Total edge irregularity strength of strong product of two paths*

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Abstract

The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 is the graph with $V(G_1) \times V(G_2)$ as the vertex set, and two distinct vertices (x_1, x_2) and (y_1, y_2) are adjacent whenever for each $i \in \{1, 2\}$ either $x_i = y_i$ or $x_i y_i \in E(G_i)$.

An edge irregular total k-labeling $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of G in such a way that for any different edges xy and $x'y'$ their weights $\varphi(x) + \varphi(xy) + \varphi(y)$ and $\varphi(x') + \varphi(x'y') + \varphi(y')$ are distinct. The total edge irregularity strength, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling.

We have determined the exact value of the total edge irregularity strength of the strong product of two paths P_n and P_m.

Keywords : irregularity strength, total edge irregularity strength, edge irregular total labeling, strong product of paths.

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1 Introduction

Bača, Jendroľ, Miller and Ryan [3] defined the notion of an *edge irregular total k-labeling* of a graph $G = (V, E)$ to be a labeling of vertices and edges of G $\varphi : V \cup E \to \{1, 2, \ldots, k\}$ such that the *edge weights* $wt_\varphi(xy) = \varphi(x) + \varphi(xy) + \varphi(y)$ are different for all edges, i.e. $wt_\varphi(xy) \neq wt_\varphi(x'y')$ for all edges $xy, x'y' \in E$. The minimum k for which the graph G has an edge irregular total k-labeling is called *the total edge irregularity strength of* G, $tes(G)$.

The original motivation for the definition of the total edge irregularity strength came from irregular assignments and the irregularity strength of graphs introduced by Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba [6]. An *irregular assignment* is a k-labeling of the edges $\phi : E \to \{1, 2, \ldots, k\}$ such that the sum of the labels of edges incident with a vertex is different for all the vertices of G, and the smallest k for which there is an irregular assignments is the *irregularity strength*, $s(G)$.

Finding the irregularity strength of a graph seems to be hard even for graphs with simple structure, see [4, 8, 11, 16]. Karoński, Luczak and Thomason [14] conjectured that the edges of every connected graph of order at least 3 can be assigned labels from $\{1, 2, 3\}$, such that for all pairs of adjacent vertices the sums of the labels of the incident edges are different.

Let us mention the following result from [3] giving a lower bound on the total edge irregularity strength of a graph:

$$tes(G) \geq \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\}, \tag{1}$$

where $\Delta(G)$ is the maximum degree of G. The authors of [3] determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs.

Recently Ivančo and Jendroľ [10] posed the following conjecture:

Conjecture 1 [10] *Let G be an arbitrary graph different from K_5. Then*

$$tes(G) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil\right\}. \tag{2}$$

Conjecture 1 has been verified for trees in [10], for complete graphs and complete bipartite graphs in [12] and [13], for the Cartesian product of two paths $P_n \square P_m$ in [15], for corona product of a path with certain graphs in [17], for large dense graphs with $\frac{|E(G)|+2}{3} \leq \frac{\Delta(G)+1}{2}$ in [5], for the categorical product of two paths $P_n \times P_m$ in [2] and for the categorical product a cycle and a path $C_n \times P_m$ in [1].

Motivated by the papers [7], [15] and [18] we investigate the total edge irregularity strength of the strong product of two paths.

The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has as vertices the pairs (x, y) where $x \in V(G_1)$ and $y \in V(G_2)$. Vertices (x_1, y_1) and (x_2, y_2) are adjacent if either x_1x_2 is an edge of G_1 and $y_1 = y_2$ or if $x_1 = x_2$ and y_1y_2 is an edge of G_2 or if x_1x_2 is an edge of G_1 and y_1y_2 is an edge of G_2. Note that the edge set of the strong product $G_1 \boxtimes G_2$ is the union of the edge sets of the Cartesian product $G_1 \square G_2$ and categorical product $G_1 \times G_2$, see e.g. [9].

For integers a and b let $[a, b]$ be an interval of integers c, $a \leq c \leq b$. If we consider graph G_1 as the path P_n with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, $E(P_n) = \{x_ix_{i+1} : i \in [1, n-1]\}$ and graph G_2 as the path P_m with $V(P_m) = \{y_1, y_2, \ldots, y_m\}$, $E(P_m) = \{y_jy_{j+1} : j \in [1, m-1]\}$ then $V(P_n \boxtimes P_m) = \{(x_i, y_j) : i \in [1, n],\ j \in [1, m]\}$ is the vertex set and $E(P_n \boxtimes P_m) = \{(x_i, y_j)(x_{i+1}, y_j) : i \in [1, n-1],\ j \in [1, m]\}$ $\cup\{(x_{i+1}, y_j)(x_i, y_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \cup \{(x_i, y_j)(x_{i+1}, y_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \cup \{(x_i, y_j)(x_i, y_{j+1}) : i \in [1, n],\ j \in [1, m-1]\}$ is the edge set of $P_n \boxtimes P_m$.

The paper adds further support to Conjecture 1 by demonstrating that the strong product $P_n \boxtimes P_m$ has total edge irregularity strength equal to $\left\lceil \frac{|E(P_n \boxtimes P_m)|+2}{3} \right\rceil$.

2 Total edge irregularity strength for small cases

In this section we discuss the total edge irregularity strength for $P_n \boxtimes P_m$ if $2 \leq m \leq n \leq 6$. It is easy to verify that $tes(P_2 \boxtimes P_2) = 3$.

Lemma 1 *Let $2 \leq m \leq 3$. Then $tes(P_3 \boxtimes P_m) = 3m - 1$.*

Proof. Since $|E(P_3 \boxtimes P_m)| = 9m - 7$ and $\Delta(P_3 \boxtimes P_m) = 3m - 1$ then from (1) it follows that $tes(P_3 \boxtimes P_m) \geq k = 3m - 1$. The existence of the optimal labeling φ_1 proves the converse inequality.

$$\varphi_1((x_i,y_j)) = \begin{cases} 1, & \text{if } i=1,\ \ j\in[1,m] \\ m-1+j, & \text{if } i=2,\ \ j\in[1,m] \\ k, & \text{if } i=3,\ \ j\in[1,m] \end{cases}$$

$$\varphi_1((x_i,y_j)(x_{i+1},y_j)) = (i-1)(m-1)+j,$$

$$\varphi_1((x_{i+1},y_j)(x_i,y_{j+1})) = (i-1)(m-3)+i+j,$$

$$\varphi_1((x_i,y_j)(x_i,y_{j+1})) = \begin{cases} 1, & \text{if } i=1,\ \ j=1 \\ k-m, & \text{if } i=2,3,\ \ j=1 \\ 2i, & \text{if } i\in[1,3],\ \ j=2 \end{cases}$$

$$\varphi_1((x_i,y_j)(x_{i+1},y_{j+1})) = k-(2-i)m.$$

□

Lemma 2 *Let* $2 \le m \le 4$. *Then* $tes(P_4 \boxtimes P_m) = \left\lceil \frac{4(4m+1)}{3} \right\rceil - (m+4)$.

Proof. According to (1) it is enough to prove that $tes(P_4 \boxtimes P_m) \le k = \left\lceil \frac{4(4m+1)}{3} \right\rceil - (m+4)$. It follows from the next construction of the labeling φ_2, where $j \in [1,m]$.

$$\varphi_2((x_i,y_j)) = \begin{cases} j, & \text{if } i=1,2 \\ 2m-1+j, & \text{if } i=3 \\ k, & \text{if } i=4 \end{cases}$$

$$\varphi_2((x_i,y_j)(x_{i+1},y_j)) =$$

$$= \varphi_2((x_{i+1},y_j)(x_i,y_{j+1})) = \begin{cases} 1, & \text{if } i=1 \\ 4m-3, & \text{if } i=2 \\ 8m-k-6+j, & \text{if } i=3 \end{cases}$$

$$\varphi_2((x_i,y_j)(x_i,y_{j+1})) = \begin{cases} 2m-1, & \text{if } i=1 \\ 4m-3, & \text{if } i=2 \\ 4(m-1), & \text{if } i=3 \\ 12m-2k-7+j, & \text{if } i=4 \end{cases}$$

$$\varphi_2((x_i,y_j)(x_{i+1},y_{j+1})) = \begin{cases} 2m, & \text{if } i=1 \\ 2m-1, & \text{if } i=2 \\ 6m-k-3+j, & \text{if } i=3. \end{cases}$$

It is easy to see that φ_2 is an edge irregular total labeling having the required property. □

Lemma 3 *Let $m \leq n$, $2 \leq m \leq 6$ and $5 \leq n \leq 6$. Then $tes(P_n \boxtimes P_m) = \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n)$.*

Proof. Again with respect to (1) it is enough to prove that $tes(P_n \boxtimes P_m) \leq \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n)$. Let $k = \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n)$ and $j \in [1, m]$. For $n = 5$ we define the labeling φ_3 and for $n = 6$ we define the labeling φ_4 in the following way:

$$\varphi_3((x_i, y_j)) = \begin{cases} j, & \text{if } i = 1, 2 \\ 3m - 2 + j, & \text{if } i = 3 \\ k - m + j, & \text{if } i = 4 \\ k, & \text{if } i = 5 \end{cases}$$

$$\varphi_3((x_i, y_j)(x_{i+1}, y_j)) =$$

$$= \varphi_3((x_{i+1}, y_j)(x_i, y_{j+1})) = \begin{cases} 1, & \text{if } i = 1 \\ 3m - 2, & \text{if } i = 2 \\ 6m - k - 3, & \text{if } i = 3 \\ 13m - 2k - 8 + j, & \text{if } i = 4 \end{cases}$$

$$\varphi_3((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 2m - 1, & \text{if } i = 1 \\ 4m - 3, & \text{if } i = 2, 3 \\ 16m - 2k - 10, & \text{if } i = 4 \\ 16m - 2k - 10 + j, & \text{if } i = 5 \end{cases}$$

$$\varphi_3((x_i, y_j)(x_{i+1}, y_{j+1})) = \begin{cases} 2m, & \text{if } i = 1 \\ m, & \text{if } i = 2 \\ 8m - k - 4, & \text{if } i = 3 \\ 15m - 2k - 8 + j, & \text{if } i = 4 \end{cases}$$

$$\varphi_4((x_i, y_j)) = \begin{cases} j, & \text{if } i = 1, 2 \\ 2m - 1 + j, & \text{if } i = 3 \\ k - m(n - i) + j, & \text{if } i = 4, 5 \\ k, & \text{if } i = 6 \end{cases}$$

$$\varphi_4((x_i, y_j)(x_{i+1}, y_j)) =$$

$$= \varphi_4((x_{i+1}, y_j)(x_i, y_{j+1})) = \begin{cases} 1, & \text{if } i = 1 \\ 2m-1, & \text{if } i = 2 \\ m(n+2)-k-4, & \text{if } i = 3, 4 \\ 10m-k-7+j, & \text{if } i = 5 \end{cases}$$

$$\varphi_4((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 2m-1, & \text{if } i = 1 \\ 6m-2-i, & \text{if } i = 2, 3 \\ m(n+5)-k-6, & \text{if } i = 4 \\ m(2n+1)-k-9, & \text{if } i = 5 \\ m(n+7)-k-9+j, & \text{if } i = 6 \end{cases}$$

$$\varphi_4((x_i, y_j)(x_{i+1}, y_{j+1})) = \begin{cases} 2m, & \text{if } i = 1 \\ 4m-2, & \text{if } i = 2 \\ m(n+4)-k-5, & \text{if } i = 3, 4 \\ m(n+6)-k-7+j, & \text{if } i = 5. \end{cases}$$

One can check that all vertex and edge labels are at most k. Moreover under the labeling φ_3 (respectively, φ_4) the weights of the edges

(i) $(x_1, y_j)(x_2, y_j)$ and $(x_2, y_j)(x_1, y_{j+1})$ admit the consecutive integers from the interval $[3, 2m+1]$,

(ii) $(x_1, y_j)(x_1, y_{j+1})$ and $(x_1, y_j)(x_2, y_{j+1})$ admit the consecutive integers from $2m+2$ to $4m-1$,

(iii) $(x_2, y_j)(x_3, y_j)$ and $(x_3, y_j)(x_2, y_{j+1})$ receive the consecutive integers from $6m-2$ to $8m-4$ (respectively, from $4m$ to $6m-2$),

(iv) $(x_2, y_j)(x_2, y_{j+1})$ and $(x_2, y_j)(x_3, y_{j+1})$ admit the consecutive integers from the interval $[4m, 6m-3]$ (respectively, from the interval $[6m-1, 8m-4]$),

(v) $(x_3, y_j)(x_4, y_j)$ and $(x_4, y_j)(x_3, y_{j+1})$ receive the consecutive integers from the interval $[8m-3, 10m-5]$,

(vi) $(x_3, y_j)(x_3, y_{j+1})$ and $(x_3, y_j)(x_4, y_{j+1})$ admit the consecutive integers from $10m-4$ to $12m-7$,

(vii) $(x_4, y_j)(x_5, y_j)$ and $(x_5, y_j)(x_4, y_{j+1})$ receive the consecutive integers from $12m-6$ to $14m-8$,

(viii) $(x_4, y_j)(x_4, y_{j+1})$ and $(x_4, y_j)(x_5, y_{j+1})$ admit the consecutive integers from the interval $[14m-7, 16m-10]$.

Under the labeling φ_3 the weights of the edges *(ix)* $(x_5, y_j)(x_5, y_{j+1})$ receive the consecutive integers from $16m-9$ to $17m-11$.

Under the labeling φ_4 the weights of the edges

(x) $(x_5, y_j)(x_6, y_j)$ and $(x_6, y_j)(x_5, y_{j+1})$ admit the consecutive integers from $16m-9$ to $18m-11$,

(xi) $(x_5, y_j)(x_5, y_{j+1})$ and $(x_5, y_j)(x_6, y_{j+1})$ receive the consecutive integers from the interval $[18m-10, 20m-13]$,

(xii) $(x_6, y_j)(x_6, y_{j+1})$ receive the consecutive integers from $20m-12$ to $21m-14$.

We can see that weights of the edges under the labeling φ_3 (respectively, φ_4) create the integer interval $[3, 17m-11]$ (respectively, $[3, 21m-14]$. Thus the labelings φ_3 and φ_4 are the desired edge irregular total k-labelings. □

3 Main Result

As $|E(P_n \boxtimes P_m)| = 4mn - 3m - 3n + 2$ then (1) implies that $tes(P_n \boxtimes P_m) \geq \left\lceil \frac{|E(G)|+2}{3} \right\rceil = \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n)$. The main result of the paper proves equality. More precisely we prove

Theorem 1 *Let $m, n \geq 2$ be positive integers and $P_n \boxtimes P_m$ be the strong product of two paths P_n and P_m. Then*

$$tes(P_n \boxtimes P_m) = \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n).$$

Proof. The cases when $2 \leq m \leq n \leq 6$ were discussed in the previous section. Because the graphs $P_m \boxtimes P_n$ and $P_n \boxtimes P_m$ are isomorphic, it is sufficient to prove the statement for $n \geq m$. Thus we suppose that $n \geq m$, $n \geq 7$, $m \geq 2$ and $k = \left\lceil \frac{4(mn+1)}{3} \right\rceil - (m+n)$.

We split the edge set of $P_n \boxtimes P_m$ into mutually disjoint subsets A_i, B_i and C_i, where
$A_i = \{(x_i, y_j)(x_{i+1}, y_j) : j \in [1, m]\} \cup \{(x_{i+1}, y_j)(x_i, y_{j+1}) : j \in [1, m-1]\}$ for all $i \in [1, n-1]$,
$B_i = \{(x_i, y_j)(x_i, y_{j+1}) : j \in [1, m-1]\}$ for $i \in [1, n]$ and
$C_i = \{(x_i, y_j)(x_{i+1}, y_{j+1}) : j \in [1, m-1]\}$ for $i \in [1, n-1]$.

Clearly, $|A_i| = 2m-1$, $|B_i| = |C_i| = m-1$ and $\sum_{i=1}^{n-1} (|A_i| + |C_i|) + \sum_{i=1}^{n} |B_i| = |E(P_n \boxtimes P_m)|$.

Now we construct the total labeling $\psi : V(P_n \boxtimes P_m) \cup E(P_n \boxtimes P_m) \to \{1, 2, \dots, k\}$ as follows:

$$\psi((x_i, y_j)) = \begin{cases} j, & \text{if } i = 1, 2 \\ m(i-2) + j, & \text{if } i \in [3, \lfloor \frac{n}{2} \rfloor] \\ k - m(n-i) + j, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, n-1] \\ k, & \text{if } i = n. \end{cases}$$

If an edge e_A belongs to A_i, an edge e_B belongs to B_i and an edge e_C belongs to C_i then we define:

$$\psi(e_A) = \begin{cases} 1, & \text{if } i = 1 \\ (2m-3)(i-2) + 3m - 2, & \text{if } i \in [2, \lfloor \frac{n}{2} \rfloor - 1] \\ (2m-3)\lfloor \frac{n}{2} \rfloor + (n-3)m - k + 4, & \text{if } i = \lfloor \frac{n}{2} \rfloor \\ (2m-3)(i+2-n) + k - 6m + 6, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, n-2] \\ k - 4m + 3 + j, & \text{if } i = n-1, \end{cases}$$

$$\psi(e_B) = \begin{cases} 2m-1, & \text{if } i = 1 \\ (2m-3)i + 2m + 2, & \text{if } i \in [2, \lfloor \frac{n}{2} \rfloor] \\ (2m-3)(i+1-n) + k - m + 1, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, n-1] \\ k - m + 1 + j, & \text{if } i = n, \end{cases}$$

$$\psi(e_C) = \begin{cases} 2m, & \text{if } i = 1 \\ (2m-3)(i-2) + 5m - 3, & \text{if } i \in [2, \lfloor \frac{n}{2} \rfloor - 1] \\ (2m-3)\lfloor \frac{n}{2} \rfloor + (n-1)m - k + 3, & \text{if } i = \lfloor \frac{n}{2} \rfloor \\ (2m-3)(i+2-n) + k - 4m + 5, & \text{if } i \in [\lfloor \frac{n}{2} \rfloor + 1, n-2] \\ k - 2m + 3 + j, & \text{if } i = n-1. \end{cases}$$

Under the total labeling ψ the weights of the edges

(i) from the set A_1 admit the first $2m-1$ integers from the interval $[3, 2m+1]$,

(ii) from the set B_1 (respectively, C_1) receive the consecutive even integers from $2m+2$ to $4m-2$ (respectively, the consecutive odd integers from $2m+3$ to $4m-1$),

(iii) from the set A_i, $2 \le i \le \lfloor \frac{n}{2} \rfloor - 1$ admit the consecutive integers from the interval $[(4m-3)i - 4m + 6, (4m-3)i - 2m + 4]$,

(iv) from the set B_i (respectively, C_i), $2 \le i \le \lfloor \frac{n}{2} \rfloor - 1$, receive the consecutive even or odd integers from $(4m-3)i - 2m + 5$ to $(4m-3)i + 1$ (respectively, the consecutive odd or even integers from $(4m-3)i - 2m + 6$ to $(4m-3)i + 2$),

(v) from the set $A_{\lfloor \frac{n}{2} \rfloor}$ admit the consecutive integers from the interval $[(4m-3)\lfloor \frac{n}{2} \rfloor - 4m + 6, (4m-3)\lfloor \frac{n}{2} \rfloor - 2m + 4]$,

(vi) from the set $B_{\lfloor \frac{n}{2} \rfloor}$ (respectively, $C_{\lfloor \frac{n}{2} \rfloor}$) receive the consecutive even or odd integers from $(4m-3)\lfloor \frac{n}{2} \rfloor - 2m + 5$ to $(4m-3)\lfloor \frac{n}{2} \rfloor + 1$ (respectively, the consecutive odd or even integers from $(4m-3)\lfloor \frac{n}{2} \rfloor - 2m + 6$ to $(4m-3)\lfloor \frac{n}{2} \rfloor + 2$),

(vii) from the set A_i, $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n-2$, admit the consecutive integers from $3k + (4m-3)i - 4mn - m + 3n + 2$ to $3k + (4m-3)i - 4mn + m + 3n$,

(viii) from the set B_i (respectively, C_i), $\lfloor \frac{n}{2} \rfloor + 1 \le i \le n-2$, receive the consecutive even or odd integers from $3k + (4m-3)i - 4mn + m + 3n + 1$ to $3k + (4m-3)i - 4mn + 3m + 3n - 3$ (respectively, the consecutive odd or even integers from $3k + (4m-3)i - 4mn + m + 3n + 2$ to $3k + (4m-3)i - 4mn + 3m + 3n - 2$),

(ix) from the set A_{n-1} admit the integers from the interval $[3k - 5m + 5, 3k - 3m + 3]$,

(x) from the set B_{n-1} (respectively, C_{n-1}) receive the consecutive even or odd integers from $3k - 3m + 4$ to $3k - m$ (respectively, the consecutive odd or even integers from $3k - 3m + 5$ to $3k - m + 1$),

(xi) from the set B_n admit the consecutive integers from $3k - m + 2$ to $3k$.

Now, it is not difficult to see that all vertex and edge labels are at most k and the edge-weights of the edges from the sets A_i, C_i, $i \in [1, n-1]$, and B_i, $i \in [1, n]$, are pairwise distinct and create the integer interval

$$\begin{array}{ll} [3, 3k] & \text{for } 4(mn+1) \equiv 0 \ (mod\ 3), \\ [3, (4m-3)\lfloor \frac{n}{2} \rfloor + 2] \cup [(4m-3)\lfloor \frac{n}{2} \rfloor + 5, 3k] & \text{for } 4(mn+1) \equiv 1 \ (mod\ 3), \\ [3, (4m-3)\lfloor \frac{n}{2} \rfloor + 2] \cup [(4m-3)\lfloor \frac{n}{2} \rfloor + 4, 3k] & \text{for } 4(mn+1) \equiv 2 \ (mod\ 3). \end{array}$$

Thus, the resulting total labeling is desired edge irregular k-labeling. This concludes the proof. □

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