Colourfully panconnected subgraphs II

Douglas R. Woodall
School of Mathematical Sciences
University of Nottingham
Nottingham NG7 2RD, UK
douglas.woodall@nottingham.ac.uk

Abstract

Let G be a connected k-colourable graph of order $n \geqslant k$. A subgraph H of G is *k-colourfully panconnected in G* if there is a k-colouring of G such that the colours are close together in H, in two different senses (called *variegated* and *panconnected*) to be made precise. Let $s_k(G)$ denote the smallest number of edges in a spanning k-colourfully panconnected subgraph H of G. It is conjectured that $s_k(G) = n - 1$ if $k \geqslant 4$ and G is not a circuit (a connected 2-regular graph) with length $\equiv 1 \pmod{k}$. It is proved that $s_k(G) = n-1$ if G contains no circuit with length $\equiv 1 \pmod{k}$, and $s_k(G) \leqslant 2n - k - 1$ whenever $k \geqslant 4$.

Keywords: Distance-k-connected; Variegated colouring; Panconnected colouring; Spanning tree.

1 Introduction

Throughout this paper, all graphs are finite, and all colourings are assumed to be proper vertex-colourings. We write $d_G(v)$ for the degree of vertex v in graph G, and $d_G(u, v)$ is the distance between vertices u and v in G. If P is a path, then its *length* $l(P)$ is the number of edges in P.

A set X of vertices in a graph G is *distance-k-connected* if, for each two vertices u, v of X, there is a sequence $u = x_0, x_1, \ldots, x_l = v$ of vertices of X such that $d_G(x_{i-1}, x_i) \leqslant k$ for each i ($1 \leqslant i \leqslant l$). This is the same as saying that $G^k[X]$ is connected, where $G^k[X]$ is the subgraph induced by X in G^k, the graph with the same vertex-set as G in which two vertices are adjacent if and only if they are distance at most k apart in G. (One can imagine that there is an animal that lives on the graph G and feeds at vertices in X. It cannot travel more than k edges without stopping at a vertex in X to feed. Then it can travel from any vertex in X to any other vertex in X if and only if X is distance-k-connected.)

The following theorem was proved by Ouyang [3, 4] with help from J. Griggs and E. Czabarka.

Theorem 1.1. [3, 4] *Let G be a connected k-chromatic graph. Then G has a k-colouring such that each colour class is distance-k-connected in G.*

This result was conjectured by Chen, Schelp and Shreve [2], who noted that 'distance-k-connected' is best possible (consider two copies of K_k connected by a long path).

If $v \in V(G)$, then a k-colouring of G is *variegated at v* if the k colours can be ordered as $c_0, c_1, \ldots, c_{k-1}$ in such a way that, for each i, colour c_i occurs on a vertex within distance i of v (so, in particular, v has colour c_0). This says that, for each i ($1 \leqslant i \leqslant k-1$), at least $i+1$ different colours occur on vertices within distance i of v (including v itself). A k-colouring of G is *variegated* if it is variegated at every vertex of G. Note that this requires all k colours to occur on vertices of G, so that although a variegated k-colouring of G is an l-colouring for each $l \geqslant k$, it is not a *variegated* l-colouring for any $l \neq k$.

A k-colouring of G is *panconnected* if, for each i ($1 \leqslant i \leqslant k$), the union of each i colour classes is distance-$(k+1-i)$-connected in G. If G is connected then this condition holds automatically when $i = k$ or $k-1$.

Borodin and Woodall [1] modified Ouyang's proof of Theorem 1.1 to prove the following slightly stronger result.

Theorem 1.2. [1] *Let G be a connected k-colourable graph with at least k vertices. Then G has a variegated panconnected k-colouring.*

As in [1], we define a subgraph H of G to be *k-colourfully panconnected (k-cp) in G* if G has a k-colouring that induces a variegated panconnected k-colouring of H. Clearly this implies that H is connected and has at least k vertices. If G is a connected k-colourable graph with at least k vertices, let

$$s_k(G) := \min\{|E(H)| : H \text{ is a spanning } k\text{-cp subgraph of } G\}.$$

This makes sense because, by Theorem 1.2, G is a spanning k-cp subgraph of itself.

If G is a connected bipartite graph of order n then it is easy to see that $s_2(G) = n-1$, and that any spanning tree will do for H. The following results are proved in [1].

Theorem 1.3. [1] *Let G be a k-colourable connected graph with order $n \geqslant k$ and minimum degree δ.*

(a) *If* $G \cong C_n$ *then*

$$s_k(G) = \begin{cases} n & \text{if } n \equiv 1 \pmod{k}, \\ n-1 & \text{otherwise.} \end{cases}$$

(b) *If* $G \cong K_{r,s}$*, so that* $n = r + s$ *and* $\delta = \min\{r, s\}$*, then*

$$s_k(G) = \begin{cases} n+\delta-2 & \text{if } k = 3, \\ n-1 & \text{if } k \geqslant 4. \end{cases}$$

(c) *If* $k \in \{3,4\}$ *then* $s_k(G) \leqslant n + \delta - 2$.

(d) *If* $k \in \{3,4\}$ *and* G *has an edge that is contained in no circuit with length* $\equiv 1 \pmod{k}$*, then* $s_k(G) = n - 1$.

(e) *If* G *is* k*-chromatic and contains a vertex that is adjacent to vertices of all other colours in every* k*-colouring of* G*, then* $s_k(G) \leqslant n$.

In [1] it is conjectured that if G is a connected k-colourable graph with order $n \geqslant k \geqslant 4$, then $s_k(G) \leqslant n$. In fact, one can make the following stronger conjecture.

Conjecture 1. *If* G *is a connected* k*-colourable graph of order* $n \geqslant k \geqslant 4$*, and* G *is not a circuit with length* $\equiv 1 \pmod{k}$*, then* $s_k(G) = n - 1$*; that is,* G *has a* k*-colourfully panconnected spanning tree.*

In this paper we will prove the following two new results. Theorem 1.4 seems very specialized, but it does, for example, show that Conjecture 1 holds if G is bipartite and k is even. Theorem 1.5 gives an upper bound on $s_k(G)$ for each $k \geqslant 4$; Theorem 1.3(c), together with a previous observation about the case $k = 2$, gives an upper bound for each $k \leqslant 4$, and so we now have two different upper bounds when $k = 4$.

Theorem 1.4. *Let* G *be a connected* k*-colourable graph of order* $n \geqslant k \geqslant 2$*, and suppose that* G *contains no circuit with length* $\equiv 1 \pmod{k}$*. Then* $s_k(G) = n - 1$.

Theorem 1.5. *Let* G *be a connected* k*-colourable graph of order* $n \geqslant k \geqslant 4$*. Then* $s_k(G) \leqslant 2n - k - 1$.

2 Proof of Theorem 1.4

The following result is taken from [1], where it is used in the proofs of parts (c)–(e) of Theorem 1.3. We will use it in proving Theorem 1.4, and we include a brief proof of it for completeness.

Lemma 2.1. *Let v_0 be a vertex in a connected k-colourable graph G. Then there is a k-colouring $\gamma_0 : V(G) \to \{0, 1, \ldots, k-1\}$, and a spanning tree T of G, such that $\gamma_0(v) \equiv d_T(v_0, v) \pmod k$ for each vertex v of G.*

Proof. Choose a k-colouring γ of G and permute colours if necessary so that $\gamma(v_0) = 0$. Set $T := \{v_0\}$.

While $V(T) \neq V(G)$, proceed as follows. If there is an edge $uw \in E(G)$ such that $u \in V(T)$, $w \in V(G) \setminus V(T)$, and $\gamma(w) \equiv \gamma(u) + 1 \pmod k$, then choose such an edge and add uw and w to T. If there is no such edge uw, then reduce the colour of every vertex in $V(G) \setminus V(T)$ by 1 (mod k). Let T and γ_0 denote the final tree T and colouring γ constructed by this procedure, with $V(T) = V(G)$; it is clear that T and γ_0 have the required property. □

We now prove Theorem 1.4. Let G satisfy the hypotheses of the theorem.

If T is a spanning tree of G with the property in Lemma 2.1, and v is any vertex of T, consider the set $\mathcal{P}(v)$ of paths in T having v as one endvertex and otherwise containing only vertices that are further from the root v_0 than v is. Let $P_1(v)$ be a longest path in $\mathcal{P}(v)$, let $P_2(v)$ be a longest path in $\mathcal{P}(v)$ having no edges in common with $P_1(v)$, and let $l_\epsilon(v) := l(P_\epsilon(v))$ ($\epsilon = 1, 2$). (Recall that $l(P)$ denotes the length of path P. If $v \neq v_0$ and $d_T(v) \leqslant \epsilon$, then $P_\epsilon(v) = \{v\}$, a 1-vertex path, and $l_\epsilon(v) = 0$ ($\epsilon = 1, 2$); and $l_2(v_0) = 0$ if $d_T(v_0) = 1$.)

Choose a vertex v_0, a spanning tree T and a k-colouring γ_0 of G, with the property in Lemma 2.1, in such a way that $l_2(v_0)$ is as large as possible.

Claim 2.1. *For each vertex v of T, $l_1(v) \leqslant l_1(v_0)$ and $l_2(v) \leqslant l_2(v_0)$.*

Proof. It is clear from the definition of l_1 that $l_1(v) \leqslant l_1(v_0)$. (Indeed, $d_T(v_0, v) + l_1(v) \leqslant l_1(v_0)$.) So suppose that $l_2(v, T) > l_2(v_0, T)$, where we temporarily specify the tree T by writing $l_\epsilon(v, T)$ and $P_\epsilon(v, T)$ in place of $l_\epsilon(v)$ and $P_\epsilon(v)$ ($\epsilon = 1, 2$). Subtract $\gamma_0(v) \pmod k$ from the colour of every vertex in G, and let T_v be the union of all paths in $\mathcal{P}(v)$, so that T_v is a subtree of T containing $P_1(v, T)$ and $P_2(v, T)$. Now use the procedure in the proof of Lemma 2.1 to extend T_v to a new spanning tree T' of G with v as its root, with an associated colouring γ_0' of G, satisfying the property in Lemma 2.1. But then $l_2(v, T') \geqslant l_2(v, T) > l_2(v_0, T)$, contrary to the choice of T. This contradiction completes the proof of Claim 2.1. □

Let $P_0 := P_1(v_0) \cup P_2(v_0)$ and let $l := l(P_0) = l_1(v_0) + l_2(v_0)$. For $\epsilon = 1, 2$, let v_ϵ denote the final vertex of $P_\epsilon(v_0)$, so that v_1, v_2 are the endvertices of P_0. If P is any path in T, let v be the (unique) vertex of P that is closest to v_0. Then

$$l(P) \leqslant l_1(v) + l_2(v) \leqslant l_1(v_0) + l_2(v_0) = l, \tag{1}$$

using Claim 2.1. In particular, T has diameter (exactly) l.

If $l_2(v) = 0$ for all v then T is a path, and it is easy to see that T is k-colourfully panconnected in G; thus we may assume that $l_2(v) > 0$ for some vertex v. It now follows from Claim 2.1 that $l_2(v_0) > 0$. Let T'' be the component of $T - v_0$ containing v_2, and let $T_1 := T - T''$ and $T_2 := T[V(T'') \cup \{v_0\}]$. Then $T_1 \cup T_2 = T$, $T_1 \cap T_2 = \{v_0\}$, and $P_\epsilon(v_0) \subseteq T_\epsilon$ ($\epsilon = 1, 2$).

For each i ($1 \leqslant i \leqslant k-1$), no vertex of T_2 with colour i is adjacent in G to any vertex of T_1 with colour $k-i$, since by the hypothesis of the theorem G contains no circuit with length $\equiv 1 \pmod k$. So for all i ($1 \leqslant i < \frac{1}{2}k$) simultaneously, interchange the colours i and $k-i$ on all vertices in T_2. The result is a proper k-colouring of G, in which every colour that is used at all is used on a vertex of P_0. If only $r < k$ colours are used, then choose $k-r$ vertices not in P_0 and recolour them with the $k-r$ unused colours, so that every colour is used on at least one vertex. Let the resulting colouring be denoted by γ. We will prove that γ is variegated and panconnected when regarded as a k-colouring of T, so that T is a k-colourfully panconnected spanning tree of G.

A path P in T will be called *monotonic* if, for each pair of vertices u, w in P at the same distance from v_0, one of u, w is in T_1 and the other is in T_2. We note the following facts about a monotonic path P: if $s \leqslant \min\{|V(P)|, k\}$, then any s consecutive vertices of P have s different colours; and if there are vertices $u, w \in P$ that have the same colour, then $d_T(u, w)$ is a multiple of k.

Claim 2.2. *The colouring γ is a variegated k-colouring of T.*

Proof. Let u be a vertex of T, let v denote the closest point of P_0 to u, and let the paths from v to v_1, v_2 and u be P_1, P_2 and P_3, respectively, where P_j has length l_j ($j = 1, 2, 3$). Let $\underline{l} := \min\{l_1, l_2\}$ and $\bar{l} := \max\{l_1, l_2\}$. Let $u \in T_\epsilon$ where $\epsilon = 1$ or 2; clearly then $v \in T_\epsilon$ also. If $\epsilon = 1$ then $l_3 \leqslant l_1 = l_1(v)$ and $l_3 \leqslant l_2(v) \leqslant l_2(v_0) \leqslant l_2$ (using Claim 2.1). If $\epsilon = 2$ then $l_3 \leqslant l_2 = l_1(v) \leqslant l_2(v_0) \leqslant l_1(v_0) \leqslant l_1$. Either way, $l_3 \leqslant \underline{l}$. Let $c_i(u)$ denote the number of colours that occur on vertices within distance i of u in T, including u itself. We must prove that for each i ($1 \leqslant i \leqslant k-1$), $c_i(u) \geqslant i + 1$.

Let T^* denote the subtree $P_1 \cup P_2 \cup P_3$ of T. By considering the vertices within distance i of u in T^*, one can see that if $0 \leqslant i \leqslant l_3 + \bar{l}$ then $c_i(u) \geqslant \min\{k, f(i)\}$, where $f(0) := 1$ and

$$f(i) - f(i-1) := \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 2l_3, \\ 2 & \text{if } 2l_3 < i \leqslant l_3 + \underline{l}, \\ 1 & \text{if } l_3 + \underline{l} < i \leqslant l_3 + \bar{l}. \end{cases} \tag{2}$$

This is because the first l_3 vertices after v along the path $P := P_3 \cup P_\epsilon$ (when traced from u to v_ϵ) may have the same colours as the first l_3 vertices of P, but otherwise, as i increases, there is no repetition of colours in T^* unless i gets so large that all colours have already occurred. It follows from (2) that $c_i(u) \geqslant i+1$ if $i \leqslant \min\{k, f(l_3+\bar{l})\} - 1$. Now, since $l_3 \leqslant \underline{l}$,

$$f(l_3+\bar{l}) = f(0) + l_3 + \bar{l} + \underline{l} - l_3 = 1 + \bar{l} + \underline{l} = 1 + l_1 + l_2 = l + 1.$$

Thus $c_i(u) \geqslant i+1$ if $i \leqslant \min\{k-1, l\}$. But $c_i(u) = k \geqslant i+1$ if $l < i \leqslant k-1$, since T has diameter l and every colour occurs in T. This proves Claim 2.2. □

Claim 2.3. *The colouring γ is a panconnected k-colouring of T.*

Proof. Let $1 \leqslant i \leqslant k$ and let X be a union of $k+1-i$ colour classes of γ. We must prove that X is distance-i-connected in T. Let a path be called *good* if it does not contain i consecutive vertices that are not in X. Clearly every monotonic path is good. If $u, w \in V(T)$, let $P(u,w)$ denote the path from u to w in T.

Let $u, w \in X$. It suffices to prove that either $P(u,w)$ is good, or there is a vertex $x \in X$ such that $P(u,x)$ and $P(x,w)$ are both good. The former holds if $P(u,w)$ is monotonic; so let us assume that it is not. Then either $u, w \in T_1$ or $u, w \in T_2$; say $u, w \in T_\epsilon$, where $\epsilon = 1$ or 2, and let $\bar{\epsilon} := 3-\epsilon$. Let v be the closest point of $P(u,w)$ to v_0, and let $a := d_T(v_0, v)$, $b := d_T(v,u)$ and $c := d_T(v,w)$. We may assume w.l.o.g. that u and w are the only vertices of X in $P(u,v) \cup P(v,w)$, and that $b \leqslant c$. If there is any vertex x of X on $P(v_{\bar{\epsilon}}, v)$, then the paths $P(u,x)$ and $P(x,w)$ are both monotonic and hence good; so assume there is no such vertex x. Then the monotonic path $P(v_{\bar{\epsilon}}, w)$ has $l_{\bar{\epsilon}}(v_0) + a + c$ consecutive vertices (all but its last vertex, in fact) that are not in X, and it follows that

$$l_2(v_0) + a + c \leqslant l_{\bar{\epsilon}}(v_0) + a + c \leqslant i - 1.$$

But $l(P(u,w)) = b + c$ and $b \leqslant l_2(v) \leqslant l_2(v_0)$, and so $l(P(u,w)) \leqslant l_2(v_0) + c \leqslant i-1$. Thus $P(u,w)$ is a good path, and Claim 2.3 is proved. □

Claims 2.2 and 2.3 together prove Theorem 1.4.

3 Proof of Theorem 1.5

In this section we prove Theorem 1.5, that if G is a connected k-colourable graph with order $n \geqslant k \geqslant 4$, then $s_k(G) \leqslant 2n-k-1 = 2(n-k)+k-1$. We know already that if G is a path or a circuit then $s_k(G) \leqslant n$, with strict inequality unless $n \equiv 1 \pmod{k}$, so that $s_k(G) = n-1 = 2n-k-1$ if

$n = k$; since $n \leqslant 2n - k - 1$ if $n \geqslant k + 1$, we may assume henceforth that G is not a path or a circuit and so has maximum degree $\Delta(G) \geqslant 3$.

We are indebted to the referee of [1] for suggesting the following simple lemma, which is included in [1] and used there several times.

Lemma 3.1. *Suppose γ is a k-colouring of a graph G that is variegated at some vertex v whose colour class is distance-k-connected in G. Suppose X is the union of i colour classes ($1 \leqslant i \leqslant k$) and $X \setminus \{v\}$ is distance-$(k+1-i)$-connected in $G - v$. Then X is distance-$(k+1-i)$-connected in G.*

Proof. If X is the colour class of v, then this holds by hypothesis. Otherwise, since γ is variegated at v, v is within distance $k+1-i$ of some vertex in $X \setminus \{v\}$. □

If $H \subseteq G$, let $\partial(H)$ be the set of all vertices of H that have neighbours in $G - V(H)$, and let $\bar{\partial}(H)$ be the set of all vertices of $G - V(H)$ that have neighbours in H: $\partial(H) := V(H) \cap N_G(V(G) \setminus V(H))$ and $\bar{\partial}(H) := N_G(V(H)) \setminus V(H)$.

We say that a k-colouring of a graph G is *strongly variegated* at a vertex v if, for each i ($2 \leqslant i \leqslant k-2$), at least $i+2$ different colours occur on vertices within distance i of v (including v itself); note that there may be only two different colours within distance 1 of v.

For a natural number j, define $\Gamma_j := \{0, 1, \ldots, j-1\}$.

Lemma 3.2. *Let G be a connected k-colourable graph with order $n \geqslant k \geqslant 4$ and maximum degree $\Delta(G) \geqslant 3$. Then G has a k-colouring γ_k, and a subtree T_k of order k, such that all vertices of T_k have different colours, and the colouring $\gamma_k|T_k$ of T_k induced by γ_k is variegated at every vertex of T_k and strongly variegated at every vertex of $\partial(T_k)$ with at most one exception.*

Proof. Let G have chromatic number $\chi = \chi(G) \geqslant 2$, and let $l := \max\{4, \chi\}$.

Claim 3.1. *G has an l-colouring γ_l, and a substar T_l of order l, such that all vertices of T_l have different colours.*

Proof. Let $\gamma : V(G) \to \Gamma_\chi$ be a χ-colouring of G. Let u be a vertex with degree at least 3 and suppose w.l.o.g. $\gamma(u) = 0$. Let T_l be a substar of G with u as its centre and leaves $u_1, \ldots, u_{l-1}$.

If $\chi = 2$ then recolour u_i with colour i ($i = 2, 3$) to obtain the colouring $\gamma_l = \gamma_4$.

If $\chi = 3$ we can choose u to have neighbours with both other colours. For, suppose that this is impossible. Choose γ and u as above so that every neighbour of u has colour 1 and u is as close as possible to a vertex with colour 2. Let $u = v_0, v_1, \ldots, v_d$ be a shortest path from u to a vertex v_d with colour 2. By assumption, $d \geqslant 2$, and $v_0, \ldots, v_{d-1}$ are coloured alternately 0 and 1, and $v_1, \ldots, v_{d-1}$ have degree 2 since otherwise we could reduce d by taking one of these vertices to be u. Moreover, all neighbours of v_d other than v_{d-1} have the same colour c, since there is only one such neighbour if v_d has degree 2, and we are assuming that no vertex with degree 3 or more has neighbours with two different colours. So if $d = 2$ then we can recolour v_1 with colour 2 and recolour v_d with a colour different from both 2 and c, and if $d > 2$ we can recolour v_1 with colour 2 and make no other changes. Then u has neighbours with both other colours, as required. Choose $T_l = T_4$ so that u_i has colour i ($i = 1, 2$), and recolour u_3 with colour 3 to form the colouring $\gamma_l = \gamma_4$.

Finally, if $l = \chi \geqslant 4$ then $\gamma_l = \gamma$. Choose u to be any vertex of colour 0 that has every other colour among its neighbours, and choose u_i to be a neighbour with colour i ($i = 1, \ldots, l-1$). Note that such a vertex u must exist, since otherwise every vertex with colour 0 could be recoloured with a different colour, which is impossible since G is χ-chromatic.

In all cases, Claim 3.1 is proved. □

We complete the proof of Lemma 3.2 as follows. For $i = l, l+1, \ldots, k-1$ in turn, we choose adjacent vertices $u_i \in \partial(T_i)$ and $w_i \in \bar{\partial}(T_i)$, recolour w_i with colour i (which is previously unused in G) to form a new colouring γ_{i+1}, and set $T_{i+1} := T_i \cup \{u_i w_i, w_i\}$. If $i > l$ and $w_{i-1} \in \partial(T_i)$ then we choose $u_i := w_{i-1}$; otherwise there is no restriction on the choice of u_i and w_i. Note that T_l has diameter 2, and the restriction $\gamma_l|T_l$ of γ_l to T_l is strongly variegated (as an l-colouring) at every vertex of T_l. Thus, for $i \geqslant l$, T_i has diameter at most $2 + i - l \leqslant i - 2$, and if the i-colouring $\gamma_i|T_i$ is variegated at u_i and strongly variegated at every other vertex of $\partial(T_i)$, then the $(i+1)$-colouring $\gamma_{i+1}|T_{i+1}$ is variegated at w_i and strongly variegated at u_i and at every vertex of $\partial(T_{i+1})$ except possibly at w_i. It follows inductively that the final colouring γ_k and tree T_k have all the required properties, and so Lemma 3.2 is proved. □

Let γ_0 and H_0 denote the colouring γ_k and subtree T_k whose existence was proved in Lemma 3.2. In the following lemma, we will form a sequence of colourings and subgraphs of G, which (ignoring a slight clash of terminology) we will call γ_i and H_i, in such a way that $H_0 \subseteq H_1 \subseteq \ldots$, until we reach a subgraph H_t with order n. Unfortunately, we have not been able to do this in such a way that every union of $k-2$ colour classes is distance-3-connected in H_t, and so we may need to add further edges afterwards in order to join the distance-3-connected components (defined in an obvious way, see below) together. To keep track of how many further edges need to be added, let $\omega(c_1, c_2, \gamma_i, H_i)$ denote the number of distance-3-connected components in H_i of the set $X := \{v \in V(H_i) : \gamma_i(v) \notin \{c_1, c_2\}\}$ (that is, $\omega(c_1, c_2, \gamma_i, H_i)$ is the number of components of the graph $H_i^3[X]$); and let $\Omega(\gamma_i, H_i)$ be the sum of $\omega(c_1, c_2, \gamma_i, H_i) - 1$ over all unordered pairs of distinct colours $c_1, c_2 \in \Gamma_k$.

Let us say that a k-colouring of a graph H is *weakly panconnected* if, for each i $(1 \leqslant i \leqslant k)$ except possibly for $i = k-2$, the union of each i colour classes is distance-$(k-i+1)$-connected in H.

The following lemma is a strengthening of Theorem 1.2, and it is proved by modifying the proof of that theorem in [1], which is in turn a modification of the proof of Theorem 1.1 used by Ouyang in [3].

Lemma 3.3. *Let G be a connected k-colourable graph with order $n \geqslant k \geqslant 4$ and maximum degree $\Delta(G) \geqslant 3$. Then G has a variegated panconnected k-colouring γ, and a spanning subgraph H, such that the colouring $\gamma|H$ of H induced by γ is variegated and weakly panconnected, and H has at most $2(n-k)+(k-1)(1-\Omega(\gamma, H))$ edges.*

Proof. Let γ_0 and H_0 denote the colouring γ_k and subtree T_k whose existence was proved in Lemma 3.2. We will form a sequence of k-colourings γ_i and subgraphs H_i of G with the following properties.

P1. $|V(H_i)| \geqslant k+i$ and $|E(H_i)| \leqslant 2(|V(H_i)|-k)+(k-1)(1-\Omega(\gamma_i, H_i))$.

P2. γ_i is a k-colouring of G that induces a variegated weakly panconnected k-colouring $\gamma_i|H_i$ of H_i.

P3. $\gamma_i|H_i$ is strongly variegated at all but at most one of the vertices in $\partial(H_i)$.

P4. For each k-colouring γ of G such that $\gamma|H_i = \gamma_i|H_i$, and for each set $X \subseteq V(G)$ that is the union of $k-2$ colour classes of γ, all vertices of $V(H_i) \cap X$ are contained in the same component of $G^3[X]$.

These properties all hold when $i = 0$, since it is easy to see from Lemma 3.2 that $\gamma_0|H_0$ is panconnected and not just weakly panconnected, so that $\Omega(\gamma_0, H_0) = 0$, and for every set X in P4, $V(H_0) \cap X$ is distance-3-connected in H_0 and so is contained in a component of $H_i^3[X]$ and hence in a component of $G^3[X]$. Since $|V(H_0)| = k$ and $|E(H_0)| = k-1$, this proves P1 and P4 and part of P2; and P3 and the rest of P2 follow directly from Lemma 3.2.

Suppose now that we have defined γ_i and H_i so that properties P1–P4 hold, for some $i \geqslant 0$. Suppose first that $\partial(H_i) = \emptyset$; then $|V(H_i)| = n$. Define $\gamma := \gamma_i$ and $H := H_i$. At this point P3 is irrelevant — it is needed only in the construction of H. P2 implies that γ is a variegated weakly panconnected k-colouring of G (as well as inducing such a colouring of H), and P4 implies that γ is actually panconnected (as a colouring of G) and not just weakly panconnected. The required upper bound on $|E(H)|$ follows from P1, and so Lemma 3.3 is proved in this case.

So suppose $\partial(H_i) \neq \emptyset$. We must show how to define γ_{i+1} and H_{i+1} so that properties P1–P4 all hold for (γ_{i+1}, H_{i+1}). Let u be the unique vertex of $\partial(H_i)$ at which $\gamma_i|H_i$ is not strongly variegated, if there is one; otherwise let u be any vertex of $\partial(H_i)$. Let $w \in N(u) \setminus V(H_i)$. Let c_u denote a colour whose closest occurrence to u in H_i is as far from u as possible (but within distance $k-1$ of u, since $\gamma_i|H_i$ is variegated at u).

Suppose first that $\gamma_i(w) = c_u$. Let $\gamma_{i+1} := \gamma_i$ and $H_{i+1} := H_i \cup \{uw, w\}$. We must verify that properties P1–P4 hold for (γ_{i+1}, H_{i+1}). It is easy to see that $\gamma_{i+1}|H_{i+1}$ is strongly variegated at u (so that P3 holds) and variegated at w. By Lemma 3.1, this implies that $\gamma_{i+1}|H_{i+1}$ is weakly panconnected since $\gamma_i|H_i$ is, which completes the proof of P2. By the same reasoning as in the proof of Lemma 3.1, if X is a union of $k-2$ colour classes of γ_i, then w is within distance 3 in H_{i+1} of some vertex of $X \cap V(H_i)$; this shows that $\Omega(\gamma_{i+1}, H_{i+1}) \leqslant \Omega(\gamma_i, H_i)$, and it implies that P4 holds, since we are assuming that P4 holds for (γ_i, H_i). Finally, P1 holds since $|V(H_{i+1})| = |V(H_i)| + 1$ and $|E(H_{i+1})| = |E(H_i)| + 1$. Thus properties P1–P4 all hold for (γ_{i+1}, H_{i+1}), with w being the exceptional vertex in P3 if there is one.

So we may assume that $\gamma_i(w) \neq c_u$. Suppose next that w has a neighbour $v \in V(H_i)$ which has colour c_u. Let $\gamma_{i+1} := \gamma_i$ and $H_{i+1} := H_i \cup \{uw, w, wv\}$. By the same reasoning as in the previous paragraph, one can verify that properties P1–P4 hold for (γ_{i+1}, H_{i+1}), with w again being the exceptional vertex in P3 if there is one. (It was wrongly stated in [1] that there is no exceptional vertex in this case. However, it is possible that $\gamma_{i+1}|H_{i+1}$ is not strongly variegated at w, if every neighbour of $\{u, v\}$ in H_i has colour $\gamma_i(w)$.)

So we may assume that $\gamma_i(w) \neq c_u$ and that w has no neighbour $v \in V(H_i)$ with colour c_u. Consider in turn each vertex of colour c_u or $\gamma_i(w)$ in $V(G) \setminus (V(H_i) \cup \{w\})$, and change its colour to be different from both c_u and $\gamma_i(w)$ if possible. Now let C be the component ('Kempe chain') containing w in the subgraph of G induced by all vertices of colour c_u and $\gamma_i(w)$. If $C \cap H_i = \emptyset$ then we can interchange colours c_u and $\gamma_i(w)$ throughout C so that w has colour c_u; if we call this new colouring γ_i', we can then proceed exactly as when $\gamma_i(w) = c_u$, except that instead of defining $\gamma_{i+1} := \gamma_i$ we define $\gamma_{i+1} := \gamma_i'$. (Note that P1–P4 hold for γ_i' since $\gamma_i'|H_i = \gamma_i|H_i$.) So we may suppose that $C \cap H_i \neq \emptyset$. Thus there exist adjacent vertices $x \in V(H_i)$ and $y \in V(G) \setminus V(H_i)$ such that $\gamma_i(x) = c_u$ and $\gamma_i(y) = \gamma_i(w)$ or *vice versa*. Note that y is adjacent in G to vertices of all other colours, since otherwise its colour would have been changed by the colour modifications at the start of this paragraph.

At this point, abusing terminology somewhat, we forget the original vertices u and w and relabel x, y as u, w, and we relabel the new colouring as γ_i. So u, w are adjacent vertices such that $u \in \partial(H_i)$, $w \in \bar{\partial}(H_i)$, $\gamma_i|H_i$

is strongly variegated at u, and w has G-neighbours with all colours other than its own. Since γ_i has not changed on any vertex of H_i, properties P1–P4 still hold. We may assume that if w has any neighbours outside H_i, then not all such neighbours have colour $\gamma_i(u)$; for, if they do, then w has neighbours in H_i with at least two colours different from $\gamma_i(u)$, and $\gamma_i|H_i$ is strongly variegated at all but at most one of these neighbours by P3, and so we can change u to be a neighbour of w with a colour different from that of the neighbours of w outside H_i.

Let c_u now denote a colour in $\Gamma_k \setminus \{\gamma_i(u), \gamma_i(w)\}$ whose closest occurrence to u in H_i is as far from u as possible, but within distance $k-2$ of u, since $\gamma_i|H_i$ is strongly variegated at u; let c'_u denote a colour in $\Gamma_k \setminus \{\gamma_i(u), \gamma_i(w), c_u\}$ whose closest occurrence to u in H_i is as far from u as possible, and if $k \geqslant 5$ let c''_u denote a colour in $\Gamma_k \setminus \{\gamma_i(u), \gamma_i(w), c_u, c'_u\}$ whose closest occurrence to u in H_i is as far from u as possible. Let w_u, w'_u be neighbours of w with colour c_u, c'_u respectively, chosen if possible to be not in H_i.

Suppose first that w has no neighbours outside H_i. Let $\gamma_{i+1} := \gamma_i$ and $H_{i+1} := H_i \cup \{uw, w, ww_u\}$. By the same argument as before, it is easy to see that properties P1–P4 hold for (γ_{i+1}, H_{i+1}). Note that $\gamma_{i+1}|H_{i+1}$ is not necessarily strongly variegated at w, but this does not matter since it is variegated at w and $w \notin \partial(H_{i+1})$.

So we may assume that w has at least one neighbour outside H_i. Recall that in this case not all such neighbours have colour $\gamma_i(u)$. Provisionally, define $\gamma_{i+1} := \gamma_i$, and form H_{i+1} from H_i by adding vertex w and all G-neighbours of w outside H_i, all edges joining w to its neighbours outside H_i, and the edges uw, ww_u and ww'_u; if exactly one of w_u, w'_u is in H_i, and $k \geqslant 5$, and all G-neighbours of w with colour c''_u are in H_i, then add also an edge joining w to a neighbour with colour c''_u; if neither w_u nor w'_u is in H_i, then if possible add a further edge joining w to a vertex $z \in N_G(w) \cap V(H_i)$ whose colour is in $\Gamma_k \setminus \{\gamma_i(u), \gamma_i(w), c_u, c'_u\}$. There are four cases.

Case 1: w_u and w'_u are both in H_i. Then in forming H_{i+1} from H_i we have added s vertices and $s+2$ edges, for some $s \geqslant 2$.

Case 2: exactly one of w_u and w'_u is in H_i. Then we have added s vertices and either $s+1$ or $s+2$ edges, for some $s \geqslant 2$.

Case 3: neither w_u nor w'_u is in H_i, and the vertex z exists. Then we have added s vertices and $s+1$ edges, for some $s \geqslant 3$.

Case 4: neither w_u nor w'_u is in H_i, and there is no vertex z. Then we have added s vertices and s edges, for some $s \geqslant k-1$.

We will prove the following facts.

F1. The colouring $\gamma_{i+1}|H_{i+1}$ is strongly variegated at all vertices in $V(H_{i+1}) \setminus V(H_i)$.

F2. The colouring $\gamma_{i+1}|H_{i+1}$ is weakly panconnected.

F3. $\Omega(\gamma_{i+1}, H_{i+1}) \leqslant \Omega(\gamma_i, H_i)$, and P4 holds for (γ_{i+1}, H_{i+1}), except possibly in Case 4, in which case it is possible to redefine γ_{i+1} if necessary so that $\Omega(\gamma_{i+1}, H_{i+1}) \leqslant \Omega(\gamma_i, H_i) + 1$ and P4 holds for (γ_{i+1}, H_{i+1}).

Together with the information about the numbers of vertices and edges given with the definitions of Cases 1–4 above, these facts imply that properties P1–P4 all hold for (γ_{i+1}, H_{i+1}), and this will finally complete the proof of Lemma 3.3.

To verify fact F1, let v be w or a neighbour of w outside H_i. Let j be an integer such that $2 \leqslant j \leqslant k-2$. Suppose first that $j \geqslant 4$. Since $\gamma_i|H_i$ is strongly variegated at u, there are at least j colours from $\Gamma_k \setminus \{c_u, c'_u\}$ that occur on vertices of H_i within distance $j-2$ of u, and so there are at least $j+2$ colours (including c_u and c'_u) that occur on vertices of H_{i+1} within distance j of v. The same conclusion holds if $j = 2$, since w and its neighbours have at least the four different colours $\gamma_i(u), \gamma_i(w), c_u, c'_u$. And it holds too if $j = 3$ and $k \geqslant 5$, since then w and its neighbours have at least five different colours; to see this in Case 1, note that in this case no neighbour of w outside H_i has colour c_u or c'_u, and recall that not all neighbours of w outside H_i have colour $\gamma_i(u)$. This proves F1.

To prove F2, we note that if $j = k-1$ or k then the union of any j colour classes is distance-$(k-j+1)$-connected, for any k-colouring of any connected graph. So suppose that $1 \leqslant j \leqslant k-3$, which implies that $2 \leqslant k-j-1 \leqslant k-2$, and let X be the union of any j colour classes of $\gamma_{i+1}|H_{i+1}$. Since $\gamma_i|H_i$ is strongly variegated at u, at least $k-j+1$ colours occur within distance $k-j-1$ of u in H_i, which means that there is a vertex of X within distance $k-j-1$ of u in H_i and hence within distance $k-j+1$ of each vertex of $V(H_{i+1}) \setminus V(H_i)$ in H_{i+1}. This proves F2.

Now consider F3. If the union of every $k-2$ colour classes of $\gamma_{i+1}|H_{i+1}$ has no more distance-3-connected components in H_{i+1} than in H_i, then it will follow that $\Omega(\gamma_{i+1}, H_{i+1}) \leqslant \Omega(\gamma_i, H_i)$, and hence that P4 holds, since we are assuming that P4 holds for (γ_i, H_i). So suppose that X is a union of $j = k-2$ colour classes of $\gamma_{i+1}|H_{i+1}$ that has more distance-3-connected components in H_{i+1} than in H_i. It is easy to see that this cannot happen if there is vertex of X within distance 1 of u in H_i, or if $w \in X$ (since there is necessarily a vertex of X within distance 2 of u in H_i), or if w is adjacent in H_{i+1} to a vertex $v \in X \cap V(H_i)$. If all of these fail, then X comprises all vertices with colours in $\Gamma_k \setminus \{\gamma_i(u), \gamma_i(w)\}$, so that w_u and $w'_u \in X$ in Cases 1 and 2 and $z \in X$ in Case 3. Thus we must be in Case 4. All vertices in $N_G(w) \setminus V(H_i)$ are within distance 3 (indeed, distance 2) of each other, and so $\Omega(\gamma_{i+1}, H_{i+1}) \leqslant \Omega(\gamma_i, H_i) + 1$, and the only problem arises if P4 fails for (γ_{i+1}, H_{i+1}). Note that P4 can only fail for the set X just described, since it is only when $\{c_1, c_2\} = \{\gamma_i(u), \gamma_i(w)\}$ that $\omega(c_1, c_2, \gamma_{i+1}, H_{i+1}) > \omega(c_1, c_2, \gamma_i, H_i)$. So suppose there is a k-colouring γ of G that induces $\gamma_{i+1}|H_{i+1}$ on H_{i+1}, such that not all vertices of $V(H_{i+1}) \cap X$ are contained in the same component of $G^3[X]$. Then, since P4 holds for (γ_i, H_i), the vertices of $V(H_{i+1}) \cap X$ are contained in precisely two different components C_1, C_2 of $G^3[X]$, where w.l.o.g. $N_G(w) \cap X \subseteq V(C_1)$ and $V(H_i) \cap X \subseteq V(C_2)$. Note that this requires that $V(C_1) \cap V(H_i) = \emptyset$ and that all H_i-neighbours of u have colour $\gamma_{i+1}(w)$.

Choose a colour $c_0 \notin \{\gamma_i(u), \gamma_i(w)\}$, let X_0 and X_w be the sets of vertices of G with colours c_0 and $\gamma_i(w)$ respectively, and let C be the component (Kempe chain) containing w in $G[X_0 \cup X_w]$. Then $X_0 \subseteq X$, $N_G(w) \cap X_0 \neq \emptyset$ (since w has neighbours of all other colours), and $G^2[V(C) \cap X_0]$ is connected, so that $V(C) \cap X_0 \subseteq V(C_1)$. Hence $V(C) \cap X_0 \cap V(H_i) = \emptyset$ and $V(C) \cap V(H_i) \subset X_w$. Suppose there exists a vertex $v \in V(C) \cap V(H_i)$. By the connectedness of C, and since $\gamma_i|H_i$ is variegated by property P2, there are vertices $v_1 \in V(C) \cap X_0 \subseteq V(C_1)$ and $v_2 \in V(H_i) \cap X \subseteq V(C_2)$ such that $d_C(v, v_1) = 1$ and $d_{H_i}(v, v_2) \leqslant 2$. Thus $d_G(v_1, v_2) \leqslant 3$, which implies that $G^3[C_1 \cup C_2]$ is connected. This contradiction shows that no such vertex v exists, and $V(C) \cap V(H_i) = \emptyset$. Thus we can interchange colours c_0 and $\gamma_i(w)$ throughout C without affecting any colours in H_i. Redefine γ_{i+1} to be the colouring so obtained. In this new γ_{i+1} it is still true that w has neighbours of all other colours, but it is no longer true that all neighbours of u in H_i have colour $\gamma_{i+1}(w) = c_0$, and so there can exist no colouring γ such that P4 fails for (γ_{i+1}, H_{i+1}). This proves fact F3, and so completes the proof of Lemma 3.3. □

We can now complete the proof of Theorem 1.5. Let γ and H be the k-colouring and spanning subgraph of G whose existence was proved in Lemma 3.3, and note that $|E(H)| \leqslant 2(n-k)+k-1-3\Omega(\gamma, H)$ since $k \geqslant 4$. If $\Omega(\gamma, H) = 0$, then every union of $k-2$ colour classes is distance-3-connected in H, and so H is the required k-colourfully panconnected subgraph of G. So suppose that $\Omega(\gamma, H) > 0$, and choose colours c_1 and c_2 such that $\omega(c_1, c_2, \gamma, H) > 1$; that is, the set $X := \{v \in V(G) : \gamma(v) \notin \{c_1, c_2\}\}$ is not distance-3-connected in H. For brevity, let $\omega := \omega(c_1, c_2, \gamma, H)$. Since γ is a panconnected k-colouring of G, X is distance-3-connected in G. Thus by adding $\omega - 1$ subpaths of G to H, each with at most three edges, we can join together the different distance-3-connected components of X in H. If the resulting graph is called H', then $\Omega(\gamma, H') \leqslant \Omega(\gamma, H) - (\omega - 1) < \Omega(\gamma, H)$ and $|E(H')| \leqslant |E(H)|+3(\omega-1) \leqslant 2(n-k)+k-1-3\Omega(\gamma, H')$. After a finite number of repetitions we arrive at a subgraph H^* for which $\Omega(\gamma, H^*) = 0$ and $|E(H^*)| \leqslant 2(n-k)+k-1 = 2n-k-1$; H^* is the required k-colourfully panconnected subgraph of G. This completes the proof of Theorem 1.5.