Orientable Embedding Distributions by Genus of Crossing-Digraph

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Abstract In this paper, the joint tree method of graph embeddings which was introduced by Liu is generalized to digraph embeddings. The genus distributions of a new type digraphs in orientable surface is determined.

Keywords Digraph, joint tree, embedding distribution, genus, surfaces.

1. Introduction

The embedding genus problems of graphs have been studied by many authors. Gross and Furst [5] first introduced the genus distributions of graphs. After that, the genus distributions of circular ladders and Mobius ladders [7], closed-end ladders and cobble-stone paths [4], bouquets of circles [6], Ringel ladder s[11] were studied. The total genus distributions of neck-laces,closed end ladders and cobblestone paths [3] and bouquets of circles [8] were also investigated.

All these results are concerning genus distributions of graphs, little is known about those of digraphs.

In this paper, we introduce a kind of digraphs and determine their genus polynomials.

Let $a = (u, v)$ be an arc of a digraph D, then a is said to *join* u *and* v. We also say that a is *incident from* u and *incident to* v, while u is *incident to* a and v is *incident from* a. Moreover, u is said to be *adjacent to* v and v is *adjacent from* u. Two vertices u and v of a digraph D are *nonadjacent* if u is neither adjacent to nor adjacent from v in D. The *outdegree* od$v$ of a vertex v in a digraph D is the number of vertices of D that are adjacent from v. Similarly, the *indegree* id$v$ is the number of vertices of D adjacent to v. The *degree* deg$v$ of a vertex v is defined by deg$v$=od$v$+id$v$.

A digraph is Eulerian if it is connected and the indegree equals the outdegree for each vertex. Bonnington et. al. [1,2] proved some basic results on digraph

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embeddings and obtained the obstructions for directed embeddings of digraphs in the plane.

A *surface* is a compact orientable 2-dimensional manifold without a boundary. It can be obtained by identifying each of pairs of edges along a given direction on a polygon with even number of edges, a surface corresponds to a cycle of letters.

The *orientable surface* of genus h, denoted by S_h, is the sphere with h handles added. A graph is said to be *embedded* in a surface S if it is drawn in S so that edges intersect only at their common end vertices. A digraph is said to be *embedded* in a surface S if it is embedded as a graph with the directions of arcs consistent in each region boundary. Two embeddings of digraph D, $f : D \to S$ and $g : D \to S$ are *the same*, if there is an orientation preserving homeomorphism $h : S \to S$ such that $hf = g$. Here each embedding is a cellular embedding. The genus g of a digraph D is the minimum integer g such that D can be embedded in S_g.

For a surface S, let $o(S)$ be the genus of S. Let A and B be sections of successive letters in cyclic orders (called *linear sequences*) and $\emptyset$ be empty set. Let e, e_1 and e_2 be distinct letters. By an *elementary transformation*, we mean one of the following three operations and their inverses on a surface S. ([10] §1.5)

OP1 $(AB) \sim (Ae)(e^- B)$, where $A \neq \emptyset$ or $B \neq \emptyset$;

OP2 $(Ae_1e_2Be_2^- e_1^-) \sim (AeBe^-)$, where $e = e_1e_2, e^- = (e_1e_2)^- = e_2^- e_1^-$;

OP3 $(Aee^- B) \sim (AB)$, where $AB \neq \emptyset$.

It can be seen that each orientable surface is equivalent to only one of the following canonical forms of surfaces:

$$O_i = \begin{cases} (a_0 a_0^-), & \text{if } i = 0 \\ (\prod_{k=1}^{i} a_k b_k a_k^- b_k^-), & \text{if } i \geq 1 \end{cases}$$

O_i means the orientable surface of genus i.

Lemma 1.1[10] *Let A,B,C,D and E be linear sequences, and let $(ABCDE)$ be a surface. For $a, b \notin (ABCDE)$, then $(AaBbCa^- Db^- E) \sim (ADCBEaba^- b^-)$.*

Lemma 1.2[10] *Let S and S′ be surfaces, if $S \sim (S'xyx^- y^-)$, and $x, y, x^-, y^- \notin S'$, then $o(S) = o(S') + 1$.*

Let G be a graph and T be a spanning tree of G. A *rotation* at a vertex v, denoted by σ_v, is a cyclic permutation of edges incident with v. Let σ be a rotation system of G, then $\sigma = \prod_{v \in V(G)} \sigma_v$. For each embedding G_σ determined by σ, a *joint tree* is obtained by splitting each co-tree edge e into two semi-edges e^+ and e^- (called *co-tree semi-edges*). For notational convenience, we use e for e^+ throughout this paper.

If G is a digraph, we just change edges to arcs and semi-edges to semi-arcs.

It has been shown in [9] that the genus distribution is independent of the choice of a spanning tree, that is

Lemma 1.3 *Let T and T' be two distinct spanning trees of a graph G and Σ be the set of rotation systems of G. For $\sigma \in \Sigma$, let G_σ be an embedding determined by σ. $\widetilde{T}_\sigma$ be the joint tree embedding corresponding to G_σ. Then there exists a bijection between $(\Sigma, \widetilde{T})$ and $(\Sigma, \widetilde{T'})$, where $(\Sigma, \widetilde{T}) = \{\widetilde{T}_\sigma | \sigma \in \Sigma\}$ and $(\Sigma, \widetilde{T'}) = \{\widetilde{T'}_\sigma | \sigma \in \Sigma\}$.*

Let G be a digraph, the sequence $g_0(G), g_1(G), g_2(G), \cdots$ is called the *genus distribution* of G, where $g_i(G)$ is the number of different embeddings of G in the orientable surface with genus i. The *genus polynomial* of G is as follows:

$$f_G(x) = \sum_{i=0}^{\infty} g_i(G) x^i$$

In this paper, we consider the genus distributions of Eulerian digraphs in orientable surfaces. The joint tree method of graph embeddings which was introduced by Liu is generalized to digraph embeddings. Then, we determine the genus distributions of crossing-digraphs U_n in orientable surface(the definition of U_n is in §2). For convenience, the braces of surfaces are omitted in the next section.

2. Genus distributions of cross-digraphs

Let U_0 be the digraph shown in Fig.1(a). The graph U_n is obtained by adding $4n$ vertices $u_1, v_1, u_3, v_3, \cdots, u_{2n-1}, v_{2n-1}, u_{2n}, v_{2n}, \cdots, u_4, v_4, u_2, v_2$ alonging the arc e and $4n$ arcs: $u_{2l-1}u_{2l}, u_{2l}u_{2l-1}, v_{2l-1}v_{2l}, v_{2l}v_{2l-1}$, denoted by a_l, b_l, c_l, d_l, respectively, for a positive integer l $(1 \leq l \leq n)$. U_1,U_2 are shown in Fig.1(b),(c)

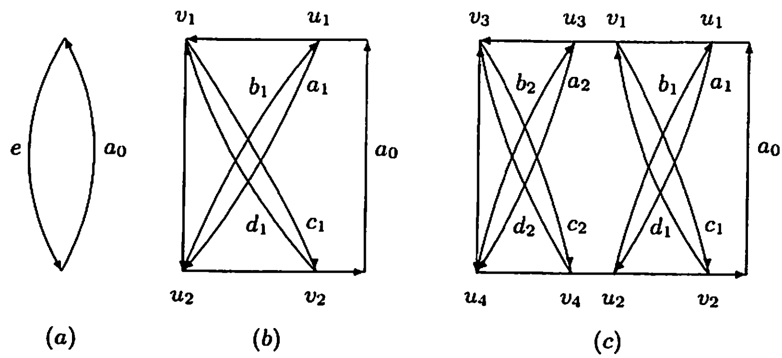

*Fig.*1 : U_0, U_1, U_2

Consider U_0. Let a_0 be a cotree edge. We can see U_0 has the unique joint tree $\widetilde{T}$. Here, $a_0a_0^-$ is its embedding surface. Then, $g_0(U_0)$=1, so $f_{U_0}(x) = 1$.

Consider U_1. Let T be a tree such that a_0, a_1, b_1, c_1, d_1 are cotree edges. Then

$\widetilde{T}$ generates 16 distinct joint trees of U_1. By calculating, we can easy get the followings: $g_0(U_1)=2$, $g_1(U_1)=10$, $g_2(U_1)=4$. Therefore $f_{U_1}(x)=2+10x+4x^2$.

Let Y be an embedding surface of U_{n-1}. Since U_n is obtained from U_{n-1} by adding four vertices and four arcs, a joint tree of Y generates 16 distinct joint trees of U_n whose embedding surfaces belong to the following 3 types:

$T1: b_na_nY_1c_nd_nd_n^-c_n^-Y_2a_n^-b_n^-$ $\quad d_nc_nY_1a_nb_nb_n^-a_n^-Y_2c_n^-d_n^-$

$T2: d_nc_nb_na_nY_1Y_2c_n^-d_n^-a_n^-b_n^-$ $\quad d_nc_nY_1a_nb_nY_2c_n^-d_n^-a_n^-b_n^-$

$d_nc_nb_na_nY_1b_n^-a_n^-Y_2c_n^-d_n^-$ $\quad d_nc_nb_na_nY_1d_n^-c_n^-Y_2a_n^-b_n^-$

$b_na_nY_1c_nd_nY_2c_n^-d_n^-a_n^-b_n^-$ $\quad b_na_nY_1c_nd_nb_n^-a_n^-d_n^-c_n^-Y_2$

$Y_1a_nb_nc_nd_nb_n^-a_n^-Y_2c_n^-d_n^-$ $\quad Y_1a_nb_nc_nd_nd_n^-c_n^-Y_2a_n^-b_n^-$

$Y_1a_nb_nc_nd_nb_n^-a_n^-d_n^-c_n^-Y_2$ $\quad d_nc_nY_1a_nb_nb_n^-a_n^-d_n^-c_n^-Y_2$

$T3: Y_1a_nb_nc_nd_nY_2c_n^-d_n^-a_n^-b_n^-$ $\quad b_na_nY_1c_nd_nb_n^-a_n^-Y_2c_n^-d_n^-$

$d_nc_nY_1a_nb_nd_n^-c_n^-Y_2a_n^-b_n^-$ $\quad d_nc_nb_na_nY_1b_n^-a_n^-d_n^-c_n^-Y_2$

Where $Y_1Y_2=Y$.

For a graph sequence G_n, let Y and X be embedding surfaces of G_{n-1} and G_n respectively. If X is generated by Y, then Y is called the mother of X.

Lemma 2.1 *Let X be an embedding surface of U_n, Y the mother of X. Then,*
(1) If X *has a form as the same as one in* $T1$, then $o(X)=o(Y)$;
(2) If X *has a form as the same as one in* $T2$, then $o(X)=o(Y)+1$;
(3) If X *has a form as the same as one in* $T3$, then $o(X)=o(Y)+2$;
Where $Y_1Y_2=Y$

Proof. Sine the arguments of the proof are similar, we shall prove for the first form in each type of X. They are checked by using OP2, OP3, Lemma1.1 and Lemma1.2 as follows:.

(1) Since $b_na_nY_1c_nd_nd_n^-c_n^-Y_2a_n^-b_n^- \sim Y_1Y_2$, therefore, $o(X)=o(Y)$;

(2) Since $d_nc_nb_na_nY_1Y_2c_n^-d_n^-a_n^-b_n^- \sim Y_1Y_2c_n^-c_nb_nb_n^-d_na_nd_n^-a_n^-$, therefore, $o(X)=o(Y)+1$;

(3) Since $Y_1a_nb_nc_nd_nY_2c_n^-d_n^-a_n^-b_n^- \sim Y_1c_nd_nY_2c_n^-d_n^-a_nb_na_n^-b_n^-$
$\sim Y_1Y_2c_nd_nc_n^-d_n^-a_nb_na_n^-b_n^-$, therefore, $o(X)=o(Y)+2$. ♯

Theorem 2.2 *The distinct embeddings of digraph U_n (for $n\geq 2$) in orientable surfaces with genus i is:*

$$g_i(U_n)=\begin{cases} 2g_i(U_{n-1}), & \text{if } i=0;\\ 2g_i(U_{n-1})+10g_{i-1}U_{n-1}), & \text{if } i=1;\\ 2g_i(U_{n-1})+10g_{i-1}(U_{n-1})+4g_{i-2}(U_{n-1}), & \text{if } 2\leq i\leq 2n-2;\\ 10g_{i-1}(U_{n-1})+4g_{i-2}(U_{n-1}), & \text{if } i=2n-1;\\ 4g_{i-2}(U_{n-1}), & \text{if } i=2n. \end{cases}$$

Proof. Let $g_{i_j}(U_n)$ be the number of distinct embeddings of U_n in orientable surfaces with genus i such that their embedding surfaces have forms in Tj. By Lemma 2.1, for $n \geq 2$

$g_{0_j}(U_n) = 0$, for $j = 2, 3$;

$g_{i_1}(U_n) = 2g_i(U_{n-1})$, for $0 \leq i \leq 2n-2$;

$g_{i_2}(U_n) = 10g_{i-1}(U_{n-1})$, for $1 \leq i \leq 2n-1$;

$g_{i_3}(U_n) = 4g_{i-2}(U_{n-1})$, for $2 \leq i \leq 2n$.

Since $g_i(U_n) = \sum_{i=0}^{3} g_{i_j}(U_n)$, the proof is done. ♯

For example, the embedding polynomials of U_n are as follows for $n = 0, 1, 2, \cdots, 5$:

$f_{U_0}(x) = 1$

$f_{U_1}(x) = 2(1 + 5x + 2x^2)$

$f_{U_2}(x) = 2^2(1 + 10x + 29x^2 + 20x^3 + 4x^4)$

$f_{U_3}(x) = 2^3(1 + 15x + 81x^2 + 185x^3 + 162x^4 + 60x^5 + 8x^6)$

$f_{U_4}(x) = 2^4(1 + 20x + 158x^2 + 620x^3 + 1249x^4 + 1240x^5 + 632x^6 + 160x^7 + 16x^8)$

$f_{U_5}(x) = 2^5(1 + 25x + 260x^2 + 1450x^3 + 4665x^4 + 8725x^5 + 9330x^6 + 5800x^7 + 2080x^8 + 400x^9 + 32x^{10})$

Lemma 2.3 *The maximum genus of digraph U_n in orientable surface is $2n$.*

Proof. It is obtained from Theorem 2.2 directly. ♯

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