The full metamorphosis of λ-fold block designs with block size four into λ-fold triple systems

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Abstract

Let (X, B) be a λ-fold block design with block size 4. If a star is removed from each block of B the resulting collection of triangles T is a partial λ-fold triple system (X, T). If the edges belonging to the deleted stars can be arranged into a collection of triangles S^*, then $(X, T \cup S^*)$ is a λ-fold triple system, called a metamorphosis of the λ-fold block design (X, B) into a λ-fold triple system. Label the elements of each block b with b_1, b_2, b_3, and b_4 (in any manner). For each $i = 1, 2, 3, 4$ define a set of triangles T_i and a set of stars S_i as follows: for each block $b = [b_1, b_2, b_3, b_4]$ belonging to B, partition b into a triangle and a star centered at b_i, and place the triangle in

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T_i and the star in S_i. Then (X, T_i) is a partial λ-fold triple system. Now if the edges belonging to the stars in S_i can be arranged into a collection of triangles S_i^*, then $(X, T_i \cup S_i^*)$ is a λ-fold triple system and we say that $M_i = (X, T_i \cup S_i^*)$ is the ith metamorphosis of (X, B). The full metamorphosis of (X, B) is the set of four metamorphoses $\{M_1, M_2, M_3, M_4\}$. The purpose of this work is to give a complete solution of the following problem: For which n and λ does there exist a λ-fold block design with block size 4 having a full metamorphosis into λ-fold triple systems?

1 Introduction

A λ-fold block design with block size k is a pair (X, B), where B is a collection of edge disjoint copies of K_k which partitions the edge set of λK_n with vertex set X. The copies of K_k are called blocks.

Let (X, B) be a λ-fold block design with block size 4 and label the elements of each block b with b_1, b_2, b_3, and b_4 (in any manner). For each $i = 1, 2, 3, 4$ define a set of triangles T_i and a set of stars S_i as follows: for each block $b = [b_1, b_2, b_3, b_4]$ belonging to B partition b into a triangle and a star centered at b_i, and place the triangle in T_i and the star in S_i. Then (X, T_i) is a partial λ-fold triple system (that is, a λ-fold block design with block size 3). Now if the stars in S_i can be arranged into a collection of triangles S_i^*, then $(X, T_i \cup S_i^*)$ is a λ-fold triple system and it is a metamorphosis of a λ-fold block design with block size 4 into a λ-fold triple system, as defined in [7]. We will refer to $M_i = (X, T_i \cup S_i^*)$ as the ith metamorphosis of (X, B). Observe that the center of the star corresponding to each block b is different in each metamorphosis M_i.

The full metamorphosis of (X, B) is a set of four metamorphoses $\{M_1, M_2, M_3, M_4\}$.

In what follows, a λ-fold block design will always mean a λ-fold block design with block size 4 and we will denote the edge sets of K_3 and K_4 by their vertex sets.

Example 1.1 *(Full metamorphosis of a block design of order* 13 *into triple systems.)*

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{13}$ and block set B, where

$B = \{[0,1,4,6],\ [1,2,5,7],\ [2,3,8,6],\ [3,7,4,9],\ [4,8,5,10],\ [5,11,9,6],\ [6,7,12,10],\ [7,8,11,0],\ [8,9,12,1],\ [2,10,9,0],\ [11,1,3,10],\ [4,2,12,11],\ [5,3,12,0]\}$. Then $M_i = T_i \cup S_i^*$ where

$T_1 = \{\{1,4,6\},\ \{2,5,7\},\ \{3,8,6\},\ \{7,4,9\},\ \{8,5,10\},\ \{11,9,6\},\ \{7,12,10\},\ \{8,11,0\},\ \{9,12,1\},\ \{10,9,0\},\ \{1,3,10\},\ \{2,12,11\},\ \{3,12,0\}\}$;

$T_2 = \{\{0,4,6\},\ \{1,5,7\},\ \{2,8,6\},\ \{3,4,9\},\ \{4,5,10\},\ \{5,9,6\},\ \{6,12,10\},\ \{7,11,0\},\ \{8,12,1\},\ \{2,9,0\},\ \{11,3,10\},\ \{4,12,11\},\ \{5,12,0\}\}$;

$T_3 = \{\{0,1,6\},\ \{1,2,7\},\ \{2,3,6\},\ \{3,7,9\},\ \{4,8,10\},\ \{5,11,6\},\ \{6,7,10\},\ \{7,8,0\},\ \{8,9,1\},\ \{2,10,0\},\ \{11,1,10\},\ \{4,2,11\},\ \{5,3,0\}\}$;

$T_4 = \{\{0,1,4\},\ \{1,2,5\},\ \{2,3,8\},\ \{3,7,4\},\ \{4,8,5\},\ \{5,11,9\},\ \{6,7,12\},\ \{7,8,11\},\ \{8,9,12\},\ \{2,10,9\},\ \{11,1,3\},\ \{4,2,12\},\ \{5,3,12\}\}$; and

$S_1^* = \{\{0,1,2\},\ \{0,4,5\},\ \{0,6,7\},\ \{1,5,11\},\ \{1,7,8\},\ \{2,3,4\},\ \{2,6,10\},\ \{2,8,9\},\ \{3,5,9\},\ \{3,7,11\},\ \{4,8,12\},\{4,10,11\},\ \{5,6,12\}\}$;

$S_2^* = \{\{0,1,3\}, \{0,8,10\}, \{1,2,4\}, \{1,6,11\}, \{1,9,10\}, \{2,3,12\}, \{2,5,11\}, \{2,7,10\}, \{3,5,8\}, \{3,6,7\}, \{4,7,8\}, \{7,9,12\}, \{8,9,11\}\}$;

$S_3^* = \{\{0,4,12\}, \{0,9,11\}, \{1,3,4\}, \{1,5,12\}, \{2,5,8\}, \{2,9,12\}, \{3,8,11\}, \{3,10,12\}, \{4,5,7\}, \{4,6,9\}, \{5,9,10\}, \{6,8,12\}, \{7,11,12\}\}$;

$S_4^* = \{\{0,2,7\},\ \{0,3,9\},\ \{0,5,6\},\ \{0,8,10\},\ \{0,11,12\},\ \{1,6,8\},\ \{1,7,9\},\ \{1,10,12\},\ \{2,6,11\},\ \{3,6,10\},\ \{4,6,9\},\ \{4,10,11\},\ \{5,7,10\}\}$.

The purpose of this paper is to find a full metamorphosis of (X, B) for each admissible n and λ. If we can give a complete solution of this problem for $\lambda = 1, 2, 3$, and 6, we can paste these solutions together to get a solution for all other values of λ. Therefore we will organize our results into four sections: $\lambda = 1, \lambda = 2, \lambda = 3$, and $\lambda = 6$, followed by a summary.

Finally, the interested reader is referred to [3, 4, 5, 6, 8, 9, 10] for related work on metamorphosis problems.

2 $\lambda = 1$

It is well-known that the spectrum for block designs with block size 4 is precisely the set of all $n \equiv 1$ or 4 (mod 12) and the spectrum for triple systems is the set of all $n \equiv 1$ or 3 (mod 6). Hence a necessary condition for the existence of a block design having a full metamorphosis into triple systems is $n \equiv 1$ (mod 12).

Example 1.1 gives a solution for $n = 13$ and the following examples give solutions for the cases $n = 25, 37$, and 73.

Example 2.1 (*Full metamorphosis of a block design of order* 25 *into triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_5 \times \mathbb{Z}_5$, and block set $B = \{[(0+i, 1+j), (0+i, 0+j), (2+i, 2+j), (1+i, 0+j)], [(0+i, 2+j), (0+i, 0+j), (4+i, 4+j), (2+i, 0+j)] \mid 0 \leq i \leq 4, 0 \leq j \leq 4\}$, where addition is done modulo 5.

Then S_i^* is a partial STS, where the starter blocks for S_1^* are $\{(0,0), (0,1), (1,4)\}$, $\{(0,0), (0,2), (2,3)\}$; for S_2^* are $\{(0,0), (0,1), (1,1)\}$, $\{(0,0), (0,2), (2,2)\}$; for S_3^* are $\{(0,0), (2,1), (4,3)\}, \{(0,0), (1,1), (4,2)\}$; and for S_4^* are $\{(0,0), (1,4), (4,0)\}, \{(0,0), (1,2), (3,0)\}$.

Example 2.2 (*Full metamorphosis of a block design of order* 37 *into triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{37}$, block set B, and starter blocks $[0, 3, 1, 24]$, $[4, 9, 0, 15]$, $[7, 17, 25, 0]$.

Then the starter blocks for S_1^* are $\{0,1,5\}, \{0,3,13\}, \{0,7,18\}$; for S_2^* are $\{0,2,5\}, \{0,6,16\}, \{0,8,17\}$; for S_3^* are $\{0,1,9\}, \{0,2,14\}, \{0,15,19\}$; and for S_4^* are $\{0,6,22\}, \{0,11,23\}, \{0,7,20\}$.

Example 2.3 (*Full metamorphosis of a block design of order* 73 *into triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{73}$, block set B, and starter blocks $[0, 1, 4, 6]$, $[20, 7, 28, 0]$, $[33, 9, 44, 0]$, $[25, 15, 47, 0]$, $[30, 12, 46, 0]$, $[31, 14, 50, 0]$.

Then the starter blocks for S_1^* are $\{0,1,11\}, \{0,4,24\}, \{0,6,25\}$, $\{0,8,30\}$, $\{0,13,31\}$, $\{0,16,33\}$; for S_2^* are $\{0,1,10\}$, $\{0,3,18\}$, $\{0,5,17\}$, $\{0,7,39\}$, $\{0,13,37\}$, $\{0,14,35\}$; for S_3^* are $\{0,2,21\}$,$\{0,3,32\}$, $\{0,4,26\}$, $\{0,8,36\}$, $\{0,11,38\}$, $\{0,16,39\}$; and for S_4^* are $\{0,2,25\}$, $\{0,5,31\}$, $\{0,6,15\}$, $\{0,7,27\}$, $\{0,12,40\}$, $\{0,14,43\}$.

Example 2.4 (*A* $4-GDD$ *of type* 4^4 *having a full metamorphosis into a* $3-GDD$ *of type* 4^4.)

Let $\mathcal{G} = (X, G, B)$ be a $4-GDD$ of type 4^4 where $X = \{1, 2, 3, ..., 16\}$, $G = \{\{0,1,2,3\}, \{4,5,6,7\}, \{8,9,10,11\}, \{12,13,14,15\}\}$, and $B = \{[0,6,9,15], [1,7,8,14], [2,5,11,13], [3,4,10,12], [0,4,11,14], [1,5,10,15], [2,7,9,12], [3,6,8,13], [7,10,13,0], [6,11,12,1], [4,8,15,2], [5,9,14,3], [5,8,12,0], [4,9,13,1], [6,10,14,2], [7,11,15,3]\}$. Then

$S_1^* = \{\{0,4,9\}, \{0,5,14\}, \{0,6,11\}, \{0,7,15\}, \{1,4,15\}, \{1,5,8\}, \{1,6,14\}, \{1,7,10\}, \{2,4,13\}, \{2,5,9\}, \{2,6,12\}, \{2,7,11\}, \{3,4,8\}, \{3,5,12\}, \{3,6,10\}, \{3,7,13\}\}$;

$S_2^* = \{\{0,4,8\}, \{0,6,10\}, \{1,5,9\}, \{1,7,11\}, \{2,5,8\}, \{2,7,10\}, \{3,4,9\}, \{3,6,11\}, \{4,10,14\}, \{4,11,12\}, \{5,10,13\}, \{5,11,15\}, \{6,8,15\}, \{6,9,13\}, \{7,8,12\}, \{7,9,14\}\}$;

$S_3^* = \{\{0,9,12\}, \{0,11,13\}, \{1,8,13\}, \{1,10,12\}, \{2,9,15\}, \{2,11,14\}, \{3,8,14\}, \{3,10,15\}, \{4,10,13\}, \{4,11,15\}, \{5,10,14\}, \{5,11,12\}, \{6,8,12\}, \{6,9,14\}, \{7,8,15\}, \{7,9,13\}\}$; and

$S_4^* = \{0,5,13\}, \{0,7,12\}, \{0,8,14\}, \{0,10,15\}, \{1,4,12\}, \{1,6,13\}, \{1,9,15\}, \{1,11,14\}, \{2,4,14\}, \{2,6,15\}, \{2,8,13\}, \{2,10,12\}, \{3,5,15\}, \{3,7,14\}, \{3,9,12\}, \{3,11,13\}\}$.

With these examples in hand we can give a general construction for all of the remaining cases.

The 12k + 1 Construction. Suppose $n = 12k + 1$, where $k \geq 4$, $k \neq 6$. Set $X = \{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 3k, 1 \leq j \leq 4\}$.

If $k \equiv 0$ or $1 \pmod 4$:

(i) On each set $\{\infty\} \cup \{(3i-2,j),(3i-1,j),(3i,j) \mid 1 \leq j \leq 4\}$, $1 \leq i \leq k$, place a block design of order 13 having a full metamorphosis into triple systems.

(ii) Take a 4-GDD of type 3^k [2] on $\{1,2,...,3k\}$ with groups $\{3i-2,3i-1,3i\}$, $1 \leq i \leq k$. For each block $\{x,y,z,w\}$ in the 4-GDD, place a copy of $\mathcal{G}$ in Example 2.4 on the set $\{x,y,z,w\} \times \{1,2,3,4\}$ with groups $\{x\} \times \{1,2,3,4\}$, $\{y\} \times \{1,2,3,4\}$, $\{z\} \times \{1,2,3,4\}$, and $\{w\} \times \{1,2,3,4\}$.

If $k \equiv 2$ or $3 \pmod 4$:

(i′) On the set $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 6,\ 1 \leq j \leq 4\}$, place a block design of order 25 having a full metamorphosis into triple systems.

(ii′) On each set $\{\infty\} \cup \{(3i-2,j),(3i-1,j),(3i,j) \mid 1 \leq j \leq 4\}$, for $3 \leq i \leq k$, place a block design of order 13 having a full metamorphosis into triple systems.

(iii′) Take a 4-GDD of type $6^1 3^{k-2}$ [1] on $\{1,2,...,3k\}$ with groups $\{1,2,3,4,5,6\}$ and $\{3i-2,\ 3i-1,3i\}$, $3 \leq i \leq k$. For each block

$\{x, y, w, z\}$ in the 4-GDD, place a copy of $\mathcal{G}$ in Example 2.4 on the set $\{x, y, z, w\} \times \{1, 2, 3, 4\}$ with groups $\{x\} \times \{1, 2, 3, 4\}$, $\{y\} \times \{1, 2, 3, 4\}$, $\{z\} \times \{1, 2, 3, 4\}$, and $\{w\} \times \{1, 2, 3, 4\}$.

Combining (i) and (ii), ((i′), (ii′) and (iii′)) gives a block design of order n with block size 4 having a full metamorphosis into triple systems. □

3 $\lambda = 2$

A necessary condition for the existence of a 2-fold block design having a full metamorphosis into a 2-fold triple system is $n \equiv 1 \pmod 3$. It is known that a pairwise balanced design (PBD) with block sizes 4 and 7 exists for all $n \equiv 1 \pmod 3$ except for $n = 10$ or 19 [1]. If we can produce 2-fold block designs of order $4, 7, 10$, and 19 having a full metamorphosis, a PBD construction will produce a 2-fold block design having a full metamorphosis into 2-fold triple systems.

Example 3.1 (*Full metamorphosis of a 2-fold block design of order* 4 *into* 2-*fold triple systems.*)

Let (X, B) be the 2-fold block design with $B = \{[1, 2, 3, 4], [2, 3, 4, 1]\}$. Then $S_1^* = \{\{1, 2, 3\}, \{1, 2, 4\}\}$, $S_2^* = \{\{1, 2, 3\}, \{2, 3, 4\}\}$, $S_3^* = \{\{1, 3, 4\}, \{2, 3, 4\}\}$, and $S_4^* = \{\{1, 2, 4\}, \{1, 3, 4\}\}$.

Example 3.2 (*Full metamorphosis of a 2-fold block design of order* 7 *into* 2-*fold triple systems.*)

Let (X, B) be the 2-fold block design with vertex set $X = \mathbb{Z}_7$ and block set $B = \{[0, 2, 3, 4], [1, 3, 5, 4], [2, 4, 5, 6], [3, 5, 6, 0], [6, 0, 4, 1], [0, 5, 1, 2], [1, 2, 6, 3]\}$. Then

$S_1^* = \{\{0, 1, 3\}, \{0, 2, 4\}, \{0, 2, 5\}, \{0, 3, 6\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 4, 6\}\}$;

$S_2^* = \{\{0, 1, 2\}, \{0, 4, 5\}, \{0, 5, 6\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 6\}\}$;

$S_3^* = \{\{0,1,4\},\{0,3,6\},\{1,2,5\},\{1,5,6\},\{2,3,6\},\{3,4,5\},\{4,5,6\}\}$; and $S_4^* = \{\{0,1,2\},\{0,3,4\},\{0,5,6\},\{1,3,4\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\}$.

Example 3.3 (*Full metamorphosis of a* 2*-fold block design of order* 10 *into* 2*-fold triple systems.*)

Let (X,B) be the 2-fold block design with vertex set $X = \mathbb{Z}_5 \times \{1,2\}$ and block set $B = \{[(0+i,1),(2+i,1),(4+i,1),(2+i,2)],[(0+i,1),(1+i,1),(2+i,2),(4+i,2)],[(0+i,1),(1+i,2),(0+i,2),(4+i,2)] \mid 0 \leq i \leq 4\}$, where additions are done modulo 5.

Then the starter blocks for S_1^* are $\{(0,1),(1,1),(0,2)\}$, $\{(0,1),(2,1),(4,2)\}$, $\{(0,1),(1,1),(2,2)\}$; the starter blocks for S_2^* are $\{(0,1),(2,1),(4,1)\}$, $\{(0,1),(0,2),(1,2)\}$, $\{(0,1),(1,2),(3,2)\}$; the starter blocks for S_3^* are $\{(0,1),(1,1),(2,2)\}$, $\{(0,1),(2,1),(0,2)\}$, $\{(0,2),(1,2),(2,2)\}$; and the starter blocks for S_4^* are $\{(0,1),(0,2),(3,2)\}$, $\{(0,1),(2,2),(4,2)\}$, $\{(0,1),(3,2),(4,2)\}$.

Example 3.4 (*Full metamorphosis of a* 2*-fold block design of order* 19 *into* 2*-fold triple systems.*)

Let (X,B) be the 2-fold block design with vertex set $X = \mathbb{Z}_{19}$, block set B, and starter blocks $[0,1,3,8],[0,16,6,2],[4,0,12,13]$.

Then the starter blocks for S_1^* are $\{0,1,3\},\{0,3,11\},\{0,4,10\}$; for S_2^* are $\{0,1,3\},\{0,5,12\},\{0,4,10\}$; for S_3^* are $\{0,2,5\},\{0,6,10\},\{0,1,8\}$; and for S_4^* are $\{0,1,5\},\{0,2,9\},\{0,5,11\}$.

Lemma 3.5 *There exists a* 2*-fold block design of order n having a full metamorphosis into* 2*-fold triple systems if and only if* $n \equiv 1 \pmod 3$.

Proof The cases $n = 4, 7, 10$, and 19 are handled in Examples 3.1, 3.2, 3.3, and 3.4. For the remaining cases take a PBD of order n with block sizes 4 and 7, and place a 2-fold block design of order 4 or 7 having a full metamorphosis into 2-fold triple systems on each block. □

4 $\lambda = 3$

A necessary condition for the existence of a 3-fold block design of order n having a full metamorphosis into 3-fold triple systems is $n \equiv 1 \pmod 4$. It is known that a PBD with block sizes $5, 9$, and 13 exists for all $n \equiv 1 \pmod 4$ except for $n = 17, 29, 33$ [2]. So we need 3-fold block designs of order $5, 9, 17, 29$, and 33 having a full metamorphosis. Note that we have a solution for $n = 13$ in Example 1.1.

Example 4.1 (*Full metamorphosis of a 3-fold block design of order* 5 *into* 3*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_5$, block set B, and starter block $[0, 1, 2, 3]$.

Then the starter block for S_1^* is $\{0, 1, 3\}$; for S_2^* is $\{0, 1, 2\}$; for S_3^* is $\{0, 1, 2\}$; and for S_4^* is $\{0, 2, 3\}$.

Example 4.2 (*Full metamorphosis of a 3-fold block design of order* 9 *into triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_9$, block set B, and starter blocks $[0, 3, 5, 8], [2, 0, 3, 4]$.

Then the starter blocks for S_1^* are $\{0, 1, 2\}, \{0, 2, 5\}$; for S_2^* are $\{0, 2, 5\}$, $\{0, 3, 7\}$; for S_3^* are $\{0, 1, 4\}, \{0, 1, 3\}$; and for S_4^* are $\{0, 1, 5\}$, $\{0, 1, 3\}$.

Example 4.3 (*Full metamorphosis of a 3-fold block design of order* 17 *into* 3*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{17}$, block set B, and starter blocks $[0, 1, 4, 6], [0, 2, 8, 12], [0, 4, 16, 7], [0, 8, 15, 14]$.

Then the starter blocks for S_1^* are $\{0, 1, 6\}$, $\{0, 1, 9\}$, $\{0, 3, 7\}$, $\{0, 2, 4\}$; for S_2^* are $\{0, 3, 9\}$, $\{0, 4, 10\}$, $\{0, 5, 10\}$, $\{0, 1, 3\}$; for S_3^* are $\{0, 3, 11\}$,

$\{0,2,4\}$, $\{0,1,8\}$, $\{0,1,5\}$; and for S_4^* are $\{0,5,12\},\{0,6,9\}$, $\{0,2,3\}$, $\{0,4,11\}$.

Example 4.4 (*Full metamorphosis of a* 3*-fold block design of order* 29 *into* 3*-fold triple systems.*)

Let (X,B) be the block design with vertex set $X=\mathbb{Z}_{29}$ and block set $B=\{[0+i,4^x+i,4^{x+1}+i,11\cdot 4^x+i] \mid 0\le i\le 28, 0\le x\le 6\}$, where addition is done modulo 29.

Then, the starter blocks of S_1^* and S_3^* are $\{0,1,3\}$, $\{0,1,5\}$, $\{0,4,9\}$, $\{0,6,16\}$, $\{0,6,17\}$, $\{0,7,15\}$, $\{0,7,16\}$ and the starter blocks of S_2^* and S_4^* are $\{0,1,3\}$, $\{0,2,15\}$, $\{0,3,7\}$, $\{0,6,17\}$, $\{0,8,17\}$, $\{0,8,18\}$, $\{0,5,15\}$.

Example 4.5 (*Full metamorphosis of a* 3*-fold block design of order* 33 *into* 3*-fold triple systems.*)

Let (X,B) be the block design with vertex set $X=\mathbb{Z}_{33}$, block set B, and starter blocks $[0,3,9,21]$, $[0,6,8,28]$, $[0,12,16,23]$, $[0,5,15,19]$, $[0,1,2,9]$, $[0,6,7,15]$, $[0,3,5,16]$, $[0,3,13,17]$.

Then the starter blocks for S_1^* are $\{0,3,9\}$, $\{0,2,16\}$, $\{0,6,16\}$, $\{0,5,12\}$, $\{0,1,16\}$, $\{0,3,8\}$, $\{0,5,20\}$, $\{0,3,12\}$; for S_2^* are $\{0,2,14\}$, $\{0,6,15\}$, $\{0,3,13\}$, $\{0,4,10\}$, $\{0,3,14\}$, $\{0,3,11\}$, $\{0,1,2\}$, $\{0,1,6\}$; for S_3^* are $\{0,1,2\}$, $\{0,2,10\}$, $\{0,2,12\}$, $\{0,4,8\}$, $\{0,4,11\}$, $\{0,5,18\}$, $\{0,6,13\}$, $\{0,7,16\}$; and for S_4^* are $\{0,4,14\}$, $\{0,4,16\}$, $\{0,13,18\}$, $\{0,7,18\}$, $\{0,7,19\}$, $\{0,8,17\}$, $\{0,8,19\}$, $\{0,9,20\}$.

Lemma 4.6 *There exists a* 3*-fold block design of order* n *having a full metamorphosis into* 3*-fold triple systems if and only if* $n\equiv 1 \pmod 4$.

Proof The cases $n=5,9,17,29$, and 33 are handled in Examples 4.1, 4.2, 4.3, 4.4, and 4.5. For the remaining cases take a PBD of order n with block sizes $5,9$ and 13 and place a 3-fold block design of order $5,9$, or 13 having a full metamorphosis into 3-fold triple systems on each block. □

5 $\lambda = 6$

A necessary condition for the existence of a 6-fold block design having a full metamorphosis into a 6-fold triple system is $n \geq 4$. A PBD of order n with block sizes $4, 5, 6$, and 7 exists for all $n \geq 4$, except $n = 8, 9, 10, 11, 12, 14, 15, 18, 19$, and 23 [2]. Note that there exist solutions for $n = 4, 7, 10, 19$ when $\lambda = 2$ and $n = 5, 9$ when $\lambda = 3$ in Examples 3.1, 3.2, 3.3, 3.4, 4.1, and 4.2. So we need to construct 6-fold block designs of order $6, 8, 11, 12, 14, 15, 18$, and 23 having a full metamorphosis.

Example 5.1 (*Full metamorphosis of a* 6*-fold block design of order* 6 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \{\infty\} \cup \mathbb{Z}_5$, block set B, and starter blocks $[\infty, 0, 1, 2], [0, \infty, 3, 1], [1, 2, 3, 0]$.

Then S_i^* is a partial cyclic STS, where the starter blocks for S_1^* are $\{\infty, 0, 1\}$, $\{\infty, 0, 2\}$, $\{0, 1, 2\}$; for S_2^* are $\{\infty, 0, 1\}$, $\{\infty, 0, 2\}$, $\{0, 1, 2\}$; for S_3^* are $\{\infty, 0, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 2\}$; and for S_4^* are $\{\infty, 0, 1\}$, $\{0, 1, 3\}$, $\{0, 1, 3\}$.

Example 5.2 (*Full metamorphosis of a* 6*-fold block design of order* 8 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \{\infty\} \cup \mathbb{Z}_7$, block set B, and starter blocks $[\infty, 0, 1, 3], [\infty, 0, 1, 3], [0, 2, 3, 4], [0, 2, 3, 4]$.

Then the starter blocks for S_1^* are $\{\infty, 1, 4\}$, $\{\infty, 1, 4\}$, $\{\infty, 1, 4\}$, $\{0, 2, 5\}$; for S_2^* are $\{\infty, 0, 2\}$, $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 3\}$; for S_3^* are $\{\infty, 0, 1\}$, $\{0, 1, 2\}$, $\{0, 1, 2\}$, $\{0, 1, 4\}$; and for S_4^* are $\{\infty, 0, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 4\}$, $\{0, 2, 4\}$.

Example 5.3 (*Full metamorphosis of a* 6*-fold block design of order* 11 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{11}$, block set B, and starter blocks $[0,1,4,6], [0,2,8,1], [0,4,5,2], [0,8,10,4], [0,5,9,8]$.

Then the starter blocks for S_1^* are $\{0,1,4\}, \{0,1,5\}, \{0,2,5\}, \{0,1,3\}, \{0,2,6\}$; for S_2^* are $\{0,4,5\}$, $\{0,2,3\}$, $\{0,3,5\}$, $\{0,4,6\}$, $\{0,3,4\}$; for S_3^* are $\{0,2,7\}$, $\{0,1,3\}$, $\{0,4,5\}$, $\{0,2,5\}$, $\{0,3,4\}$; and for S_4^* are $\{0,3,6\}$, $\{0,1,6\}$, $\{0,1,2\}$, $\{0,2,4\}$, $\{0,3,7\}$.

Example 5.4 (*Full metamorphosis of a* 6*-fold block design of order* 12 *into* 6*-fold triple systems.*)

Let (X, B) bé the block design with vertex set $X = \{\infty\} \cup \mathbb{Z}_{11}$, block set B, and starter blocks $[\infty,0,1,4], [\infty,0,2,5], [0,1,4,6]$, $[0,1,2,8]$, $[0,2,4,5]$, $[0,4,8,10]$.

Then the starter blocks for S_1^* are $\{\infty,0,1\}$, $\{\infty,0,3\}$, $\{\infty,0,4\}$, $\{0,1,3\}$, $\{0,1,5\}$, $\{0,2,6\}$; for S_2^* are $\{\infty,0,2\}$, $\{0,1,2\}, \{0,1,4\}, \{0,2,6\}$, $\{0,4,8\}, \{0,5,10\}$; for S_3^* are $\{\infty,0,5\}$, $\{0,1,3\}$, $\{0,1,3\}$, $\{0,1,3\}$, $\{0,2,4\}$, $\{0,4,8\}$; and for S_4^* are $\{\infty,0,3\}$, $\{0,2,5\}$, $\{0,2,5\}$, $\{0,5,10\}$, $\{0,3,7\}$, $\{0,5,10\}$.

Example 5.5 (*Full metamorphosis of a* 6*-fold block design of order* 14 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \{\infty\} \cup \mathbb{Z}_{13}$, block set B, and starter blocks $[6,0,1,4]$, $[12,0,2,8]$, $[11,0,3,4]$, $[9,0,6,8]$, $[5,0,3,12]$, $[\infty,1,0,4]$, $[0,\infty,7,2]$.

Then the starter blocks for S_1^* are $\{0,2,7\}$, $\{0,1,4\}$, $\{0,2,8\}$, $\{0,1,4\}$, $\{0,2,7\}$, $\{\infty,0,2\}$, $\{\infty,0,6\}$; for S_2^* are $\{0,2,6\}$, $\{0,1,6\}$, $\{0,1,5\}$, $\{0,1,4\}$, $\{0,2,5\}$, $\{\infty,0,1\}$, $\{\infty,0,3\}$; for S_3^* are $\{0,3,6\}$, $\{0,1,4\}$, $\{0,2,5\}$, $\{0,1,3\}$, $\{0,2,8\}$, $\{0,1,6\}$, $\{\infty,0,4\}$; and for S_4^* are $\{0,1,5\}$, $\{0,2,6\}$, $\{0,4,5\}$, $\{0,1,4\}$, $\{0,4,6\}$, $\{0,2,5\}$, $\{\infty,0,6\}$.

Example 5.6 (*Full metamorphosis of a* 6*-fold block design of order* 15 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{15}$, block set B, and starter blocks $[0,2,5,6], [0,4,10,12], [0,5,8,9], [0,1,3,10], [0,1,6,8], [0,1,3,4], [0,2,6,10]$.

Then the starter blocks for S_1^* are $\{0,1,2\}$, $\{0,1,4\}$, $\{0,2,8\}$, $\{0,3,8\}$, $\{0,3,9\}$, $\{0,4,9\}$, $\{0,5,10\}$; for S_2^* are $\{0,1,2\}$, $\{0,1,3\}$, $\{0,2,5\}$, $\{0,2,8\}$, $\{0,3,7\}$, $\{0,4,8\}$, $\{0,4,9\}$; for S_3^* are $\{0,2,7\}$, $\{0,1,3\}$, $\{0,2,5\}$, $\{0,1,2\}$, $\{0,3,7\}$, $\{0,3,9\}$, $\{0,4,9\}$; and for S_4^* are $\{0,1,2\}$, $\{0,1,7\}$, $\{0,2,8\}$, $\{0,3,7\}$, $\{0,3,8\}$, $\{0,4,8\}$, $\{0,4,9\}$.

Example 5.7 (*Full metamorphosis of a* 6*-fold block design of order* 18 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \{\infty\} \cup \mathbb{Z}_{17}$, block set B, and starter blocks $[0,13,4,1]$, $[0,3,12,5]$, $[0,9,2,15]$, $[0,10,6,11]$, $[0,13,1,16]$, $[0,5,3,14]$, $[0,15,9,8]$, $[\infty,0,3,7]$, $[0,\infty,1,7]$.

Then the starter blocks for S_1^* are $\{0,1,3\}, \{0,5,11\}, \{0,4,9\}$, $\{0,4,12\}$, $\{0,2,3\}$, $\{0,1,4\}$, $\{0,2,10\}$, $\{\infty,0,1\}$, $\{\infty,0,7\}$; for S_2^* are $\{0,2,5\}$, $\{0,1,7\}$, $\{0,2,8\}$, $\{0,2,9\}$, $\{0,4,9\}$, $\{0,5,13\}$, $\{0,3,7\}$, $\{\infty,0,3\}$, $\{\infty,0,7\}$; for S_3^* are $\{0,4,8\}$, $\{0,1,6\}$, $\{0,5,11\}$, $\{0,2,9\}$, $\{0,3,7\}$, $\{0,4,9\}$, $\{0,1,3\}$, $\{0,1,3\}$, $\{\infty,0,6\}$; and for S_4^* are $\{0,1,6\}$, $\{0,2,8\}$, $\{0,3,7\}$, $\{0,2,10\}$, $\{0,1,6\}$, $\{0,5,6\}$, $\{0,1,4\}$, $\{0,3,10\}$, $\{\infty,0,2\}$.

Example 5.8 (*Full metamorphosis of a* 6*-fold block design of order* 23 *into* 6*-fold triple systems.*)

Let (X, B) be the block design with vertex set $X = \mathbb{Z}_{23}$ and block set $B = \{[0+i, 2^x+i, 4\times 2^x+i, 17\times 2^x+i] \mid 0 \le i \le 23, 0 \le x \le 10\}$, where addition is done modulo 23.

Then the starter blocks for S_1^*, S_2^*, S_3^* and S_4^* are $\{0, 6 \times 2^j, 13 \times 2^j\}$, $0 \leq j \leq 10$.

Lemma 5.9 *There exists a* 6*-fold block design of every order* $n \geq 4$ *having a full metamorphosis into* 6*-fold triple systems.*

Proof The cases $n = 6, 8, 11, 12, 14, 15, 18$, and 23 are handled in Examples 5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, and 5.8. The cases $n = 4, 5, 7, 9, 10$, and 19 are handled in Examples 3.1, 3.2, 3.3, 3.4, 4.1 and 4.2. For the remaining cases take a PBD of order n with block sizes $4, 5, 6$ and 7, and place a 6-fold block design of order $4, 5, 6$, or 7 having a full metamorphosis into 6-fold triple systems on each block. □

6 Summary

We can paste together solutions for $\lambda = 1, 2, 3$, and 6 to obtain solutions for all other values of λ. The following table gives a summary of the results in this paper.

λ (mod 6)	Spectrum of λ-fold block designs having a full metamorphosis into λ-fold triple systems
1 or 5	$n \equiv 1 \pmod{12}$
2 or 4	$n \equiv 1 \pmod{3}$
3	$n \equiv 1 \pmod{4}$
0	$n \geq 4$

References

[1] A. E. Brouwer, Optimal packings of K_4's into K_n, J. Combinatorial Theory A 26 (1979), 278-297.

[2] C. J. Colbourn, J. H. Dinitz (Eds), The CRC Handbook of Combinatorial Designs (1996), CRC Press, Boca Raton, FL.

[3] S. Küçükçifçi and C. C. Lindner, *The metamorphosis of λ-fold block designs with block size four into λ-fold kite systems*, JCMCC 40 (2002), 241-252.

[4] S. Küçükçifçi and C. C. Lindner, *The metamorphosis of λ-fold block designs with block size four into $K_4 \setminus e$ designs, $\lambda \geq 2$*, Util. Math. 63 (2003), 239-254.

[5] S. Küçükçifçi, C. C. Lindner, and A. Rosa *The metamorphosis of λ-fold block designs with block size four into a maximum packing of λK_n with 4-cycles*, Discrete Math. 278 (2004), no. 1-3, 175-193.

[6] S. Küçükçifçi *The metamorphosis of λ-fold block designs with block size four into maximum packings of λK_n with kites*, Util. Math. 68 (2005), 165-195.

[7] C. C. Lindner and A. Rosa, *The metamorphosis of λ-fold block designs with block size four into λ-fold triple systems*, J. Statist. Plann. Inference 106 (2002), no. 1-2, 69-76.

[8] C. C. Lindner and A. Rosa, *The metamorphosis of block designs with block size four into $(K_4 \setminus e)$-designs*, Util. Math. 61 (2002), 33-46.

[9] C. C. Lindner and Anne Street, *The metamorphosis of λ-fold block designs with block size four into λ-fold 4-cycle systems*, Bulletin of the ICA 28 (2000), 7-18.

[10] E. S. Yazici, *Metamorphosis of 2-fold 4-cycle systems into maximum packings of 2-fold 6-cycle systems*, Australas. J. Combin. 32 (2005), 331–338.