On the m-Hull Number of the Join and Composition of Graphs

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ABSTRACT

In this paper, the m-hull sets in the join and composition of two connected graphs are characterized and their m-hull numbers are shown to be direct consequences of these characterizations.

1 Introduction

Given a connected graph $G = (V(G), E(G))$ and vertices u and v of G, we call any u-v path of length $d_G(u,v)$ (length of the shortest path connecting u and v) as *u-v geodesic*. Any path that does not contain an edge joining two non-consecutive vertices u and v is called a *u-v monophonic path* (or simply, *m-path*). The *monophonic closure* of a subset S of $V(G)$ is $J_G[S] = \bigcup_{u,v\in S} J_G[u,v]$, where $J_G[u,v]$ is the set containing u and v and all vertices lying on some u-v m-path.

A subset C of $V(G)$ is said to be *m-convex* if, for every pair of vertices $x, y \in C$, the vertex set of every x-y m-path is contained in C. It is easy to verify that S is m-convex if and only if $J_G[S] = S$.

The *m-convex hull* $[S]_m$ of a subset S of $V(G)$ is the smallest m-convex set in G containing S. It can be formed from the sequence $\{J_G^p[S]\}$, where p is a nonnegative integer, $J_G^0[S] = S$, $J_G^1[S] = J[S]$, and $J_G^p[S] = J[J_G^{p-1}[S]]$ for $p \geq 2$. For some p, we must have $J_G^q[S] = J_G^p[S]$ for all $q \geq p$. Further, if p is the smallest nonnegative integer such that $J_G^q[S] = J_G^p[S]$ for all $q \geq p$, then $J_G^p[S] = [S]_m$. If $[S]_m = V(G)$, then we call S an *m-hull set* (monophonic hull set). The *m-hull number* of G, denoted by $mh(G)$ is the minimum cardinality of an m-hull set in G. Any m-hull set S with $|S| = mh(G)$ is called a *minimum m-hull set* in G.

The next remark that follows is from the definition of the m-convex hull of a set.

Remark 1.1 Let G be a connected graph and $S \subseteq V(G)$. Then $[S]_m = S$ if and only if S is an m-convex set in G.

Lemma 1.2 *Let G be a connected graph. Then $mh(G) = |V(G)|$ if and only if G is a complete graph.*

Proof. Let G be a connected graph and suppose that $mh(G) = |V(G)|$. Suppose further that G is not complete.Then there exist $a, b \in V(G)$ such that $d_G(a, b) \neq 1$. Let $c \in J_G[a, b] \backslash \{a, b\}$ and consider $S = V(G) \backslash \{c\}$. Then S is not m-convex in G; hence $J_G[S] \neq S$. Thus, $S \subset J_G[S]$. It follows that $J_G[S] = V(G)$. This implies that $V(G) = J_G[S] \subseteq [S]_m$. Therefore, $[S]_m = V(G)$; that is, S is an m-hull set in G. Hence, $mh(G) \leq |S| = |V(G)| - 1$, contrary to our assumption. Therefore, G must be a complete graph.

For the converse, suppose that G is a complete graph. Then every subset C of $V(G)$ is m-convex in G; hence, $[C]_m = C$ by Remark 1.1 Thus, $[C]_m = V(G)$ if and only if $C = V(G)$. This implies that $C = V(G)$ is the only m-hull set in G. Thus, $mh(G) = |V(G)|$. ■

2 Join of Two Graphs

Our first goal is to characterize m-hull sets in $G + K_n$, where G is a connected graph and K_n the complete graph of order n.

Lemma 2.1 *Let G be a connected non-complete graph, $S \subseteq V(G + K_n)$ and $S_1 = S \cap V(G)$. For every $p \geq 0$, if $y \in J^p_{G+K_n}[S] \cap V(G)$, then $y \in J^p_G[S_1]$.*

Proof. The conclusion clearly holds for $p = 0$. Let $y \in J_{G+K_n}[S] \cap V(G)$. If $y \in S_1$, then $y \in J_G[S_1]$. Suppose $y \notin S_1$. Then there exist $s, t \in S$ such that $y \in J_{G+K_n}[s, t]$. Since $y \neq s$ and $y \neq t$, it follows that $d_{G+K_n}(s, t) = 2$. This means that $s, t \in S_1$. Since every s-t m-path in $G + K_n$ containing y is an s-t m-path in G, it follows that $y \in J_G[s, t]$. Thus, $y \in J_G[S_1]$. This shows that the assertion of the Lemma holds for $p = 1$.

Next,suppose that the assertion holds for $p \geq 1$. Assume that $y \in J^{p+1}_{G+K_n}[S] \cap V(G)$. If $y \in J^p_{G+K_n}[S]$, then $y \in J^p_G[S_1]$ by assumption. So suppose that $y \notin J^p_{G+K_n}[S]$. Then there exist $u, v \in J^p_{G+K_n}[S]$ such that $y \in J_{G+K_n}[u, v]$. Since $y \neq u$ and $y \neq v$, it follows that $d_{G+K_n}(u, v) = 2$. Hence, $u, v \in V(G)$. By the assumption, $u, v \in J^p_G[S_1]$. This implies that $y \in J^{p+1}_G[S_1]$. Therefore, the assertion also holds for $p + 1$. This completes the proof of the lemma. ■

Theorem 2.2 *Let G be a connected non-complete graph. A subset $S \subseteq V(G + K_n)$ is an m-hull set in $G + K_n$ if and only if $S \cap V(G)$ is an m-hull set in G.*

Proof. Let G be a connected non-complete graph. Suppose $S \subseteq V(G+K_n)$ is an m-hull set in $G+K_n$, say $[S]_m = J^p_{G+K_n}[S] = V(G+K_n)$, where p is the smallest non-negative integer satisfying the first equality. Let $S_1 = S \cap V(G)$ and let $x \in V(G) \setminus J^{p-1}_G[S_1]$. Since S is an m-hull set in $G+K_n$, there exist $a, b \in J^{p-1}_{G+K_n}[S]$ such that $x \in J_{G+K_n}[a,b]$. By the contrapositive of Lemma 2.1, $x \notin J^{p-1}_{G+K_n}[S]$; hence, $x \neq a$ and $x \neq b$. This implies that $d_{G+K_n}(a,b) = 2$, that is, $a, b \in V(G)$. Thus, $a, b \in J^{p-1}_G[S_1]$ by Lemma 2.1. Therefore, $x \in J^p_G[S_1]$. This shows that $V(G) \setminus J^{p-1}_G[S_1] \subseteq J^p_G[S_1]$. Since $J^{p-1}_G[S_1] \subseteq J^p_G[S_1]$, it follows that $V(G) \subseteq J^p_G[S_1]$. Therefore, S_1 is an m-hull set in $G+K_n$.

For the converse, suppose that S_1 is an m-hull set in G, say $J^p_G[S_1] = V(G)$, where p is the smallest nonnegative integer satisfying the equality. Since $J^p_G[S_1] \subseteq J^p_{G+K_n}[S_1]$, it follows that $V(G) \subseteq J^p_{G+K_n}[S_1]$. If $\langle S_1 \rangle$ were a complete subgraph of G, then $J^r_G[S_1] = S_1$ for all nonnegative integer r. Thus, $J^r_G[S_1] = S_1 \neq V(G)$ contrary to our assumption. Therefore, $\langle S_1 \rangle$ is non-complete. Choose $u, v \in S_1$ such that $d_G(u,v) \neq 1$. Then $d_{G+K_n}(u,v) = 2$ and $[\, u, w, v \,]$ is a u-v m-path for every $w \in V(K_n)$. It follows that $V(K_n) \subseteq J_{G+K_n}[u,v] \subseteq J_{G+K_n}[S_1]$. Therefore, $V(G+K_n) \subseteq J^p_{G+K_n}[S_1]$, that is, S_1 is an m-hull set in $G+K_n$. Since $S_1 \subseteq S$, S is also a m-hull set in G. This completes the proof of the theorem. ∎

Corollary 2.3 *Let G be a connected non-complete graph. A subset $S \subseteq V(G+K_n)$ is a minimum m-hull set in $G+K_n$ if and only if $S \subseteq V(G)$ and S is a minimum m-hull set.*

Proof. Let G be a connected non-complete graph. Suppose $S \subseteq V(G+K_n)$ is a minimum m-hull set in $G+K_n$. Then $S_1 = S \cap V(G)$ is an m-hull set in G by Theorem 2.2. Suppose that $S^* = S \cap V(K_n) \neq \emptyset$. Then $S_1 = S \setminus S^*$ is an m-hull set in $G+K_n$ by Theorem 2.2 and $|S_1| < |S|$. This contradicts the fact that S is a minimum m-hull set in $G+K_n$. Therefore, $S \subseteq V(G)$. Again, by Theorem 2.2, S is a minimum m-hull set in G, that is, $mh(G) = |S|$.

The converse can be proved in a similar manner. ∎

The following result gives the m-hull number of $G+K_n$ which is a direct consequence of the above Corollary and Lemma 1.2

Corollary 2.4 *Let G be a connected graph of order p and K_n the complete graph of order n. Then*

$$mh(G+K_n) = \begin{cases} p+n & \text{if } G = K_p, \\ mh(G) & \text{if } G \neq K_p. \end{cases}$$

Example 2.5 Let n and m be positive integers. Then

1. $mh(K_n + K_m) = n + m$ for all $n \geq 2$;
2. $mh(C_n + K_m) = 2$ for all $n \geq 4$;
3. $mh(W_n + K_m) = 2$ for all $n \geq 4$.

The next result characterizes m-hull sets in the join of any non-complete graphs.

Theorem 2.6 *Let G and H be non-complete graphs. A subset S of $V(G+H)$ is an m-hull set in $G + H$ if and only if there exist $a, b \in S$ with $d_{G+H}(a, b) = 2$. In this case, either $a, b \in V(G)$ or $a, b \in V(H)$.*

Proof. Let G and H be non-complete graphs and $S \subseteq V(G + H)$ an m-hull set in $G + H$. If $S = V(G + H)$, then we are done. So suppose $S \neq V(G + H)$. Since S is a m-hull set in $G + H$, it follows that $S \neq J_{G+H}[S]$ (otherwise, $J^p_{G+H}[S] = S \neq V(G + H)$ for all $p \geq 1$, contrary to our assumption). Let $x \in J_{G+H}[S] \setminus S$. Then there exist $a, b \in S$ such that $x \in J_{G+H}[a, b]$. Since $x \neq a$ and $x \neq b$, $d_{G+H}(a, b) = 2$. In this case, $a, b \in V(G)$ or $a, b \in V(H)$.

Conversely, suppose that there exist $a, b \in S$ with $d_{G+H}(a, b) = 2$. Assume that $a, b \in V(G)$. Then $V(H) \subseteq J_{G+H}[a, b] \subseteq J_{G+H}[S]$. Pick $u, v \in V(H) = J_{G+H}[S]$ such that $uv \notin E(H)$. Then $d_{G+H}(u, v) = 2$; hence $V(G) \subseteq J^2_{G+H}[S]$. Since $J_{G+H}[S] \subseteq J^2_{G+H}[S]$, $V(G + H) \subseteq J^2_{G+H}[S]$. This shows that S is a m-hull set in $G + H$. ■

As a quick consequence of Theorem 2.6, we have

Corollary 2.7 *Let G and H be non-complete graphs. Then $mh(G + H) = 2$.*

3 Compositon of Two Graphs

The concept that follows appeared and was illustrated in [5].

Definition 3.1 *Let G be a connected graph and $A \subseteq V(G)$. A point a in A is called a monophonic interior point of A if $a \in J_G[A \setminus \{a\}]$. The set of all monophonic interior points of A is denoted by A°.*

Lemma 3.2 [5]. *Let G be a connected graph and K_n be the complete graph of order n. If $P = [(u_1, v_1), (u_2, v_2), \ldots, (u_r, v_r)]$, $r \geq 2$ is an m-path in $G[K_n]$, then we have the following possibilities:*

i. *If u_i's are distinct, then $[u_1, u_2, \ldots, u_r]$ is an m-path in G.*

ii. *If u_i's are not distinct, then $r = 2$.*

Lemma 3.3 *Let G be a connected graph and K_n the complete graph of order n, and $C \subseteq V(G[K_n])$. Then $(J^k_{G[K_n]}[C])_G = J^k_G[C_G]$.*

Proof. We prove this by induction on k.

Let $a \in (J_{G[K_n]}[C])_G$. Then there exists $b \in V(K_n)$ such that $(a,b) \in (J_{G[K_n]}[C])$. Thus, we can find an m-path $[(u_1,v_1),\dots,(u_s,v_s)]$ in $G[K_n]$ such that $(u_1,v_1),(u_s,v_s) \in C$ and $(a,b) = (u_r,v_r)$ for some $r, 1 \leq r \leq s$. If $s = 2$, then clearly, $[u_1,u_2]$ is an m-path in G. Thus $a \in C_G$. Suppose $s \geq 3$, then by Lemma 3.2, $[u_1,\dots,u_s]$ is an m-path in G containing a. Thus, in either case, $a \in J_G[C_G]$.

Conversely, let $a \in J_G[C_G]$. Then there exists an m-path $[u_1,\dots,u_s]$ in G such that $u_1,u_s \in C_G$ and $a = u_r$, for some r, $1 \leq r \leq s$. Let $v,v' \in V(K_n)$ such that $(u_1,v),(u_s,v') \in C$. If $v = v'$, then $[(u_1,v),\dots,(u_s,v)]$ is an m-path in $G[K_n]$ containing (a,v). If $v \neq v'$, then $[(u_1,v),\dots,(u_s,v),(u_s,v')]$ is an m-path in $G[K_n]$ containing (a,v). Thus, $(a,v) \in (J_{G[K_n]}[C])$. Consequently, $a \in (J_{G[K_n]}[C])_G$.

This shows that the assertion holds for $k = 1$. Next, suppose the assertion holds for $k = n > 1$. Now $(J^{n+1}_{G[K_n]}[C])_G = J(J^n_{G[K_n]}[C])_G$. By induction hypothesis, $J(J^n_{G[K_n]}[C])_G = J(J^n_G[C_G])$. Thus $(J^{n+1}_{G[K_n]}[C])_G = J^{n+1}_G[C_G]$. Therefore, the assertion holds for all integer $k \geq 1$. ∎

Theorem 3.4 [5]. *Let G be a connected graph and K_n the complete graph of order n. Then $C = \bigcup_{a\in S}(\{a\} \times T_a)$, where $S \subseteq V(G)$ and $T_a \subseteq V(K_n)$, is monophonic in $G[K_n]$ if and only if S is monophonic in G and $T_a = V(K_n)$ whenever $a \in S \setminus S^\circ$.*

The next result describes completely the m-hull sets in $G[K_n]$.

Theorem 3.5 *Let G be a connected graph, K_n the complete graph of order n, and $C \subseteq V(G[K_n])$. Then $C = \bigcup_{a\in S}(\{a\} \times T_a)$ is an m-hull set in $G[K_n]$ if and only if S is an m-hull set in G and $T_a = V(K_n)$ whenever $a \in S \setminus (V(G))^\circ$, where $(V(G))^\circ$ is the set of all monophonic interior points of G.*

Proof. Suppose $C \subseteq V(G[K_n])$ is an m-hull set in $G[K_n]$. Then $J^l_{G[K_n]}[C] = V(G[K_n])$ for some positive integer l. This implies that $J^{l-1}_{G[K_n]}[C]$ is monophonic in $G[K_n]$. Consequently, $(J_{G[K_n]}{}^{l-1}[C])_G$ is monophonic in G. By Lemma 3.3, $(J^{l-1}_{G[K_n]}[C])_G = J^{l-1}_G[C_G] = J^{l-1}_G[S]$. Hence, $J^{l-1}_G[S]$ is monophonic in G. Thus, $J^l_G[S] = J(J^{l-1}_G[S]) = V(G)$, i.e., S is an m-hull set in G.

Next, $J^{l-1}_{G[K_n]}[C]$ monophonic in $G[K_n]$ implies $(J^{l-1}_{G[K_n]}[C])_G$ is monophonic in G and $T_a = V(K_n)$ whenever $a \in (J^{l-1}_{G[K_n]}[C])_G \setminus (J^{l-1}_{G[K_n]}[C])^\circ_G$ by Theorem 3.4. This implies that $J^{l-1}_G[S]$ is monophonic in G and $T_a = V(K_n)$ whenever $a \in J^{l-1}_G[S] \setminus (J^{l-1}_G[S])^\circ$. Consequently, $J^l_G[S] = V(G)$ and $T_a = V(K_n)$

whenever $a \in J_G^{l-1}[S] \setminus (J_G^{l-1}[S])^\circ$. Note that $S \subseteq J_G^{l-1}[S]$ and $(J_G^{l-1}[S])^\circ \subseteq V(G)^\circ$. It follows that $J_G^l[S] = V(G)$ and $T_a = V(K_n)$ whenever $a \in S \setminus V(G)^\circ$.

For the converse, suppose S is an m-hull set in G and $T_a = V(K_n)$ whenever $a \in S \setminus V(G)^\circ$. Then $V(G) = J_G^l[S] = (J_{G[K_n]}^l[C])_G$. Let $(a,b) \in V(G[K_n])$. If $a \in S \setminus V(G)^\circ$, then $(a,b) \in C \subseteq J_{G[K_n]}^l[C]$. Suppose $a \in (V(G))^\circ$. Then there exist $u, u' \in V(G) \setminus \{a\}$ such that $a \in J_G[u,u']$. Let $[u_1, \ldots, u_r, \ldots, u_s]$ be an m-path in G such that $u_1 = u$, $u_r = a$, and $u_s = u'$. Let $v, v' \in V(K_n)$ such that $(u,v), (u'v') \in J_{G[K_n]}^l[C]$. Consider the following cases:

Case 1 $v = v' = b$. Then $[(u_1,v), \ldots, (u_r,v), \ldots, (u_s,v)]$ is an m-path in $G[K_n]$ containing (a,b).

Case 2 $v \neq b$ and $v' = b$. Then $[(u_1,v), \ldots, (u_r,v), (u_r,b), \ldots, (u_s,b)]$ is an m-path in $G[K_n]$ containing (a,b).

Case 3 $v = b$ and $v' \neq b$. This case is similar to Case 2.

Case 4 $v \neq b$ and $v' \neq b$. Then $[(u_1,v), \ldots, (u_r,v), (u_r,b), \ldots, (u_s,b), (u_s,v')]$ is an m-path in $G[K_n]$ containing (a,b).

Thus $(a,b) \in J_{G[K_n]}^{l+1}[C]$.

Therefore $V(G[K_n]) = J_{G[K_n]}^{l+1}[C]$. Accordingly, C is an m-hull set in $G[K_n]$. ■

The next result gives formula for computing the m-hull number of $G[K_n]$.

Corollary 3.6 *Let G be a connected graph, K_n the complete graph of order n, and $(V(G))^\circ$ be the set of all monophonic interior points of $V(G)$. Then*

$$mh(G[K_n]) = min\{n\,|S \setminus (V(G))^\circ| + |S \cap (V(G))^\circ| : S \text{ is an } m\text{-hull set in } G\}.$$

We now give the m-hull number of the composition $G[H]$, where G and H are non-complete graphs.

Theorem 3.7 *Let G and H be non-complete graphs. Then $mh(G[H]) = 2$.*

Proof. Let $a \in V(G)$ and $b, b' \in V(H)$ such that $d_H(b,b') = 2$. We show that $S = \{(a,b), (a,b')\}$ is an m-hull set in $G[H]$. Let $(u,v) \in V(G[H])$ such that $u \neq a$ and $[u_1, u_2, \ldots, u_r]$ be an a-u geodesic, where $u_1 = a$ and $u_r = u$.

Claim: For each $(u,v) \in V(G[H])$, there exists $l(u)$ such that $(u,v) \in J^l[S]$.

Since (u_1,b), $(u_1,b') \in S$, $(u_2,b), (u_2,b') \in J[S]$. This implies that (u_3,b), $(u_3,b') \in J^2[S]$. Continuing in this manner, we obtain (u_{r-1},b), $(u_{r-1},b') \in J^{r-2}[S]$, $r \geq 2$. Consequently, $(u_r,v) = (u,v) \in J^{r-1}[S]$.

Let $n = max\{l(u) : u \in V(G)\}$ and $(u,v) \in V(G[H])$. Then by the Claim, $(u,v) \in J^{r-1}[S] \subseteq J^n[S]$. Thus, $V(G[H]) = J^n[S]$. Accordingly, S is an m-hull set in $G[H]$. Thus, $mh(G[H]) = 2$. ■

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