} + \mathcal{P}(\mathcal{S})$ describes a set of the corresponding edge-weights of T. Thus, T admits a super $(c + \frac{3n+1}{2}, 1)$-EAT labeling. □

Let us remark that the previous lemma was proved in [17] by different construction. We described only one convenient vertex labeling f which will be useful in the next theorem.

Theorem 3. *Let T be a caterpillar of order n, $n \ge 3$ odd. If T admits a super $(a, 1)$-EAT labeling, then mT also admits a super $(b, 1)$-EAT labeling for every $m \ge 2$.*

Proof. Assume that a caterpillar T of order n, $n \ge 3$ odd, with vertices and edges denoted as in Lemma 7 admits a super $(a,1)$-EAT labeling. We denote by $V(mT)=\bigcup_{j=1}^{m}\{V_1^j(T)\cup V_2^j(T)\}$ the vertex set of a disjoint union of m copies of the caterpillar T where $V_1^j(T)=\{v_l^j : 1\le l\le t+s\}$, $V_2^j(T)=\{u_l^j : 1\le l\le n-t-s\}$, $1\le j\le m$. Let $E(mT)=\bigcup_{j=1}^{m}\{e_i^j : 1\le i\le n-1\}$ be the edge set of mT. Evidently every edge e_i^j has one endpoint in $V_1^j(T)$ and other one in $V_2^j(T)$.

Let us distinguish two cases:

Case 1. m is odd

We extend the vertex labeling f from Lemma 7 onto a labeling g_1 such that for every $1\le l\le t+s$ we put

$$g_1(v_l^j)=\begin{cases} m[f(v_l)-1]+\frac{m+3}{2}-j & \text{if } 1\le j\le \frac{m+1}{2}\\ m[f(v_l)-1]+\frac{3m+3}{2}-j & \text{if } \frac{m+3}{2}\le j\le m\end{cases}$$

and for every $1\le l\le n-t-s$ we put

$$g_1(u_l^j)=\begin{cases} m[f(u_l)-1]+2j-1 & \text{if } 1\le j\le \frac{m+1}{2}\\ m[f(u_l)-1]+2j-m-1 & \text{if } \frac{m+3}{2}\le j\le m.\end{cases}$$

It is a routine procedure to verify that as $f(v_l)\in\{1,2,\dots,t\}\cup\{n-s+1,n-s+2,\dots,n\}$ and $f(u_l)\in\{t+1,t+2,\dots,n-s\}$ then the vertex labeling g_1 is a bijective function from $V(mT)$ onto the set $\{1,2,\dots,mn\}$. Moreover for the edge-weights we have

$$w_{g_1}(e_i^j)=mw_f(e_i)+\tfrac{1-3m}{2}+j \quad \text{for } 1\le i\le n-1 \quad \text{and } 1\le j\le m.$$

It follows from Lemma 7 that

$$w_f(e_i)=\begin{cases} c+(i-1) & \text{if } 1\le i\le \frac{n-1}{2}\\ c+i & \text{if } \frac{n+1}{2}\le i\le n-1\end{cases}$$

thus the edge-weights of the corresponding edges in each copy of mT produce a sequence $\mathcal{N}_j=\{w_{g_1}(e_i^j) : w_{g_1}(e_i^j)=c_j+m(i-1), 1\le i\le \frac{k+1}{2}\}\cup\{w_{g_1}(e_i^j) : w_{g_1}(e_i^j)=c_j+mi, \frac{k+3}{2}\le i\le k+1\}$ for $c_j=mc+\frac{1-3m}{2}+j$, $k=n-2$ and $1\le j\le m$. According to Lemma 2, it follows that for each sequence $\mathcal{N}_j$, $1\le j\le m$, there exists a permutation of the elements of the arithmetic sequence $\mathcal{S}_j=\{r_j+m(i-1) : 1\le i\le k+1\}$ for $k=n-2$ and

$$r_j=\begin{cases} \frac{m(2n+1)-j}{2}+1 & \text{if } j \text{ is odd}\\ mn+m+\frac{2-j}{2} & \text{if } j \text{ is even}\end{cases}$$

such that $\mathcal{N}_j+\mathcal{P}(\mathcal{S}_j)$, $1\le j\le m$, is an arithmetic sequence with the first term

$$a_j = \begin{cases} \frac{m(2c+3n-3)+3+j}{2} & \text{if } j \text{ is odd} \\ \frac{m(2c+3n-2)+3+j}{2} & \text{if } j \text{ is even} \end{cases}$$

and common difference m.

If $\bigcup_{j=1}^{m} S_j = \{mn+1, mn+2, \ldots, 2mn-m\}$ is the set of the edge labels of mT, then $\bigcup_{j=1}^{m}\{\mathcal{N}_j + \mathcal{P}(S_j)\} = \bigcup_{j=1}^{m}\{a_j + m(i-1) : 1 \le i \le n-1\} = \{m(c+\frac{3n-3}{2})+2, m(c+\frac{3n-3}{2})+3, \ldots, m(c+\frac{5n-5}{2}), m(c+\frac{5n-5}{2})+1\}$ is the set of the edge-weights and we arrive at the desired result.

Case 2. m is even

We extend the vertex labeling f onto a labeling g_2 in the following way, where for every $1 \le l \le t+s$

$$g_2(v_l^j) = \begin{cases} m[f(v_l)-1] + \frac{m+2}{2} - j & \text{if } 1 \le j \le \frac{m}{2} \\ m[f(v_l)-1] + \frac{3m+2}{2} - j & \text{if } \frac{m}{2}+1 \le j \le m \end{cases}$$

and for every $1 \le l \le n-t-s$

$$g_2(u_l^j) = \begin{cases} m[f(u_l)-1] + 2j - 1 & \text{if } 1 \le j \le \frac{m}{2} \\ m[f(u_l)-1] + 2j - m & \text{if } \frac{m}{2}+1 \le j \le m. \end{cases}$$

Again it is not difficult to verify that as $f(v_l) \in \{1, 2, \ldots, t\} \cup \{n-s+1, n-s+2, \ldots, n\}$ and $f(u_l) \in \{t+1, t+2, \ldots, n-s\}$ then the vertex labeling $g_2 : V(mT) \to \{1, 2, \ldots, mn\}$ is a bijective function. For the edge-weights we have

$$w_{g_2}(e_i^j) = \begin{cases} mw_f(e_i) - \frac{3m}{2} + j & \text{if } 1 \le j \le \frac{m}{2} \\ mw_f(e_i) - \frac{3m}{2} + j + 1 & \text{if } \frac{m}{2}+1 \le j \le m. \end{cases}$$

Now, we define the arithmetic sequences $S_j = \{r_j + m(i-1) : 1 \le i \le k+1\}$ for $k = n-2$, $1 \le j \le m$, where

$$\left.\begin{array}{lll} \text{for} & k' = m-1 \equiv 5 \pmod 6, & k' \ge 5 \\ \text{for} & k' = m-1 \equiv 1 \pmod 6, & k' \ge 1 \\ \text{for} & k' = m-1 \equiv 3 \pmod 6, & k' \ge 3 \end{array}\right\} r_j = mn + 1 - r + h_j.$$

We are using the labeling h from the proof of Lemma 2 for $d = 1$ and for every $k' = m-1$.

We will use a similar argument applied in *Case 1* that the edge-weights of the corresponding edges in each copy of mT produce a sequence $\mathcal{N}_j = \{w_{g_2}(e_i^j) : w_{g_2}(e_i^j) = c_j + m(i-1), 1 \le i \le \frac{k+1}{2}\} \cup \{w_{g_2}(e_i^j) : w_{g_2}(e_i^j) = c_j + mi, \frac{k+3}{2} \le i \le k+1\}$ for $k = n-2$ and

$$c_j = \begin{cases} \frac{m}{2}(2c-3) + j & \text{if } 1 \le j \le \frac{m}{2} \\ \frac{m}{2}(2c-3) + j + 1 & \text{if } \frac{m}{2}+1 \le j \le m, \end{cases}$$

where c is an edge-weight of the edge e_1 under the labeling f.

With respect to Lemma 2, for each sequence $\mathcal{N}_j$, $1 \leq j \leq m$, there exists a permutation of the elements of the arithmetic sequence $\mathcal{S}_j = \{r_j + m(i-1) : 1 \leq i \leq k+1\}$, $1 \leq j \leq m$, such that $\mathcal{N}_j + \mathcal{P}(\mathcal{S}_j)$, $1 \leq j \leq m$, is an arithmetic sequence with first term $c_j + r_j + \frac{(k+1)m}{2}$ and a common difference m. If $\bigcup_{j=1}^{m} \mathcal{S}_j = \{mn+1, mn+2, \ldots, 2mn-m\}$ is a set of edge labels of mT, then $\bigcup_{j=1}^{m}\{\mathcal{N}_j + \mathcal{P}(\mathcal{S}_j)\} = \{m(c+\frac{3n-3}{2})+2, m(c+\frac{3n-3}{2})+3, \ldots, m(c+\frac{5n-5}{2}), m(c+\frac{5n-5}{2})+1\}$ determines the set of the edge-weights of mT and the resulting total labeling is super $(b,1)$-EAT. □

5 Open questions

We have not yet found a construction that will produce a super (a,d)-EAT labeling of mG, for $d \in \{0,2\}$ and m even. So, we propose the following open problem.

Open Problem 1. *Let G be a graph of order n and size $n-1$. For the graph mG determine if there is a super (a,d)-EAT labeling, for $d \in \{0,2\}$ and m even.*

In Theorem 2 we proved that if G is an α-graph of order n and size $n-1$ and $||V_1|-|V_2|| \leq 1$, where $\{V_1, V_2\}$ is the bipartition of the vertex set of G, then mG is super $(a,3)$-EAT, for every $m \geq 1$. In [2] it is exhibited a super $(13,3)$-EAT labeling of a caterpillar which does not satisfy the restriction for the cardinalities of bipartite sets V_1 and V_2 because in this case $|V_1| = 2$ and $|V_2| = 2n-1$. What we can say on a super $(a,3)$-EAT labeling of mG in the case when a graph G of order n and size $n-1$ does not satisfy the restriction for the cardinalities of the bipartite sets V_1 and V_2? At this time we have no answer. Therefore for the further investigation we propose:

Open Problem 2. *Let T be a caterpillar of order n and $||V_1|-|V_2|| > 1$, where $\{V_1, V_2\}$ is the bipartition of its vertex set. For the graph mT determine if there is a super $(a,3)$-EAT labeling.*

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