MAXIMAL SEPARATION ON 2-D ARRAYS

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Abstract. Given m, n and $2 \leq l \leq mn$, we study the problem of separating l symbols on an $m \times n$ array such that the minimum ℓ^1 distance between any two of the l symbols is as large as possible. This problem is similar in nature to the well-known Tammes' problem where one tries to achieve the largest angular separation for a given number of points on a 2-D or higher dimensional sphere. It is also closely related to the well-studied problem of constructing optimal interleaving schemes for correcting error bursts in multi-dimensional digital data where a burst can be an arbitrarily shaped connected region in the array. Moreover, the interest in studying this problem also arises from considerations of minimizing the risk of multiple nearby node failures in a distributed data storage system (or a similar industrial network) in the event of a relatively large scale random disruption. We derive bounds on the maximum possible distance of separation for general m, n and l, and provide also optimal constructions in several special cases including small and large l values, small m (or n) values, and $n - 1 \geq (l - 1)(m - 1)$.

Key words. ℓ^1-distance; Interleaving; Packing; Separation; Sphere packing.

1. Introduction. In this paper, we consider the discrete geometry problem of separating $2 \leq l \leq mn$ symbols on an given $m \times n$ array such that the minimum ℓ^1 distance between any two symbols is as large as possible. The motivation to study this problem arises from constructing optimal interleaving schemes for correcting error bursts in multi-dimensional digital data where an error burst can be an arbitrarily shaped connected region in the array. In an interleaving scheme [1, 8, 9, 12, 13, 15, 16], the data's code symbols are scrambled to ensure that error bursts spread across multiple codewords. The code symbols, once de-interleaved, have errors small enough to be corrected easily. Note that, for single random error correction codes, an arbitrarily shaped error burst of size t in an interleaved array can be corrected if and only if any cluster of size t contains at most one symbol from each codeword, or equivalently, the ℓ^1 distance between any two symbols from any same codeword in the array is at least t. Therefore, to maximize its burst error correcting power, any two such symbols should be separated as much as possible so that an arbitrary error burst of size t can be corrected for the largest value of t. Thus the separation problem we study in this paper may be considered a variation of the interleaving problem by focusing on one specially selected codeword. Suppose the array contains m rows and n columns, and the codeword under consideration consists of $l \geq 2$ symbols, the question is then how to place the l symbols on the $m \times n$ array such that the minimum ℓ^1 distance between any two of these l symbols is as large as possible.

The interest in studying this problem also arises from considerations of minimizing the risk of simultaneous multiple nearby (cluster) node failures in a distributed data storage system (or a similar facility network). To increase

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reliability against random individual node failures, distributed storage systems often partition the data, introduce redundancy through erasure coding techniques and store the encoded data across the nodes [5, 11]. However, to further reduce the risk of cluster node failures in the event of a relatively large scale damage, whether caused by a natural disaster, enemy attack, or any other reason, it is desirable to further separate the nodes so that such an event would affect only a small number of nodes.

Furthermore such a problem also occurs when one wishes to minimize unwanted collaborations due to proximity among a group of contestants. In this sense our separation problem is similar in nature to the well-known Tammes' problem [2, 3, 4, 14] where one tries to maximize the angular separation between l points on a 2-D or higher dimensional sphere. (This is also known as the "inimical dictators" problem: where should l inimical dictators build their palaces on a planet so as to be as far away from each other as possible?)

Let $P_i(x_i, y_i)$, $1 \leq i \leq l$, be the cell positions of the symbols in the array with $0 \leq x_i < m$, $0 \leq y_i < n$, and $(x_i, y_i) \neq (x_j, y_j)$ for any $i \neq j$. For simplicity, we also write $\vec{x} = [x_1, x_2, \cdots, x_l]$, $\vec{y} = [y_1, y_2, \cdots, y_l]$. The distance of separation of the symbols (in ℓ^1 metric) is then given by $d(\vec{x}, \vec{y}) =: \min_{1 \leq i \neq j \leq l} |x_i - x_j| + |y_i - y_j|$. Thus, given m, n, and l, our goal is to determine (and realize) the maximum possible distance of separation $\mathcal{D}(m, n; l) =: \max\{d(\vec{x}, \vec{y}) : \vec{x} \in \mathbb{Z}_m^l, \vec{y} \in \mathbb{Z}_n^l\}$.

Note that optimal interleaving aims to maximize the distance of separation for all codewords in an array, while optimal separation achieves maximum distance of separation for a single codeword, ignoring the other codewords in the array. Consequently, separation can achieve greater distances than interleaving. As shown in Fig. 1.1, a 5×5 array with 5 symbols can achieve a maximum separation of distance 4, whereas a 5×5 array with 5 codewords and 5 symbols for each codeword can achieve a maximum interleaving distance of 3.

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(a)

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

(b)

FIG. 1.1. *Comparison of separation and interleaving:* (a) *Optimal separation of* 5 *symbols on a* 5×5 *array with maximum distance of separation* 4. (b) *Optimal interleaving of a* 5×5 *array with maximum interleaving distance* 3.

Finally, it is easy to see that the above separation problem is equivalent to its complementary packing problem [6, 7, 10] of finding, for any given m, n, and d, the maximum number of symbols $\mathcal{L}(m, n; d)$ that can be placed on an $m \times n$ array such that the ℓ^1 distance between any two symbols is at least d. Then $\mathcal{L}(m, n; d) = \max\{l : \mathcal{D}(m, n; l) \geq d\}, \mathcal{D}(m, n; l) = \max\{d : \mathcal{L}(m, n; d) \geq l\}$.

Note that by symmetry, we also have $\mathcal{D}(m, n; l) = \mathcal{D}(n, m; l)$, $\mathcal{L}(m, n; d) = \mathcal{L}(n, m; d)$. Thus unless otherwise stated, we assume $m \leq n$ in the rest of the paper. Additionally, these measures satisfy the following combinatorial properties:

(1) $\mathcal{L}(m, n_1 + n_2; d) \leq \mathcal{L}(m, n_1; d) + \mathcal{L}(m, n_2; d)$.

(2) If $\mathcal{D}(m, n_1; l_1) \leq d$, $\mathcal{D}(m, n_2; l_2) \leq d$, then $\mathcal{D}(m, n_1 + n_2; l_1 + l_2) \leq d$.

Some early results on the packing problem can be found in [6, 7, 10] where the k-packing number of an $m \times n$ array, $P_k(P_{m,n})$, corresponds to $\mathcal{L}(m, n; d)$ defined above with $d = k + 1$. For $k = 1$, it is known that $P_1(P_{m,n}) = \mathcal{L}(m, n; 2) = \lceil mn/2 \rceil$. For $k = 2$, with the aid of a computer program for finding the packing numbers for the cases when $m, n \leq 18$, and when $m \leq 8$ and $n \leq 25$, the 2-packing problem is solved in [6]; in particular, it is verified that $P_2(P_{m,n}) = \mathcal{L}(m, n; 3) = \lceil mn/5 \rceil$ for $m, n \geq 9$. The problem for $k = 3$ is further examined in [7], and again with the aid of computer search, the 3-packing problem is solved for the cases $m \leq 18$ only.

Our primary goal is to determine the maximum possible distance of separation $\mathcal{D}(m, n; l)$ for general values of m, n and l. In the following sections, we derive two types of bounds on $\mathcal{D}(m, n; l)$ for general m, n and l, and provide also optimal constructions in several special cases including small and large l values, small m values and $n - 1 \geq (l - 1)(m - 1)$.

2. Zigzag bounds and constructions.

2.1. Preliminaries. We start with the simplest case of $l = 2$. Obviously, to maximize the distance of separation, the two symbols must be placed at two diagonally opposite corners of the array, namely, $(0, 0)$ and $(m - 1, n - 1)$, or $(0, n - 1)$ and $(m - 1, 0)$. Thus we have

LEMMA 2.1 (Case $l = 2$). $\mathcal{D}(m, n; 2) = m + n - 2$.

Another simple case is $m = 1$. In this case, we have

LEMMA 2.2 (Case $m = 1$). $\mathcal{D}(1, n; l) = \lfloor (n - 1)/(l - 1) \rfloor$, $\mathcal{L}(1, n; d) = 1 + \lfloor (n - 1)/d \rfloor$.

Proof. For $m = 1$, we have $x_i = 0$ for all $i = 1, 2, \cdots, l$. Without loss of generality, we assume $0 \leq y_1 < y_2 < \cdots < y_l \leq n - 1$. Then we have $y_{i+1} - y_i \geq d(\vec{x}, \vec{y})$ for all $i = 1, 2, \cdots, l - 1$. Adding all these up, we get $(l - 1)d(\vec{x}, \vec{y}) \leq \sum_{i=1}^{l-1}(y_{i+1} - y_i) = y_l - y_1 \leq n - 1$. This yields $d(\vec{x}, \vec{y}) \leq \lfloor (n - 1)/(l - 1) \rfloor$, and thus $\mathcal{D}(1, n; l) \leq \lfloor (n - 1)/(l - 1) \rfloor$.

On the other hand, it is easy to see that the maximum distance of separation $d = \mathcal{D}(1, n; l) = \lfloor (n - 1)/(l - 1) \rfloor$ can be achieved by taking $y_i = (i - 1)d$, $1 \leq i \leq l$. □

REMARK 2.3. *If $(l - 1) | (n - 1)$, then $d = \mathcal{D}(1, n; l)$ can be achieved only with $y_i = (i - 1)d$, $1 \leq i \leq l$, that is, $\vec{y} = [0, d, 2d, \cdots, (l - 1)d]$.*

2.2. Upper bound. Next, by compressing the $m \times n$ array into a 1-D array, we prove

THEOREM 2.4. $\mathcal{D}(m, n; l) \leq (m - 1) + \lfloor (n - 1)/(l - 1) \rfloor$.

Proof. Let $\vec{x} \in \mathbb{Z}_m^l$, $\vec{y} \in \mathbb{Z}_n^l$ and $d = d(\vec{x}, \vec{y})$. Without loss of generality, we assume $y_1 \leq y_2 \leq \cdots \leq y_l$. Since $|x_i - x_j| + |y_i - y_j| \geq d$ for all $i \neq j$, and $|x_i - x_j| \leq m - 1$, we have $|y_i - y_j| \geq d - (m - 1)$ for all $i \neq j$, and thus $(l-1)(d-m+1) \le \sum_{i=1}^{l-1} |y_{i+1} - y_i| = y_l - y_1 \le n-1$. This shows $d \le (m-1) + \lfloor (n-1)/(l-1) \rfloor$. Theorem 2.4 now follows. □

2.3. Maximal separation in the case $n-1 \ge (l-1)(m-1)$. Our next theorem shows that the upper bound established in Theorem 2.4 can actually be achieved when $n-1 \ge (l-1)(m-1)$.

THEOREM 2.5. *Assume $l \ge 2$ and $n-1 \ge (l-1)(m-1)$. Then $\mathcal{D}(m,n;l) = (m-1) + \lfloor (n-1)/(l-1) \rfloor$.*

Proof. Let $\delta = \lfloor (n-1)/(l-1) \rfloor$ and $d = (m-1) + \delta$. Note that the assumption $n-1 \ge (l-1)(m-1)$ implies $\delta \ge m-1$. Now consider the zigzag construction with $\vec{x} = [0, m-1, 0, m-1, \cdots]$ and $\vec{y} = [0, \delta, 2\delta, \cdots]$ (that is $y_i = (i-1)\delta$ for all i, and $x_i = 0$ for all i even and $x_i = m-1$ for all i odd). Then for $i \ne j$ and $i-j$ odd, we have $|x_i - x_j| = m-1$ and $|y_i - y_j| = |i-j|\delta \ge \delta$, and thus $|x_i - x_j| + |y_i - y_j| \ge (m-1) + \delta = d$. On the other hand, for $i \ne j$ and $i-j$ even, we have $|x_i - x_j| = 0$, $|y_i - y_j| = |i-j|\delta \ge 2\delta$, and thus $|x_i - x_j| + |y_i - y_j| \ge 2\delta \ge (m-1) + \delta = d$. This shows $d(\vec{x}, \vec{y}) \ge d$ and thus $\mathcal{D}(m,n;l) \ge d$. □

REMARK 2.6. *If $n-1 \ge (l-1)(m-1)$ and $(l-1)|(n-1)$, then the maximal separation $\mathcal{D}(m,n;l) \le (m-1) + \lfloor (n-1)/(l-1) \rfloor$ can be achieved only with $\vec{x} = [0, m-1, 0, m-1, \cdots]$ (or by upside-down flipping, $\vec{x} = [m-1, 0, m-1, 0, \cdots]$) and $\vec{y} = [0, \delta, 2\delta, \cdots]$ where $\delta = (n-1)/(l-1)$.*

REMARK 2.7. *From the proof of Theorem 2.5, it is clear that, in the case $n-1 \le (l-1)(m-1)$ (and thus $\delta = \lfloor (n-1)/(l-1) \rfloor \le m-1$), the distance of separation for the same zigzag scheme $\vec{x} = [0, m-1, 0, m-1, \cdots]$, $\vec{y} = [0, \delta, 2\delta, \cdots]$ is then given by 2δ. The corresponding lower bound $\mathcal{D}(m,n;l) \ge 2\lfloor (n-1)/(l-1) \rfloor$ for the case $n-1 \le (l-1)(m-1)$, however, is in general not optimal. This is not surprising, since for relatively large values of l, an effective separation scheme would have to utilize also the interior cells of the array, instead of the boundary cells exclusively as in case of the zigzag scheme, see again the example in Fig. 1.1(a). In such cases, the sphere packing bounds presented in the next section will provide more accurate estimates.*

3. Sphere packing bounds.

3.1. Discrete 2-D spheres. The concept of discrete spheres plays a key role in the study of interleaving schemes for correcting burst errors in 2-D and higher dimensional data [1, 8, 9, 15, 16, 17]. In this section we show how they can be used to obtain alternate bounds on $\mathcal{D}(m,n;l)$.

Given $d \in \mathbb{N}$, the discrete 2-D sphere $\mathcal{S}_{2,d}$ with diameter d can be defined as the following subset of $\mathbb{Z}^2$

$$\mathcal{S}_{2,d} = \begin{cases} \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| + |x_2| < d/2\} & \text{if } d \text{ is odd,} \\ \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| + |x_2 - 1/2| < d/2\} & \text{if } d \text{ is even.} \end{cases}$$

Fig. 3.1 shows the 2-D spheres $\mathcal{S}_{2,d}$ with $d = 1, 2, 3, 4, 5$. Geometrically, $\mathcal{S}_{2,d}$ can be constructed recursively by appending all neighbors of $\mathcal{S}_{2,d-2}$, starting with $\mathcal{S}_{2,1} = \{(0,0)\}$ if d is odd, and $\mathcal{S}_{2,2} = \{(0,0),(0,1)\}$ if d is even. Additionally, any translation $\mathcal{S}_{2,d}(\mathbf{C}) = \mathcal{S}_{2,d} + \mathbf{C} = \{\mathbf{x} + \mathbf{C} \in \mathbb{Z}^2 : \mathbf{x} \in \mathcal{S}_{2,d}\}$ of $\mathcal{S}_{2,d}$ by $\mathbf{C} \in \mathbb{Z}^2$ will also be referred to as a sphere with diameter d.

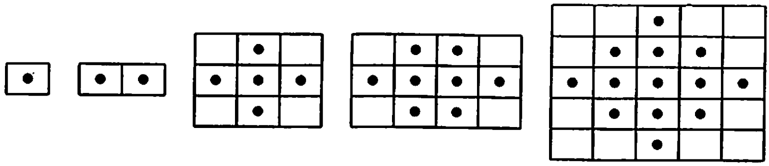


FIG. 3.1. *2-D spheres* $\mathcal{S}_{2,d}$ *with* $d = 1, 2, 3, 4, 5$.

LEMMA 3.1 ([1, 9]). *Let* $d \in \mathbb{N}$. *Then for any* $\mathbf{x} \in \mathcal{S}_{2,d}$, $\mathbf{y} \in \mathcal{S}_{2,d}$, *it holds that* $d(\mathbf{x}, \mathbf{y}) < d$; *and for any* $\mathbf{x} \notin \mathcal{S}_{2,d}$, *there exists* $\mathbf{y} \in \mathcal{S}_{2,d}$ *such that* $d(\mathbf{x}, \mathbf{y}) \geq d$; *also if* $d(\mathbf{x}, \mathbf{y}) \geq d$, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$, *then* $\mathcal{S}_{2,d}(\mathbf{x}) \cap \mathcal{S}_{2,d}(\mathbf{y}) = \emptyset$; *moreover, by counting the elements in* $\mathcal{S}_{2,d}$, *we have* $|\mathcal{S}_{2,d}| = \lceil d^2/2 \rceil$.

3.2. Sphere packing bounds. An especially important property of these discrete 2-D spheres $\mathcal{S}_{2,d}$ is that they tile up $\mathbb{Z}^2$ for any $d \in \mathbb{N}$. By labeling each cell by a distinct integer in the same way for each of these spheres $\mathcal{S}_{2,d}$, one naturally obtains a lattice interleaving scheme on $\mathbb{Z}^2$ with interleaving distance d using $|\mathcal{S}_{2,d}|$ distinct integers.

LEMMA 3.2 ([1, 9]). *Let* $d \in \mathbb{N}$, $b = 2\lceil d/2 \rceil - 1$, $k = |\mathcal{S}_{2,d}| = \lceil d^2/2 \rceil$. *By labeling cell* (i, j) *by integer* $a_{ij} = j - bi \pmod{k}$, *one obtains a lattice interleaver on* $\mathbb{Z}^2$ *with interleaving distance* d *using* $|\mathcal{S}_{2,d}|$ *distinct integers.*

Using such lattice interleavers, it is clear that, inside any $m \times n$ subarray of $\mathbb{Z}^2$, at least one of the $k = |\mathcal{S}_{2,d}|$ integers will appear at least $l = \lceil mn/k \rceil$ times. Thus we have

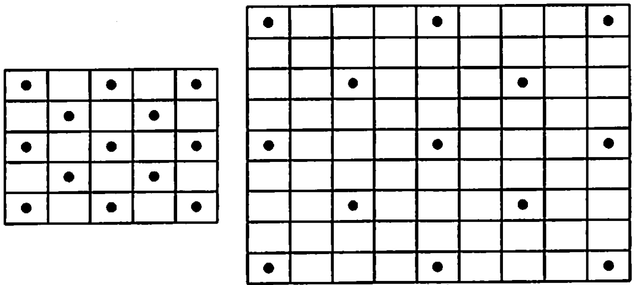
THEOREM 3.3. $\mathcal{L}(m, n; d) \geq \lceil mn/|\mathcal{S}_{2,d}| \rceil$.

COROLLARY 3.4. $\mathcal{D}(m, n; l) \geq d$ *for any* d *satisfying* $(l-1)|\mathcal{S}_{2,d}| < mn$.

On the other hand, suppose l symbols are placed at the cells $P_i(x_i, y_i)$, $i = 1, 2, \ldots, l$, on an $m \times n$ array such that the ℓ^1 distance between any two of them is at least d, then the l spheres $\mathcal{S}_{2,d}(P_i)$, centered at P_i, $1 \leq i \leq l$, are all disjoint. While some elements of these spheres may fall outside the given $m \times n$ array, each of these spheres is contained in a $d \times d$ array on $\mathbb{Z}^2$, with the center lying inside the given $m \times n$ array. It follows that the union of all these spheres $\mathcal{S}_{2,d}(P_i)$, $i = 1, 2, \ldots, l$ is a subset of an expanded $(m + d - 1) \times (n + d - 1)$ array containing the original $m \times n$ array on $\mathbb{Z}^2$. This leads to

THEOREM 3.5. *Let* $d = \mathcal{D}(m, n; l)$ *or* $l = \mathcal{L}(m, n; d)$. *Then we have* $l|\mathcal{S}_{2,d}| \leq (m + d - 1)(n + d - 1)$.

In fact, it is easy to see that, unless $d = 1$, the union of the spheres $\mathcal{S}_{2,d}(P_i)$, $i = 1, 2, \ldots, l$, cannot cover the whole expanded $(m + d - 1) \times (n + d - 1)$ array. Thus the upper bound in Theorem 3.5 may be improved by examining the boundary effects more closely. The details can be quite complicated in general, especially in determining $\mathcal{L}(m, n; d)$. However, in the case of relatively large values of l, so that the maximum possible distance of separation $\mathcal{D}(m, n; l)$ is relatively small compared with m and n, the majority of the symbols will be sufficiently away from the boundary. In such cases, one would expect the upper bound provided by Theorem 3.5 to be quite accurate. See Fig. 3.2 for an example.



(a) (b)

FIG. 3.2. *(a) A* 5×5 *array with* 13 *symbols and maximum possible distance of separation* $\mathcal{D}(5,5;13) = 2$. *The optimality can be verified by Theorem 3.5 (see also Theorem 4.8 and Corollary 4.9). (b) A* 9×9 *array with* 13 *symbols and maximum possible distance of separation* $\mathcal{D}(9,9;13) = 4$, *obtained by periodically extending the* 5 × 5 *array in Fig. 1.1(a), or by expanding the* 5 × 5 *array in (a) by inserting empty rows and columns. In this case, the upper bound* $\mathcal{D}(9,9;13) \le 4$ *can also be verified by Theorem 3.5 (with strict inequality).*

4. Small and large l values. With the general bounds on $\mathcal{D}(m,n;l)$ and $\mathcal{L}(m,n;d)$ at hand, we now turn our attention to finding the exact values for $\mathcal{D}(m,n;l)$ and $\mathcal{L}(m,n;d)$ in several special cases. In this section, we consider special l values of $l = 3,4,5$ and $l > \lceil mn/2 \rceil$, and the next section will be devoted to small m values of $m = 2,3,4$.

4.1. Case $l = 3$. In this subsection, we consider the case $l = 3$ and prove

THEOREM 4.1. *Let $l = 3$ and $m \le n$. Then*

$$\mathcal{D}(m,n;3) = \begin{cases} \lfloor 2(m+n-2)/3 \rfloor & \text{if } n \le 2m-1, \\ m-1+\lfloor (n-1)/2 \rfloor & \text{if } n \ge 2m-1. \end{cases}$$

Note that by Theorems 2.4 and 2.5, we only have to prove Theorem 4.1 for the case $n \le 2m-1$. This is achieved by the following two lemmas.

LEMMA 4.2 (New upper bound for $l = 3$). $\mathcal{D}(m,n;3) \le \lfloor 2(m+n-2)/3 \rfloor$.

Proof. Let $\vec{x} \in \mathbb{Z}_m^3$, $\vec{y} \in \mathbb{Z}_n^3$ and $d = d(\vec{x},\vec{y})$. Then for all $1 \le i < j \le 3$, we have $|x_i - x_j| + |y_i - y_j| \ge d$. Adding all these inequalities, we obtain $3d \le \sum_{i<j} |x_i - x_j| + |y_i - y_j| = 2(\max_i x_i - \min_i x_i) + 2(\max_i y_i - \min_i y_i) \le 2(m-1) + 2(n-1)$. Thus $d \le 2(m+n-2)/3$. Lemma 4.2 now follows. □

LEMMA 4.3 (Construction). *Let $l = 3$ and $m \le n \le 2m-1$. Then* $\mathcal{D}(m,n;3) \ge \lfloor 2(m+n-2)/3 \rfloor$.

Proof. Let $d = \lfloor 2(m+n-2)/3 \rfloor$, $\delta_1 = \lfloor (2n-m-1)/3 \rfloor = d-(m-1)$, $\delta_2 = \lfloor (2m-n-1)/3 \rfloor = d-(n-1)$. Note that by assumption $m \le n \le 2m-1$, we have $0 \le \delta_1 < n$, $0 \le \delta_2 < m$. Next, let $P_1(0,0)$, $P_2(m-1,\delta_1)$, $P_3(\delta_2, n-1)$. Then we have $|P_1P_2| = m-1+\delta_1 = d$, $|P_1P_3| = n-1+\delta_2 = d$, and $|P_2P_3| = m+n-2-\delta_1-\delta_2 = 2(m+n-2)-2d \ge 3\lfloor 2(m+n-2)/3 \rfloor - 2d = 3d-2d = d$. This shows $\mathcal{D}(m,n;3) \ge d = \lfloor 2(m+n-2)/3 \rfloor$ in this case. □

4.2. Case $l = 4$. Next for $l = 4$, we prove the following

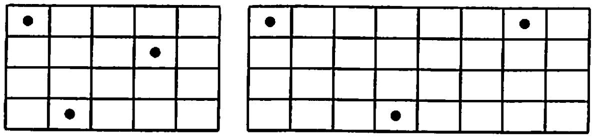


FIG. 4.1. *Left:* $\mathcal{D}(4,4;3) = \mathcal{D}(4,5;3) = 4$ *with* $\vec{x} = [0,3,1]$, $\vec{y} = [0,1,3]$. *Right:* $\mathcal{D}(4,7;3) = \mathcal{D}(4,8;3) = 6$ *with* $\vec{x} = [0,3,0]$, $\vec{y} = [0,3,6]$.

THEOREM 4.4. *Let* $l = 4$ *and* $m \le n$. *Then*

$$\mathcal{D}(m,n;4) = \begin{cases} \lfloor (m+n-2)/2 \rfloor & \text{if } n \le 3m-2, \\ m-1+\lfloor (n-1)/3 \rfloor & \text{if } n \ge 3m-2. \end{cases}$$

Again by Theorems 2.4 and 2.5, it remains to prove Theorem 4.4 for the case $m \le n \le 3m-2$. First we have

LEMMA 4.5 (New upper bound for $l = 4$). $\mathcal{D}(m,n;4) \le \lfloor (m+n-2)/2 \rfloor$.

Proof. Let $\vec{x} \in \mathbb{Z}_m^4$ and $\vec{y} \in \mathbb{Z}_n^4$. Denote $P_i(x_i, y_i)$, $i = 1,2,3,4$. Without loss of generality, we assume $x_1 \le x_2 \le x_3 \le x_4$. Furthermore, by reflection (upside-down flipping) if necessary, we may also assume $y_1 \le y_4$. Based on the values of y_2 and y_3, we separate the following cases:

(a) At least one of y_2 and y_3 is between y_1 and y_4. Let $i = 2$ or $i = 3$ with $y_1 \le y_i \le y_4$. Then we have $|P_1P_i| + |P_iP_4| = |P_1P_4| \le m+n-2$, hence $d(\vec{x},\vec{y}) \le \min\{|P_1P_i|, |P_iP_4|\} \le \lfloor (m+n-2)/2 \rfloor$.

(b) One of y_2 and y_3 is less than y_1 and the other is greater that y_4. Let $i, j \in \{2,3\}$ such that $y_i \le y_1$, $y_j \ge y_4$. Then we have $|P_1P_i| + |P_iP_4| + |P_1P_j| + |P_jP_4| = 2(x_4 - x_1) + 2(y_j - y_i) \le 2(m+n-2)$. This implies $d(\vec{x},\vec{y}) \le \min\{|P_1P_i|, |P_iP_4|, |P_1P_j|, |P_jP_4|\} \le \lfloor (m+n-2)/2 \rfloor$.

(c) $y_2 \le y_1$, $y_3 \le y_1$. In this case, we have $|P_1P_2| + |P_2P_3| = |P_1P_3|$ if $y_3 \le y_2$, and $|P_2P_3| + |P_3P_4| = |P_2P_4|$ if $y_3 \ge y_2$. In either case, we have $d(\vec{x},\vec{y}) \le \lfloor (m+n-2)/2 \rfloor$.

(d) $y_2 \ge y_4$, $y_3 \ge y_4$. Then we have $|P_1P_2| + |P_2P_3| = |P_1P_3|$ if $y_2 \le y_3$, and $|P_2P_3| + |P_3P_4| = |P_2P_4|$ if $y_2 \ge y_3$. Again, in either case, we have $d(\vec{x},\vec{y}) \le \lfloor (m+n-2)/2 \rfloor$.

Lemma 4.5 now follows. □

LEMMA 4.6 (Construction for $l = 4$ and $n \le 3m-2$). *Assume* $l = 4$, $m \le n \le 3m-2$. *Then we have* $\mathcal{D}(m,n;4) = \lfloor (m+n-2)/2 \rfloor$.

Proof. Let $d = \lfloor (m+n-2)/2 \rfloor$, $\delta = \lfloor (n-m)/2 \rfloor$. Then from $m \le n \le 3m-2$, we have $0 \le \delta < n$. Let $P_1(0,0)$, $P_2(m-1,\delta)$, $P_3(0,n-1-\delta)$ and $P_4(m-1,n-1)$. Then we have $|P_1P_2| = |P_3P_4| = m-1+\delta = d$, $|P_1P_4| = m+n-2 \ge 2d$, $|P_1P_3| = |P_2P_4| = n-1-\delta = \lceil (m+n-2) \rceil \ge d$, and finally $|P_2P_3| = m+n-2-2\delta = 2m-2 = d+(3m-2-n)/2 \ge d$ for $n-m$ even, and $|P_2P_3| = m+n-2-2\delta = 2m-1 = d+(3m+1-n)/2 \ge d$ for $n-m$ odd. Therefore $\mathcal{D}(m,n;4) \ge d = \lfloor (m+n-2)/2 \rfloor$ for $m \le n \le 3m-2$. □

4.3. Case $l = 5$. Next for $l = 5$, we prove

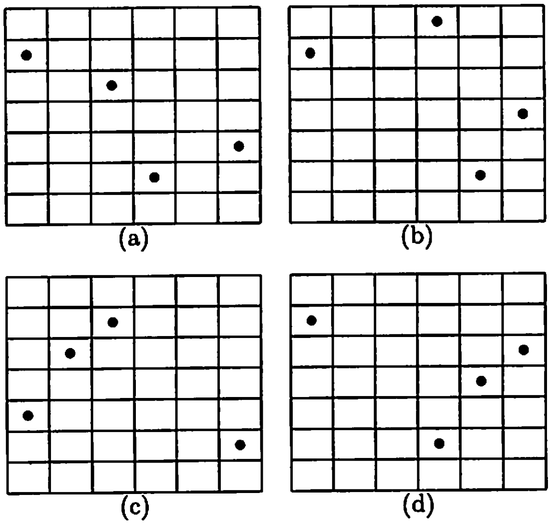


FIG. 4.2. *Relative positions of the 4 symbols in the array with* $x_1 \le x_2 \le x_3 \le x_4$ *and* $y_1 \le y_4$*: (a)* $y_1 \le y_2 \le y_4$*; (b)* $y_2 \le y_1$ *and* $y_3 \ge y_4$*; (c)* $y_2, y_3 \le y_1$*; (d)* $y_2, y_3 \ge y_4$.

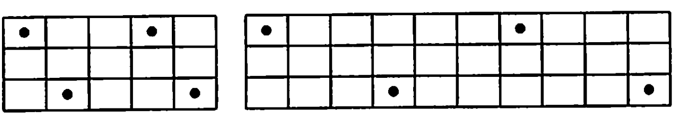


FIG. 4.3. *Left:* $\mathcal{D}(3,5;4) = 3$ *with* $\vec{x} = [0,2,0,2]$, $\vec{y} = [0,1,3,4]$. *Right:* $\mathcal{D}(3,10;4) = 5$ *with* $\vec{x} = [0,2,0,2]$, $\vec{y} = [0,3,6,9]$.

THEOREM 4.7. *Let* $l = 5$ *and* $m \le n$. *Then*

$$\mathcal{D}(m,n;5) = \begin{cases} \lfloor 2(m + \lceil n/2 \rceil - 2)/3 \rfloor & \text{if } n \le 4m-3, \\ (m-1) + \lfloor (n-1)/4 \rfloor & \text{if } n \ge 4m-3. \end{cases}$$

Proof. Again by Theorems 2.4 and 2.5, it suffices to prove Theorem 4.7 for the case $m \le n \le 4m-3$. Note that one of the two $m \times \lceil n/2 \rceil$ subarrays consisting of the left and right $\lceil n/2 \rceil$ columns must contain at least 3 symbols. Thus we have $\mathcal{D}(m,n;5) \le \mathcal{D}(m, \lceil n/2 \rceil; 3)$. The upper bound is then established by using Theorem 4.1 to verify that $\mathcal{D}(m, \lceil n/2 \rceil; 3) = \lfloor 2(m + \lceil n/2 \rceil - 2)/3 \rfloor$ for $m \le n \le 4m-3$.

Next we show that the above bound can be achieved by using essentially the same construction in Lemma 4.3. First, for n odd, we write $n = 2n' - 1$ so $n' = \lceil n/2 \rceil$. Then $m \le n \le 4m-3$ implies $n' \le 2m-1$ and $m \le 2n'-1$. By Lemma 4.3, $\mathcal{D}(m,n';3) = d = \lfloor 2(m+n'-2)/3 \rfloor$ is achieved by using $P_1(0,0)$, $P_2(m-1,\delta_1)$, $P_3(\delta_2, n'-1)$ where $\delta_1 = \lfloor (2n'-m-1)/3 \rfloor$, $\delta_2 = \lfloor (2m-n'-1)/3 \rfloor$ with $0 \le \delta_1 < n'$, $0 \le \delta_2 < m$. Let $P_4(m-1, 2n'-1-\delta_1)$, $P_5(0, 2n'-1)$. By symmetry, we now have $|P_3P_4| = |P_3P_2|$, $|P_3P_5| = |P_3P_1|$ and $|P_4P_5| = |P_2P_1|$.

Furthermore, we have $|P_1P_4| = |P_2P_5| \geq |P_1P_2|$, $|P_1P_5| \geq |P_2P_4| = 2n' - 1 - 2\delta_1 = 2(m + n' - 2) - 2d + 1 \geq d + 1$. This shows $\mathcal{D}(m, n; 5) \geq d = \lfloor 2(m + n' - 2)/3 \rfloor$.

Finally, for n even, we have $\mathcal{D}(m, n; 5) \geq \mathcal{D}(m, n - 1; 5) = \lfloor 2(m + \lceil (n - 1)/2 \rceil - 2)/3 \rfloor = \lfloor 2(m + \lceil n/2 \rceil - 2)/3 \rfloor$. Theorem 4.7 now follows. □

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FIG. 4.4. *Left:* $\mathcal{D}(4, 7; 5) = \mathcal{D}(4, 8; 5) = 4$ *with* $\vec{x} = [0, 3, 1, 3, 0]$, $\vec{y} = [0, 1, 3, 5, 6]$. *Right:* $\mathcal{D}(5, 5; 5) = \mathcal{D}(5, 6; 5) = 4$ *with* $\vec{x} = [0, 4, 2, 4, 0]$, $\vec{y} = [0, 0, 2, 4, 4]$.

4.4. Case $l > \lceil mn/2 \rceil$. When l is sufficiently large, the high density of symbols will necessarily lead to the occurrence of a cluster of connected symbols in the array, thus forcing $\mathcal{D}(m, n; l) = 1$. The next theorem shows this is the case when $l > \lceil mn/2 \rceil$.

THEOREM 4.8 (Case of too many symbols: $l > \lceil mn/2 \rceil$). *Let $l > \lceil mn/2 \rceil$. Then $\mathcal{D}(m, n; l) = 1$.*

Proof. Consider a path that consists of horizontal or vertical moves at each step and visits each cell of the array exactly once (see Fig. 4.5 for an example). Suppose $\mathcal{D}(m, n; l) \geq 2$. Then for each of the l symbols except the one (if any) at the end of the path, the cell next to the symbol along the path must be vacant. Thus we have $2(l - 1) + 1 \leq mn$, or equivalently $l \leq \lceil mn/2 \rceil$, contradicting the assumption $l > \lceil mn/2 \rceil$. □

⇒	→	→	→	↓
↓	←	←	←	←
→	→	→	→	↓
↓	←	←	←	←
→	→	→	→	□

⇒	→	→	→	↓
→	→	→	↓	↓
↑	→	□	↓	↓
↑	↑	←	←	↓
↑	←	←	←	←

FIG. 4.5. *Each path starts with the upper-left cell and visits each cell of the array exactly once.*

COROLLARY 4.9 ([6, 10]). $\mathcal{L}(m, n; 2) = \lceil mn/2 \rceil$.

Proof. By Theorem 4.8, we have $\mathcal{L}(m, n; 2) \leq \lceil mn/2 \rceil$. The equality can be achieved by the standard chessboard scheme, see Fig. 4.6 for an example. □

5. Small m values.

5.1. Case $m = 2$. First we consider the case $m = 2$. By Theorem 2.5, we have $\mathcal{D}(2, n; l) = 1 + \lfloor (n - 1)/(l - 1) \rfloor$ if $n - 1 \geq (l - 1)(m - 1)$, that is, $l \leq n$. On the other hand, for $l > n = \lceil mn/2 \rceil$, Theorem 4.8 shows $\mathcal{D}(2, n; l) = 1$. Combining these, we have

THEOREM 5.1. *Let $m = 2$ and $l \geq 2$. Then $\mathcal{D}(2, n; l) = 1 + \lfloor (n - 1)/(l - 1) \rfloor$.*

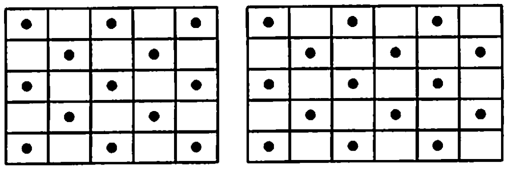

FIG. 4.6. *Chessboard schemes for achieving* $\mathcal{L}(m,n;2) = \lceil mn/2 \rceil$.

COROLLARY 5.2. $\mathcal{L}(2,n;1) = mn$, *and for* $d \geq 2$, *we have* $\mathcal{L}(2,n;d) = 1 + \lfloor (n-1)/(d-1) \rfloor$.

5.2. Case $m = 3$. Next, for $m = 3$, we have

THEOREM 5.3 ($m = 3$). *Let* $m = 3$, $l \geq 2$. *Then*

$$\mathcal{D}(3,n;l) = \begin{cases} 1 & \text{if } l > \lceil 3n/2 \rceil, \\ 2 & \text{if } \lceil 2n/3 \rceil < l \leq \lceil 3n/2 \rceil, \\ 3 & \text{if } \lceil n/2 \rceil < l \leq \lceil 2n/3 \rceil, \\ 2 + \lfloor (n-1)/(l-1) \rfloor & \text{if } l \leq \lceil n/2 \rceil. \end{cases}$$

Proof. Note that by Theorem 4.8, we have $\mathcal{D}(3,n;l) = 1$ for $l > \lceil 3n/2 \rceil$. Next, by Theorem 2.5, we have $\mathcal{D}(3,n;l) = 2 + \lceil (n-1)/(l-1) \rceil \geq 4$ for $n \geq 2l - 1$, that is, $l \leq \lceil n/2 \rceil$. Additionally, by Theorem 2.4 and Corollary 4.9, we have $2 \leq \mathcal{D}(3,n;l) \leq 3$ in the remaining cases $\lceil n/2 \rceil < l \leq \lceil 3n/2 \rceil$. Thus, to finish the proof of Theorem 5.3, it suffices to show $\mathcal{L}(3,n;3) = \lceil 2n/3 \rceil$.

To show this, we first note that for $n = 1, 2, 3$, the result $\mathcal{L}(3,n;3) = \lceil 2n/3 \rceil$ holds trivially. Next, for $n > 3$, by using $\mathcal{L}(3,n;3) \leq \mathcal{L}(3,n-3;3) + \mathcal{L}(3,3;3)$, the upper bound $\mathcal{L}(3,n;3) \leq \lceil 2n/3 \rceil$ follows easily by induction. Finally we note that the equality $\mathcal{L}(3,n;3) = \lceil 2n/3 \rceil$ can be achieved by using the periodic construction in Fig. 5.1. □

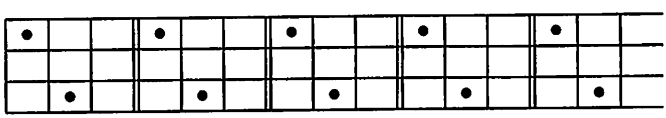

FIG. 5.1. *Periodic packing scheme for* $3 \times n$ *arrays with distance of separation* $d = 3$.

COROLLARY 5.4. *Let* $m = 3$. *Then we have* $\mathcal{L}(3,n;1) = 3n$, $\mathcal{L}(3,n;2) = \lceil 3n/2 \rceil$, $\mathcal{L}(3,n;3) = \lceil 2n/3 \rceil$, *and* $\mathcal{L}(3,n;d) = 1 + \lfloor (n-1)/(d-2) \rfloor$ *for* $d \geq 4$.

5.3. Case $m = 4$. We now consider the case of $m = 4$. By Theorem 4.8 (and Corollary 4.9), we have $\mathcal{D}(4,n;l) = 1$ for $l > \mathcal{L}(4,n;2) = 2n$. Also by Theorem 2.5, we have $\mathcal{D}(4,n;l) = 3 + \lfloor (n-1)/(l-1) \rfloor$ for $n \geq 3l - 2$ (or equivalently, $l \leq \lceil n/3 \rceil$). This implies $\mathcal{L}(4,n;d) = 1 + \lfloor (n-1)/(d-3) \rfloor$ for $6 \leq d \leq m + n - 2$. For the remaining cases $\lceil n/3 \rceil < l \leq 2n$, we have by Theorem 2.4, $2 \leq \mathcal{D}(4,n;l) \leq 5$.

LEMMA 5.5 ($4 \times n$ arrays with $n \leq 7$ and $d = 3$). $\mathcal{L}(4,1;3) = \mathcal{L}(4,2;3) = 2$, $\mathcal{L}(4,3;3) = 3$, $\mathcal{L}(4,4;3) = 4$, $\mathcal{L}(4,5;3) = 5$, $\mathcal{L}(4,6;3) = \mathcal{L}(4,7;3) = 6$.

Proof. Using earlier results for $m \leq 3$, we have $\mathcal{L}(4,1;3) = \mathcal{L}(1,4;3) = 2$, $\mathcal{L}(4,2;3) = \mathcal{L}(2,4;3) = 2$, $\mathcal{L}(4,3;3) = \mathcal{L}(3,4;3) = 3$. Next, for $n = 4,5,6$, we have $\mathcal{L}(4,4;3) \leq \mathcal{L}(4,2;3)+\mathcal{L}(4,2;3) = 4$, $\mathcal{L}(4,5;3) \leq \mathcal{L}(4,2;3)+\mathcal{L}(4,3;3) = 5$, $\mathcal{L}(4,6;3) \leq \mathcal{L}(4,3;3)+\mathcal{L}(4,3;3) = 6$. For $n = 7$, we will prove $\mathcal{L}(4,7;3) \leq 6$ by showing $\mathcal{D}(4,7;7) \leq 2$.

Suppose $\mathcal{D}(4,7;7) \geq 3$. We divide the 4×7 array into two subarrays of size 2×7 each. Note that one of these 2×7 arrays must contain ≥ 4 symbols. Without loss of generality, we assume the top 2×7 subarray contains ≥ 4 symbols. Since $\mathcal{L}(2,7;3) = 4$, the top 2×7 subarray must then contain exactly 4 symbols. Furthermore, by Remark 2.3, $\mathcal{L}(2,7;3) = 4$ can only be achieved with one of the two schemes shown in Fig. 5.2. In either case, to preserve a minimum distance of separation of $d = 3$, the entire third row must then be vacant. This forces the remaining 3 symbols to be placed on the last row with $\mathcal{D}(1,7;3) = 2 < 3 = d$. This contradiction shows $\mathcal{D}(4,7;7) \leq 2$, and thus $\mathcal{L}(4,7;3) \leq 6$.

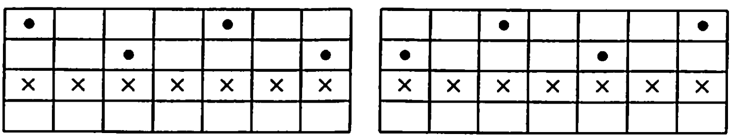

FIG. 5.2. $\mathcal{L}(2,7;3) = 4$.

Finally, it can be easily checked that the equalities are achieved by the scheme in Fig. 5.3. □

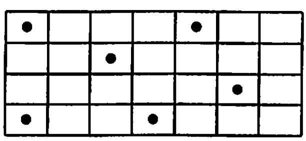

FIG. 5.3. *Optimal separation on $4 \times n$ arrays with $n \leq 7$ and $d = 3$.*

LEMMA 5.6. *Let $n' = \lfloor n/7 \rfloor$ and $r = n - 7n'$ so that $0 \leq r \leq 6$. Then*

$$\mathcal{L}(4,n;3) = \begin{cases} 6n'+2 & \text{if } r = 1, \\ 6n'+r & \text{if } r \neq 1. \end{cases}$$

Proof. By using $\mathcal{L}(4,n;3) \leq \mathcal{L}(4,n-7;3) + \mathcal{L}(4,7;3) = \mathcal{L}(4,n-7;3) + 6$ for $n > 7$, the upper bound follows from Lemma 5.5 by induction. On the other hand, the lower bound can be achieved by periodically extending the 4×7 array in Fig. 5.3 □

REMARK 5.7. *The above result for $\mathcal{L}(4,n;3)$ can be expressed more compactly in the form*

$$\mathcal{L}(4,n;3) = \lceil 6n/7 \rceil + \chi_{7|(n-1)}(n)$$

where $\chi_{7|(n-1)}(n) = 1$ *if* $7 \mid (n-1)$ *and* $\chi_{7|(n-1)}(n) = 0$ *if* $7 \nmid (n-1)$.

LEMMA 5.8. $\mathcal{L}(4,n;4) = \lfloor (3n+4)/5 \rfloor$ *for* $1 \le n \le 10$.

Proof. By examining the 4×10 array (and its subarrays) in Fig. 5.4, it is easy to see that $\mathcal{L}(4,n;4) \ge \lfloor (3n+4)/5 \rfloor$ for $1 \le n \le 10$. Next we show $\mathcal{L}(4,n;4) \le \lfloor (3n+4)/5 \rfloor$ for $1 \le n \le 10$.

Using the earlier results for $m \le 3$, we have $\mathcal{L}(4,1;4) = \mathcal{L}(1,4;4) = 1$, $\mathcal{L}(4,2;4) = \mathcal{L}(2,4;4) = 2$, $\mathcal{L}(4,3;4) = \mathcal{L}(3,4;4) = 2$. Next for $n = 4$, we have $\mathcal{L}(4,4;4) \le \mathcal{L}(4,1;4) + \mathcal{L}(4,3;4) = 3$. For $n = 5$, Theorem 4.4 shows $\mathcal{D}(4,5;4) = 3$ and thus $\mathcal{L}(4,5;4) \le 3$. Finally for $6 \le n \le 10$, we have $\mathcal{L}(4,6;4) \le \mathcal{L}(4,3;4) + \mathcal{L}(4,3;4) = 4$, $\mathcal{L}(4,7;4) \le \mathcal{L}(4,3;4) + \mathcal{L}(4,4;4) = 5$, $\mathcal{L}(4,8;4) \le \mathcal{L}(4,3;4) + \mathcal{L}(4,5;4) = 5$, $\mathcal{L}(4,9;4) \le \mathcal{L}(4,4;4) + \mathcal{L}(4,5;4) = 6$, and $\mathcal{L}(4,10;4) \le \mathcal{L}(4,5;4) + \mathcal{L}(4,5;4) = 6$. □

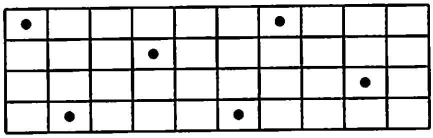

FIG. 5.4. *Optimal separation on* $4 \times n$ *arrays with* $n \le 7$ *and* $d = 4$.

LEMMA 5.9. *Let* $m = 4$ *and* $d = 4$. *Then* $\mathcal{L}(4,n;4) = \lfloor (3n+4)/5 \rfloor$.

Proof. By periodically extending the 4×10 array in Fig. 5.4, we can see that $\mathcal{L}(4,n;4) \ge \lfloor (3n+4)/5 \rfloor$ for all $n \ge 1$. On the other hand, the upper bound $\mathcal{L}(4,n;4) \le \lfloor (3n+4)/5 \rfloor$ follows from Lemma 5.8 for $n \le 10$ and induction for $n > 10$ by using $\mathcal{L}(4,n;4) \le \mathcal{L}(4,n-10;4) + \mathcal{L}(4,10;4) = \mathcal{L}(4,n-10;4) + 6$. □

LEMMA 5.10. *Let* $m = 4$ *and* $d = 5$. *Then* $\mathcal{L}(4,n;5) = \lfloor (2n+4)/5 \rfloor$.

Proof. By using the periodic construction in Fig. 5.5, we can see that $\mathcal{L}(4,n;5) \ge \lfloor (2n+4)/5 \rfloor$ holds for all $n \ge 1$. To establish the upper bound, we first note that for $n = 3,4,5$, $\mathcal{L}(4,1;5) = \mathcal{L}(1,4;5) = 1$, $\mathcal{L}(4,2;5) = \mathcal{L}(2,4;5) = 1$, $\mathcal{L}(4,3;5) = \mathcal{L}(3,4;5) = 2$. Next, for $n = 4$, we have $\mathcal{L}(4,4;5) \le \mathcal{L}(4,2;5) + \mathcal{L}(4,2;5) = 2$. For $n = 5$, by Theorem 4.1, we have $\mathcal{D}(4,5;3) = \lfloor 2(4+5-2)/3 \rfloor = 4 < d = 5$. This shows $\mathcal{L}(4,5;5) \le 2$. Finally, for $n > 5$, let $n' = \lfloor (n-1)/5 \rfloor$ so that $n = 5n' + r$ with $1 \le r \le 5$. Then $\mathcal{L}(4,n;5) \le \mathcal{L}(4,n-5;5) + \mathcal{L}(4,5;5) \le \cdots \le \mathcal{L}(4,r;5) + n'\mathcal{L}(4,5;5) = \lfloor (2r+4)/5 \rfloor + 2n' = \lfloor (2n+4)/5 \rfloor$. □

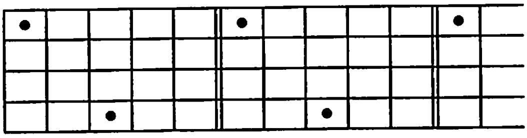

FIG. 5.5. *Optimal separation on* $4 \times n$ *arrays with* $d = 5$.

Summarizing the above results for $m = 4$, we have

THEOREM 5.11 ($m = 4$). *Let $m = 4$, $l \geq 2$. Then*

$$\mathcal{L}(4,n;d) = \begin{cases} 2n & \text{if } d = 2, \\ \lceil 6n/7 \rceil + \chi_{7|(n-1)} & \text{if } d = 3, \\ \lfloor (3n+4)/5 \rfloor & \text{if } d = 4, \\ \lfloor (2n+4)/5 \rfloor & \text{if } d = 5, \\ 1 + \lfloor (n-1)/(d-3) \rfloor & \text{if } d \geq 6, \end{cases}$$

and

$$\mathcal{D}(4,n;l) = \begin{cases} 1 & \text{if } l > 2n, \\ 2 & \text{if } \lceil 6n/7 \rceil + \chi_{7|(n-1)} < l \leq 2n, \\ 3 & \text{if } \lfloor (3n+4)/5 \rfloor < l \leq \lceil 6n/7 \rceil + \chi_{7|(n-1)}, \\ 4 & \text{if } \lfloor (2n+4)/5 \rfloor < l \leq \lfloor (3n+4)/5 \rfloor, \\ 5 & \text{if } \lceil n/3 \rceil < l \leq \lfloor (2n+4)/5 \rfloor, \\ 3 + \lfloor (n-1)/(l-1) \rfloor & \text{if } l \leq \lceil n/3 \rceil. \end{cases}$$

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