Intuitionistic Fuzzy Subcoalgebras of Coalgebras

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Abstract

In this paper, we apply the concepts of intuitionistic fuzzy sets to coalgebras. We give the definition of intuitionistic fuzzy subcoalgebras and investigate some properties of intuitionistic fuzzy subcoalgebras. Considering the applications of intuitionistic fuzzy subcoalgebras, we discuss their properties under homomorphisms of coalgebras.

Keywords: Intuitionistic fuzzy sets; Intuitionistic fuzzy subcoalgebras; Fuzzy subcoalgebras

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1 Introduction

Presently science and technology is featured with complex process and phenomena for which complete information is not always available. For such cases, mathematical models are developed to handle various types of systems containing elements of uncertainty. A large part of these models are based on an extension of the ordinary set theory, namely, the so called

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fuzzy sets. It is well known that the notion of fuzzy sets was introduced by Zadeh [18] as a method of representing uncertainty. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines.

After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [5, 6] is one among them. Recently, many researchers have applied this concept to algebraic structures, for example (see [1, 2, 3, 4, 7, 8, 10, 11, 13, 15, 16]). These algebraic structures can be regarded as universal algebras which makes n-tuples elements into one. In 2000, Rutten [17] introduced the notion of universal coalgebras as the dual of universal algebras. The family of labelled transition systems and coalgebras defined in [12] are the universal coalgebras. The theory of universal coalgebras turned out to be suited, moreover, as models for certain types of automata and more generally, for (transition and dynamical) systems. As an application of the theory of universal coalgebras, we studied fuzzy subcoalgebras of coalgebras in [9]. Now, it is natural to consider intuitionistic fuzzy subcoalgebras for further study.

2 Preliminaries

In this section, some relevant definitions and notations are reproduced.

Definition 2.1. ([12]) A k-coalgebra is a triple (C, Δ, ε), where C is a k-vector space, $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are morphisms of k-vector spaces such that the following diagrams are commutative:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ {\scriptstyle\Delta}\downarrow & & {\scriptstyle I\otimes\Delta}\downarrow \\ C\otimes C & \xrightarrow{\Delta\otimes I} & C\otimes C\otimes C \end{array}$$

$$\begin{array}{ccccc} & {}^{\sim}\swarrow & C & \searrow^{\sim} & \\ C\otimes k & & \downarrow{\scriptstyle\Delta} & & k\otimes C \\ & {\scriptstyle I\otimes\varepsilon}\nwarrow & C\otimes C & \nearrow{\scriptstyle\varepsilon\otimes I} & \end{array}$$

Remark 2.1. Let (C, Δ, ε) be a coalgebra. For an element $c \in C$, we denote $\Delta(c) = \sum_{i=1,n} c_{i1} \otimes c_{i2}$. In this paper, we require that $\{c_{i1}\}$, $\{c_{i2}\}$ are linearly independent respectively. In this case, the decomposition of $\Delta(c)$ is unique.

Definition 2.2. Let C be a k-coalgebra. An intuitionistic fuzzy set (IFS for short) of C defined as an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in C\}$, where the functions $\mu_A : C \to [0,1]$ and $\nu_A : C \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in C$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in C$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in C\}$.

In this paper, we use the symbols $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Definition 2.3. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a coalgebra C. Then

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in C$,
(2) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle | x \in C\}$,
(3) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle | x \in C\}$,
(4) $\Box A = \{\langle x, \mu_A(x), \mu_A^c(x)\rangle | x \in C\}$,
(5) $\Diamond A = \{\langle x, \nu_A^c(x), \nu_A(x)\rangle | x \in C\}$.

Definition 2.4. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy sets of a coalgebra C. The intuitionistic sum of $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ is defined to the intuitionistic fuzzy set $A + B = (\mu_{A+B}, \nu_{A+B})$ of C given by

$$\mu_{A+B}(x) = \begin{cases} \sup\limits_{x=a+b} \{\mu_A(a) \wedge \mu_B(b)\} & \text{if } x = a + b \\ 0 & \text{otherwise} \end{cases},$$

$$\nu_{A+B}(x) = \begin{cases} \inf\limits_{x=a+b} \{\nu_A(a) \vee \nu_B(b)\} & \text{if } x = a + b \\ 1 & \text{otherwise} \end{cases}.$$

Definition 2.5. Let f be a mapping from a coalgebra C to a coalgebra C'. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy sets in C and C' respectively, then the preimage of $B = (\mu_B, \nu_B)$ under f is defined to be an intuitionistic fuzzy set $f^{-1}(B) = (\mu_{f^{-1}(B)}, \nu_{f^{-1}(B)})$ where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$ for any $x \in C$ and the image of $A = (\mu_A, \nu_A)$ under f is defined to be an intuitionistic fuzzy set $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ where

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\mu_A(x)\} & \text{if } y \in f(C) \\ 0 & \text{if } y \notin f(C) \end{cases},$$

$$\nu_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\nu_A(x)\} & \text{if } y \in f(C) \\ 1 & \text{if } y \notin f(C) \end{cases}.$$

3 Intuitionistic fuzzy subcoalgebras

Definition 3.1. ([12]) Let (C, Δ, ε) be a coalgebra. A k-subspace D of C is called a *subcoalgebra* if $\Delta(D) \subseteq D \otimes D$.

Definition 3.2. Let $A = (\mu_A, \nu_A)$ be an intuitionstic fuzzy set of C. For any $x \in C, \Delta(x) = \sum_{i=1,n} x_{i1} \otimes x_{i2}$. Then $A = (\mu_A, \nu_A)$ is called *intuitionistic fuzzy subcoalgebra* of C, if it satisfies the following conditions:

(1) $\mu_A(\alpha x + \beta y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(\alpha x + \beta y) \leq \nu_A(x) \vee \nu_A(y)$, for any $x, y \in C$ and $\alpha, \beta \in k$,

(2) $\mu_A(x) \leq \mu_A(x_{i1}) \wedge \mu_A(x_{i2})$ and $\nu_A(x) \geq \nu_A(x_{i1}) \vee \nu_A(x_{i2})$, for any $x \in C$ and all i.

Example 3.1. Let C be a vector space with basis $\{g_i, d_i | i \in N^*\}$. We define $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ by

$$\Delta(g_i) = g_i \otimes g_i, \quad \Delta(d_i) = g_i \otimes d_i + d_i \otimes g_{i+1},$$

$$\varepsilon(g_i) = 1, \quad \varepsilon(d_i) = 0.$$

Then (C, Δ, ε) is a coalgebra ([12]).

Let $x \neq 0$. Then $x = \sum a_i g_i + \sum b_i d_i$ where $a_i, b_i \neq 0$. We define $A = (\mu_A, \nu_A)$ where $\mu_A(x) = (\wedge \mu_A(g_i)) \wedge (\wedge \mu_A(d_i))$, $\mu_A(g_i) = \frac{i}{i+1}$, $\mu_A(d_i) = \frac{1}{i+1}$ and $\nu_A(x) = (\vee \nu_A(g_i)) \vee (\vee \nu_A(d_i))$, $\nu_A(g_i) = \frac{1}{i+1}$, $\nu_A(d_i) = \frac{i-1}{i+1}$. If $x = 0$, we define $\mu_A(0) = 1$ and $\nu_A(0) = 0$. Then μ is an intuitionistic fuzzy subcoalgebra of C.

Indeed, let $x = \sum a_i g_i + \sum b_i d_i$ and $y = \sum k_i g_i + l_i d_i$, where $a_i, b_i, k_i, l_i \neq 0$. Then for $\alpha, \beta \in k$, $\alpha x + \beta y = \sum(\alpha a_i + \beta k_i) g_i + \sum(\alpha b_i + \beta l_i) d_i$, we have $\mu_A(\alpha x + \beta y) = (\wedge \mu_A(g_i)) \wedge (\wedge \mu_A(d_i)) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(\alpha x + \beta y) = (\vee \nu_A(g_i)) \vee (\vee \nu_A(d_i)) \leq \max\{\nu_A(x), \nu_A(y)\}$. Since $\Delta(x) = \sum a_i \Delta(g_i) + \sum b_i \Delta(d_i) = \sum a_i (g_i \otimes g_i) + \sum b_i (g_i \otimes d_i + d_i \otimes g_{i+1})$, we have $\mu_A(x) = (\wedge \mu_A(g_i)) \wedge (\wedge \mu_A(d_i)) \leq \min\{\mu_A(g_i), \mu_A(g_i)\}$, $\mu_A(x) = (\wedge \mu_A(g_i)) \wedge (\wedge \mu_A(d_i)) \leq \min\{\mu_A(g_i), \mu_A(d_i)\}$, $\mu_A(x) = (\wedge \mu_A(g_i)) \wedge (\wedge \mu_A(d_i)) \leq \min\{\mu_A(d_i), \mu_A(g_{i+1})\}$ and $\nu_A(x) = (\vee \nu_A(g_i)) \vee (\vee \nu_A(d_i)) \geq \max\{\nu_A(g_i), \nu_A(g_i)\}$, $\nu_A(x) = (\vee \nu_A(g_i)) \vee (\vee \nu_A(d_i)) \geq \max\{\nu_A(g_i), \nu_A(d_i)\}$, $\nu_A(x) = (\vee \nu_A(g_i)) \vee (\vee \nu_A(d_i)) \geq \max\{\nu_A(d_i), \nu_A(g_{i+1})\}$. So $\mu_A(x) \leq \min\{\mu_A(x_{i1}), \mu_A(x_{i2})\}$ and $\nu_A(x) \geq \max\{\nu_A(x_{i1}), \nu_A(x_{i2})\}$. Hence $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C.

Lemma 3.1. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C, then so is $\Box A = (\mu_A, \mu_A^c)$.*

Lemma 3.2. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C, then so is $\Diamond A = (\nu_A^c, \nu_A)$.*

Theorem 3.1. *$A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy subcoalgebras .*

Definition 3.3. For any $t \in [0,1]$ and fuzzy subset μ of C, the set $U(\mu, t) = \{x \in C | \mu(x) \geq t\}$ (resp. $L(\mu, t) = \{x \in C | \mu(x) \leq t\}$) is called an upper (resp. lower) t-level cut of μ.

Theorem 3.2. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C, then the sets $U(\mu_A, t)$ and $L(\nu_A, t)$ are subcoalgebras of C, for every $t \in \Im\mu_A \cap \Im\nu_A$.*

Proof. Let $x, y \in U(\mu_A, t)$ and $\alpha, \beta \in k$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra, we have $\mu_A(x) \geq t, \mu_A(y) \geq t$ and $\mu_A(\alpha x + \beta y) \geq \mu_A(x) \wedge \mu_A(y) \geq t$. Therefore, $\alpha x + \beta y \in U(\mu_A, t)$. Let $x \in U(\mu_A, t)$. Because $\mu_A(x_{i1}) \wedge \mu_A(x_{i2}) \geq \mu_A(x) \geq t$, we have $\mu_A(x_{i1}) \geq t$ and $\mu_A(x_{i2}) \geq t$. Hence, $\Delta(U(\mu_A, t)) \subseteq U(\mu_A, t) \otimes U(\mu_A, t)$.

Similarly, let $x, y \in L(\nu_A, t)$ and $\alpha, \beta \in k$. Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra, we have $\nu_A(x) \leq t, \nu_A(y) \leq t$ and $\nu_A(\alpha x + \beta y) \leq \nu_A(x) \vee \nu_A(y) \leq t$. Therefore, $\alpha x + \beta y \in L(\nu_A, t)$. Let $x \in L(\nu_A, t)$. Because $\nu_A(x_{i1}) \vee \nu_A(x_{i2}) \leq \nu_A(x) \leq t$, we have $\nu_A(x_{i1}) \leq t$ and $\nu_A(x_{i2}) \leq t$. Hence, $\Delta(L(\nu_A, t)) \subseteq L(\nu_A, t) \otimes L(\nu_A, t)$. □

Theorem 3.3. *If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set of C such that all non-empty level sets $U(\mu_A, t)$ and $L(\nu_A, t)$ are subcoalgebras of C, then $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subcoalgebra of C.*

Proof. Let $x, y \in C$ and $\alpha, \beta \in k$. We may assume that $\mu_A(y) \geq \mu_A(x) = t_1$ and $\nu_A(y) \leq \nu_A(x) = t_0$, then $x, y \in U(\mu_A, t_1)$ and $x, y \in L(\nu_A, t_0)$. Since $U(\mu_A, t_1)$ and $L(\nu_A, t_0)$ are subspaces, we get $\alpha x + \beta y \in U(\mu_A, t_1)$ and $\alpha x + \beta y \in L(\nu_A, t_0)$. So $\mu_A(\alpha x + \beta y) \geq t_1 = \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(\alpha x + \beta y) \leq t_0 = \nu_A(x) \vee \nu_A(y)$.

Let $x \in C$. Suppose that $\mu_A(x) = t_0$, we have $x \in U(\mu_A, t_0)$. Since $\Delta(x) \in U(\mu_A, t_0) \otimes U(\mu_A, t_0)$, we have $\mu_A(x_{i1}) \geq t_0$ and $\mu_A(x_{i2}) \geq t_0$, so $\mu_A(x) = t_0 \leq \mu_A(x_{i1}) \wedge \mu_A(x_{i2})$. Also, suppose that $\nu_A(x) = t_1$, we have $x \in L(\nu_A, t_1)$. Since $\Delta(x) \in L(\nu_A, t_1) \otimes L(\nu_A, t_1)$, we have $\nu_A(x_{i1}) \leq t_1$ and $\nu_A(x_{i2}) \leq t_1$, so $\nu_A(x) = t_1 \geq \nu_A(x_{i1}) \vee \nu_A(x_{i2})$. □

Theorem 3.4. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic fuzzy subcoalgebras of coalgebra C such that $\mu_A(0) = \mu_B(0)$ and $\nu_A(0) = \nu_B(0)$. Then $A + B = (\mu_{A+B}, \nu_{A+B})$ is also an intuitionistic fuzzy subcoalgebra of C.*

Proof. Let $x, y \in C$ and $\alpha \in k$. Firstly suppose that $\mu_{A+B}(x+y) < \mu_{A+B}(x) \wedge \mu_{A+B}(y)$ and $\nu_{A+B}(x+y) > \nu_{A+B}(x) \vee \nu_{A+B}(y)$, then $\mu_{A+B}(x+y) < \mu_{A+B}(x)$, $\mu_{A+B}(x+y) < \mu_{A+B}(y)$ and $\nu_{A+B}(x+y) > \nu_{A+B}(x)$, $\nu_{A+B}(x+y) > \nu_{A+B}(y)$. Choose a number $t, t' \in [0,1]$ such that $\mu_{A+B}(x+y) < t < \mu_{A+B}(x)$, $\mu_{A+B}(x+y) < t < \mu_{A+B}(y)$ and $\nu_{A+B}(x+y) > t' > \nu_{A+B}(x)$, $\nu_{A+B}(x+y) > t' > \nu_{A+B}(y)$. Then there exist $a, b, c, d \in C$ with $x = a+b, y = c+d$ such that $\mu_A(a) > t, \mu_A(b) > t$, $\mu_B(c) > t, \mu_B(d) > t$ and $\nu_A(a) < t', \nu_A(b) < t', \nu_B(c) < t', \nu_B(d) < t'$, we have

$$\begin{aligned} \mu_{A+B}(x+y) &= \sup_{x+y=m+n} \left\{ \mu_A(m) \wedge \mu_B(n) \right\} \\ &\geq \sup_{x+y=a+b+c+d} \left\{ \mu_A(a+b) \wedge \mu_B(c+d) \right\} \\ &> t > \mu_{A+B}(x+y) \end{aligned}$$

and

$$\begin{aligned} \nu_{A+B}(x+y) &= \inf_{x+y=m+n} \left\{ \nu_A(m) \vee \nu_B(n) \right\} \\ &\leq \inf_{x+y=a+b+c+d} \left\{ \nu_A(a+b) \vee \nu_B(c+d) \right\} \\ &< t' < \nu_{A+B}(x+y). \end{aligned}$$

These are contradictions. So $\mu_{A+B}(x+y) \geq \mu_{A+B}(x) \wedge \mu_{A+B}(y)$ and $\nu_{A+B}(x+y) \leq \nu_{A+B}(x) \vee \nu_{A+B}(y)$.

Secondly, let $\alpha \in k$ and $x \in C$. We will show $\mu_{A+B}(\alpha x) \geq \mu_{A+B}(x)$ and $\nu_{A+B}(\alpha x) \leq \nu_{A+B}(x)$. If $\alpha \neq 0$, then

$$\begin{aligned} \mu_{A+B}(\alpha x) &= \sup_{\alpha x=a+b} \left\{ \mu_A(a) \wedge \mu_B(b) \right\} \\ &= \sup_{x=\frac{1}{\alpha}a+\frac{1}{\alpha}b} \left\{ \mu_A(\frac{1}{\alpha}a) \wedge \mu_B(\frac{1}{\alpha}b) \right\} \\ &= \mu_{A+B}(x). \end{aligned}$$

and

$$\begin{aligned} \nu_{A+B}(\alpha x) &= \inf_{\alpha x=a+b} \left\{ \nu_A(a) \vee \nu_B(b) \right\} \\ &= \inf_{x=\frac{1}{\alpha}a+\frac{1}{\alpha}b} \left\{ \nu_A(\frac{1}{\alpha}a) \vee \nu_B(\frac{1}{\alpha}b) \right\} \\ &= \nu_{A+B}(x). \end{aligned}$$

If $\alpha = 0$, then

$$\begin{aligned}
\mu_{A+B}(\alpha x) &= \mu_{A+B}(0) = \sup_{0=a+b}\Big\{\mu_A(a) \wedge \mu_B(b)\Big\} \\
&\geq \mu_A(0) \wedge \mu_B(0) \geq \sup_{x=c+d}\Big\{\mu_A(c) \wedge \mu_B(d)\Big\} \\
&= \mu_{A+B}(x)
\end{aligned}$$

and

$$\begin{aligned}
\nu_{A+B}(\alpha x) &= \nu_{A+B}(0) = \inf_{0=a+b}\Big\{\nu_A(a) \vee \nu_B(b)\Big\} \\
&\leq \nu_A(0) \vee \nu_B(0) \leq \inf_{x=c+d}\Big\{\nu_A(c) \vee \nu_B(d)\Big\} \\
&= \nu_{A+B}(x).
\end{aligned}$$

Finally, let $x = a + b \in C$. Then

$$\begin{aligned}
\sum_{i=1,n} x_{i1} \otimes x_{i2} &= \Delta(x) = \Delta(a+b) = \Delta(a) + \Delta(b) \\
&= \sum_{s=1,n} a_{s1} \otimes a_{s2} + \sum_{t=1,n} b_{t1} \otimes b_{t2}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mu_{A+B}(x) &= \sup_{x=a+b}\Big\{\mu_A(a) \wedge \mu_B(b)\Big\} \\
&\leq \sup_{x=a+b}\Big\{\mu_A(a_{s1}) \wedge \mu_A(a_{s2}) \wedge \mu_B(b_{t1}) \wedge \mu_B(b_{t2})\Big\} \\
&= \sup_{x=a+b}\Big\{\mu_A(a_{s1}) \wedge \mu_B(0) \wedge \mu_A(0) \wedge \mu_B(b_{t1})\Big\} \wedge \\
&\quad \sup_{x=a+b}\Big\{\mu_A(a_{s2}) \wedge \mu_B(0) \wedge \mu_A(0) \wedge \mu_B(b_{t2})\Big\} \\
&\leq (\mu_{A+B}(a_{s1}) \wedge \mu_{A+B}(b_{t1})) \wedge (\mu_{A+B}(a_{s2}) \wedge \mu_{A+B}(b_{t2})) \\
&\leq \mu_{A+B}(x_{i1}) \wedge \mu_{A+B}(x_{i2}).
\end{aligned}$$

and

$$
\begin{aligned}
\nu_{A+B}(x) &= \inf_{x=a+b}\left\{\nu_A(a)\vee\nu_B(b)\right\}\\
&\geq \inf_{x=a+b}\left\{\nu_A(a_{s1})\vee\nu_A(a_{s2})\vee\nu_B(b_{t1})\vee\nu_B(b_{t2})\right\}\\
&= \inf_{x=a+b}\left\{\nu_A(a_{s1})\vee\nu_B(0)\vee\nu_A(0)\vee\nu_B(b_{t1})\right\}\vee\\
&\quad \inf_{x=a+b}\left\{\nu_A(a_{s2})\vee\nu_B(0)\vee\nu_A(0)\vee\nu_B(b_{t2})\right\}\\
&\geq (\nu_{A+B}(a_{s1})\vee\nu_{A+B}(b_{t1}))\vee(\nu_{A+B}(a_{s2})\vee\nu_{A+B}(b_{t2}))\\
&\geq \nu_{A+B}(x_{i1})\vee\nu_{A+B}(x_{i2}).
\end{aligned}
$$

Therefore $A+B=(\mu_{A+B},\nu_{A+B})$ is an intuitionistic fuzzy subcoalgebra of C. □

Theorem 3.5. *Let $A_i=(\mu_{A_i},\nu_{A_i})$ be a family of intuitionistic fuzzy subcoalgebras of C. Then $\bigcap_{i\in N} A_i$ is an intuitionistic fuzzy subcoalgebra.*

However, the union of two intuitionistic fuzzy subcoalgebras can not be fuzzified. Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be intuitionistic fuzzy subcoalgebras of C. Then $A\cup B$ can not be an intuitionistic fuzzy subcoalgebra of C. In fact intuitionistic fuzzy subcoalgebras are intuitionistic fuzzy subspaces of coalgebra C, whereas the union of intuitionistic fuzzy subspaces is not an intuitionistic fuzzy subspace in general. The counterexample is Example 3.2.

Example 3.2. Here, we also use the coalgebra in Example 3.1 and assume the basis is $\{g_1,g_2,d_1\}$. Let $x\neq 0$. Then $x=\sum a_ig_i+b_1d_1$ where $a_i,b_1\neq 0$. We define $A=(\mu_A,\nu_A)$ where $\mu_A(x)=(\wedge\mu_A(g_i))\wedge\mu_A(d_1)$ in which $\mu_A(g_1)=1$, $\mu_A(g_2)=\frac{1}{3}$, $\mu_A(d_1)=\frac{1}{4}$ and $\nu_A(x)=(\vee\nu_A(g_i))\vee\nu_A(d_1)$ in which $\nu_A(g_1)=0$, $\nu_A(g_2)=\frac{1}{3}$, $\nu_A(d_1)=\frac{1}{4}$.

We also define $B=(\mu_B,\nu_B)$ where $\mu_B(x)=(\wedge\mu_B(g_i))\wedge\mu_B(d_1)$ in which $\mu_B(g_1)=\frac{1}{2}$, $\mu_B(g_2)=1$, $\mu_B(d_1)=\frac{1}{5}$ and $\nu_B(x)=(\vee\nu_B(g_i))\vee\nu_B(d_1)$ in which $\nu_B(g_1)=\frac{1}{2}$, $\nu_B(g_2)=0$, $\nu_B(d_1)=\frac{3}{5}$. If $x=0$, we define $\mu_A(0)=1,\nu_A(0)=0$ and $\mu_B(0)=1,\nu_B(0)=0$. Then A,B are intuitionistic fuzzy subcoalgebras of C. We have $(\mu_A\cup\mu_B)(g_1+g_2)=\mu_A(g_1+g_2)\vee\mu_B(g_1+g_2)=\min\{\mu_A(g_1),\mu_A(g_2)\}\vee\min\{\mu_B(g_1),\mu_B(g_2)\}=\frac{1}{3}\vee\frac{1}{2}=\frac{1}{2}$, however, $\min\{(\mu_A\cup\mu_B)(g_1),(\mu_A\cup\mu_B)(g_2)\}=\min\{\mu_A(g_1)\vee\mu_B(g_1),\mu_A(g_2)\vee\mu_B(g_2)\}=\min\{1,1\}=1$, so $(\mu_A\cup\mu_B)(g_1+g_2)<\min\{(\mu_A\cup\mu_B)(g_1),(\mu_A\cup\mu_B)(g_2)\}$, this shows that $A\cup B$ is not an intuitionistic fuzzy subcoalgebra of C.

4 The homomorphisms of coalgebras

Let C be a k-coalgebra and I be a coideal of C. Let C/I be a quotient space, $p : C \to C/I$ by $p(x) = \bar{x}$ is canonical map, where $\bar{x} = x + I \in C/I$, we know that there exists a unique structure of coalgebra on C/I.

Theorem 4.1. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set of coalgebra C and I be a coideal of C. Define $B = (\mu_B, \nu_B) : C/I \to [0,1]$ by $\mu_B(\bar{m}) = \sup_{x \in \bar{m}}\{\mu_A(x)\}$ and $\nu_B(\bar{m}) = \inf_{x \in \bar{m}}\{\nu_A(x)\}$, for any $\bar{m} \in C/I$. Then $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy subcoalgebra of C/I.*

Proof. Let $\bar{x}, \bar{y} \in C/I$. Then

$$\begin{aligned}
\mu_B(\bar{x}+\bar{y}) &= \sup_{z \in \overline{x+y}}\{\mu_A(z)\} \geq \sup_{a \in \bar{x}, b \in \bar{y}}\{\mu_A(a+b)\} \\
&\geq \sup_{a \in \bar{x}, b \in \bar{y}}\{\mu_A(a) \wedge \mu_A(b)\} = \sup_{a \in \bar{x}}\{\mu_A(a)\} \wedge \sup_{b \in \bar{y}}\{\mu_A(b)\} \\
&= \mu_B(\bar{x}) \wedge \mu_B(\bar{y}).
\end{aligned}$$

and

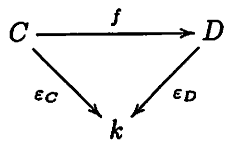
$$\begin{aligned}
\nu_B(\bar{x}+\bar{y}) &= \inf_{z \in \overline{x+y}}\{\nu_A(z)\} \leq \inf_{a \in \bar{x}, b \in \bar{y}}\{\nu_A(a+b)\} \\
&\leq \inf_{a \in \bar{x}, b \in \bar{y}}\{\nu_A(a) \vee \nu_A(b)\} = \inf_{a \in \bar{x}}\{\nu_A(a)\} \vee \inf_{b \in \bar{y}}\{\nu_A(b)\} \\
&= \nu_B(\bar{x}) \vee \nu_B(\bar{y}).
\end{aligned}$$

Let $\alpha \in k$ and $\bar{x} \in C/I$. Then $\mu_B(\alpha\bar{x}) \geq \mu_B(\bar{x})$ and $\nu_B(\alpha\bar{x}) \leq \nu_B(\bar{x})$. Let $\bar{x} \in C/I$. Then $\overline{\Delta}(\bar{x}) = \sum \overline{x_{(j1)}} \otimes \overline{x_{(j2)}} = \sum \overline{m_{(i1)}} \otimes \overline{m_{(i2)}} = \overline{\Delta}(\bar{m})$, where $m \in \bar{x}$. Therefore, $\mu_B(\bar{x}) = \sup_{m \in \bar{x}}\{\mu_A(m)\} \leq \sup_{m_{i1} \in \overline{x_{j1}}}\{\mu_A(m_{(i1)})\} \wedge \sup_{m_{i2} \in \overline{x_{j2}}}\{\mu_A(m_{(i2)})\} = \mu_B(\overline{x_{j1}}) \wedge \mu_B(\overline{x_{j2}})$, and $\nu_B(\bar{x}) = \inf_{m \in \bar{x}}\{\nu_A(m)\} \geq \inf_{m_{i1} \in \overline{x_{j1}}}\{\nu_A(m_{(i1)})\} \vee \inf_{m_{i2} \in \overline{x_{j2}}}\{\nu_A(m_{(i2)})\} = \nu_B(\overline{x_{j1}}) \vee \nu_B(\overline{x_{j2}})$.

Hence $B = (\mu_B, \nu_B)$ is an intuitionistic fuzzy subcoalgebra of C/I. □

Definition 4.1. ([12]) Let C and D be two coalgebras. The k-linear map $f : C \to D$ is a *morphism of coalgebra*, if the following diagrams are commutative:

$$\begin{array}{ccc}
C & \xrightarrow{f} & D \\
{\scriptstyle \Delta_C}\downarrow & & {\scriptstyle \Delta_D}\downarrow \\
C \otimes C & \xrightarrow{f \otimes f} & D \otimes D
\end{array}$$



Proposition 4.1. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subcoalgebra of D and $f : C \to D$ be a morphism of coalgebra. Then $f^{-1}(A) = (\mu_{f^{-1}(A)}, \nu_{f^{-1}(A)})$ is an intuitionistic fuzzy subcoalgebra of C.*

Proof. Obviously, $f^{-1}(A) = (\mu_{f^{-1}(A)}, \nu_{f^{-1}(A)})$ is an intuitionistic fuzzy subspace. Let $x \in C$. $\Delta_D(f(x)) = \sum_{i=1,n} f(x)_{i1} \otimes f(x)_{i2} = \sum_{j=1,n} f(x_{j1}) \otimes f(x_{j2})$, so

$$\begin{aligned}\mu_{f^{-1}(A)}(x) = \mu_A(f(x)) &\leq \mu_A(f(x)_{i1}) \wedge \mu_A(f(x)_{i2}) \\ &= \mu_A(f(x_{j1})) \wedge \mu_A(f(x_{j2})) \\ &= \mu_{f^{-1}(A)}(x_{j1}) \wedge \mu_{f^{-1}(A)}(x_{j2})\end{aligned}$$

$$\begin{aligned}\nu_{f^{-1}(A)}(x) = \nu_A(f(x)) &\geq \nu_A(f(x)_{i1}) \vee \nu_A(f(x)_{i2}) \\ &= \nu_A(f(x_{j1})) \vee \nu_A(f(x_{j2})) \\ &= \nu_{f^{-1}(A)}(x_{j1}) \vee \nu_{f^{-1}(A)}(x_{j2})\end{aligned}$$

So $f^{-1}(A) = (\mu_{f^{-1}(A)}, \nu_{f^{-1}(A)})$ is an intuitionistic fuzzy subcoalgebra of C. □

Proposition 4.2. *Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subcoalgebra of C and suppose that $f : C \to D$ be a morphism of coalgebra. Then $f(A) = (\mu_{f(A)}, \nu_{f(A)})$ is an intuitionistic fuzzy subcoalgebra of D.*

Proof. Let $x, y \in D$. It easily get $\mu_{f(A)}(x + y) \geq \mu_{f(A)}(x) \wedge \mu_{f(A)}(y)$ and $\nu_{f(A)}(x + y) \leq \nu_{f(A)}(x) \vee \nu_{f(A)}(y)$. We omit here.

Let $x \in D$. There exists $m \in C$, such that $f(m) = x$. We have $\sum_{i=1,n} x_{i1} \otimes x_{i2} = \Delta_D(x) = \Delta_D(f(m)) = (f \otimes f)\Delta_C(m) = \sum_{j=1,n} f(m_{j1}) \otimes f(m_{j2})$, So $m_{j1} \in f^{-1}(x_{i1}), m_{j2} \in f^{-1}(x_{i2})$. Then

$$\begin{aligned}\mu_{f(A)}(x) &= \sup_{m \in f^{-1}(x)} \{\mu_A(m)\} \\ &\leq \sup_{m_{j1} \in f^{-1}(x_{i1}), m_{j2} \in f^{-1}(x_{i2})} \{\mu_A(m_{j1}) \wedge \mu_A(m_{j2})\} \\ &= \sup_{m_{j1} \in f^{-1}(x_{i1})} \{\mu_A(m_{j1})\} \wedge \sup_{m_{j2} \in f^{-1}(x_{i2})} \{\mu_A(m_{j2})\} \\ &= \mu_{f(A)}(x_{i1}) \wedge \mu_{f(A)}(x_{i2})\end{aligned}$$

and

$$
\begin{aligned}
\nu_{f(A)}(x) &= \inf_{m\in f^{-1}(x)}\{\nu_A(m)\}\\
&\geq \inf_{m_{j1}\in f^{-1}(x_{i1}),m_{j2}\in f^{-1}(x_{i2})}\{\nu_A(m_{j1})\vee\nu_A(m_{j2})\}\\
&= \inf_{m_{j1}\in f^{-1}(x_{i1})}\{\nu_A(m_{j1})\}\vee\inf_{m_{j2}\in f^{-1}(x_{i2})}\{\nu_A(m_{j2})\}\\
&= \nu_{f(A)}(x_{i1})\vee\nu_{f(A)}(x_{i2}).
\end{aligned}
$$

Hence $f(A)=(\mu_{f(A)},\nu_{f(A)})$ is an intuitionistic fuzzy subcoalgebra of D. □

4 Conclusions

Algebraic structures play an important role in mathematics with wide range of applications in many disciplines such as theoretical physics, computer science, engineering, information sciences and coding theory. In this paper, we have applied the concept of intuitionistic fuzzy set theory to generalize results concerning subcoalgebras. Deschrijver and Kerre [14] have pointed out intuitionistic fuzzy sets and interval-valued fuzzy sets are same. It is clear that the most of these result can be simply extended to interval-valued fuzzy subcoalgebras. Our introduced concepts and results can be applied in engineering and computer science.

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