On the Laplacian Spectral Radius of Unicyclic Graphs with Fixed Diameter *

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Abstract

The set of unicyclic graphs with n vertices and diameter d is denoted by $\mathcal{U}_{(n,d)}$. For $3 \leq i \leq d$, let $P_{n-d-1}(i)$ be the graph obtained from path $P_{d+1} : v_1 v_2 \cdots v_{d+1}$ by adding $n-d-1$ pendant edges at v_i, and $U^{0}_{n-d-2}(i)$ be the graph obtained from $P_{n-d-1}(i)$ by joining v_{i-2} and a pendant neighbor of v_i. In this paper, we determine all unicyclic graphs in $\mathcal{U}_{(n,d)}$ whose largest Laplacian eigenvalue is greater than $n-d+2$. For $n-d \geq 6$ and $G \in \mathcal{U}_{(n,d)}$, we prove further that the largest Laplacian eigenvalue $\mu(G) \leq \max\{\, \mu(U^{0}_{n-d-2}(i)) \mid 3 \leq i \leq d\,\}$, and conjecture that $U^{0}_{n-d-2}(\lceil \frac{d}{2} \rceil + 1)$ is the unique graph which has the greatest value of the greatest Laplacian eigenvalue in $\mathcal{U}_{(n,d)}$. We also prove that the conjecture is true for $3 \leq d \leq 6$.

Mathematics Subject Classifications(2000): 05C50
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1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v \in V$, the degree of v, written by $d(v)$, is the number of edges incident with v. Let $A(G)$ be the adjacency matrix of G and let $D(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Denote by $\phi(G, x)$ or simply $\phi(G)$ the characteristic polynomial of $L(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's theorem, it follows that its eigenvalues are non-

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negative real numbers. We denote the largest eigenvalue of $L(G)$ by $\mu(G)$ and call it the Laplacian spectral radius of G.

The investigation on the Laplacian spectral radius of graphs is an important topic in the theory of graph spectra. Recently, the problem concerning graphs with maximal or minimal Laplacian spectral radius of a given class of graphs has been studied by many authors. Denote by $\mathcal{T}_n$ the set of trees on n vertices. Gutman [8] proved that the star has the greatest Laplacian spectral radius in $\mathcal{T}_n$. Petrović and Gutman [12] proved the path has the smallest Laplacian spectral radius in $\mathcal{T}_n$. Zhang and Li [14] and Guo [5] gave the first four trees in $\mathcal{T}_n$, ordered according to their Laplacian radii. Yu, Lu and Tian [13] determined the fifth to eighth trees in the above ordering. Guo [5] found the sharp upper bound for Laplacian spectral radii of trees in terms of the matching number and number of vertices, and characterized the graph attained the upper bound. Hong and Zhang [9] determined the tree with largest Laplacian spectral radius among all the trees with n vertices and k pendant vertices. Guo [7] determined the first four graphs with the largest Laplacian spectral radius among all unicyclic graphs on n vertices.

The diameter of a connected graph is the maximum distance between pairs of vertices in V. Denote by $\mathcal{U}_{(n,d)}$ the set of unicyclic graphs with n vertices and diameter d. Denote by $P_{d+1} : v_1v_2\cdots v_{d+1}$ the path on $d+1$ vertices. We use $U_s^t(i)$ to denote the unicyclic graph in $\mathcal{U}_{(n,d)}$, shown in Fig. 1, where s and t are all nonnegative integers.

The rest of this paper is organized as follows. In section 2 we introduce some notations and lemmas which will be used later on. In section 3 we determine all unicyclic graphs in $\mathcal{U}_{(n,d)}$ whose Laplacian spectral radius is greater than $n-d+2$. In section 4 we prove further that $\mu(G) \le \max\{\, \mu(U_{n-d-2}^{\,0}(i)) \mid 3 \le i \le d\}$ for $n-d \ge 6$ and $G \in \mathcal{U}_{(n,d)}$. Section 5 contains a conjecture that $U_{n-d-2}^{\,0}(\lceil \frac{d}{2} \rceil + 1)$ is the unique graph which has the greatest Laplacian spectral radius in $\mathcal{U}_{(n,d)}$. We also prove that the conjecture is true for $3 \le d \le 6$.

2. Preliminary

Let x be a unit eigenvector of G corresponding to $\mu(G)$. Then $\mu(G) = x^T L(G) x$. It will be convenient to associate with x a labelling of G in which vertex v is labelled x_v. We use $\Delta_{r,s}^t(i)$, $\nabla_s^t(i)$ and $\Box_{s,t}(i)$ to denote the unicyclic graphs in $\mathcal{U}_{(n,d)}$ shown in Figs. 2-4, respectively, where r, s and t are all nonnegative integers. The terminology not defined here can be found in [2, 3].

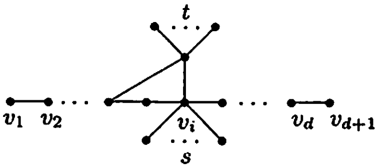

Fig. 1 $U_s^t(i)$

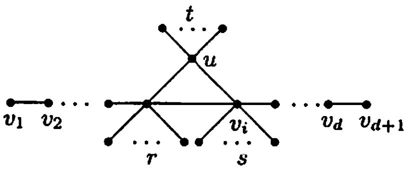

Fig. 2 $\triangle_{r,s}^t(i)$

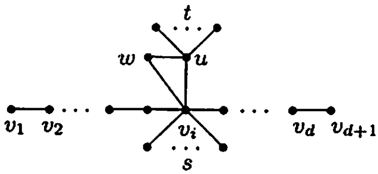

Fig. 3 $\nabla_s^t(i)$

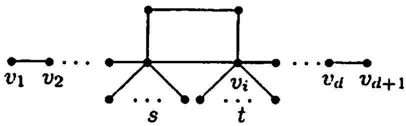

Fig. 4 $\square_{s,t}(i)$

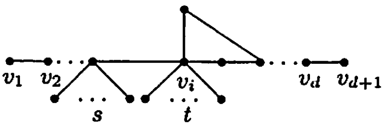

Fig. 5 $U_{s,t}^+(i)$

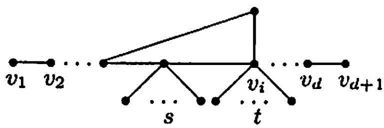

Fig. 6 $U_{s,t}^-(i)$

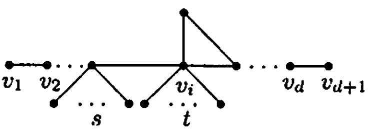

Fig. 7 $U_{s,t}(i)$

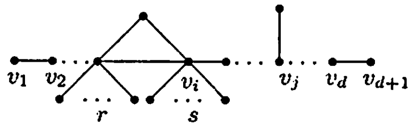

Fig. 8 $\triangle_{r,s}^0(i,j)$

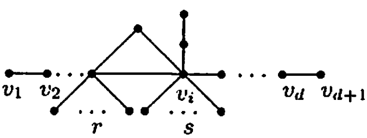

Fig. 9 $\triangle_{r,s}^{0,*}(i)$

Since $\mathcal{U}_{(n,2)} = \{K_{1,n-1} + e\}$, we assume $d \geq 3$ in the following. Now we introduce some lemmas which will be used later on.

Lemma 1 [4, 10]. *Let G have at least one edge, and let Δ be the maximum degree in G. Then $\mu(G) \geq \Delta+1$. For G a connected graph on $n > 1$ vertices, the equality holds if and only if $\Delta = n-1$.*

Lemma 2 [1]. *Let G be a graph. Then*

$$\mu(G) \leq \max\{\, d(u) + d(v) \mid uv \in E(G)\,\}.$$

Lemma 3 [11]. *Let G be a graph and $m(u)$ be the average of the degrees of the vertices of G adjacent to u. Then*

$$\mu(G) \leq \max\{\, d(u) + m(u) \mid u \in V(G)\,\}.$$

Lemma 4 [4]. *Let G be a bipartite graph. Then $D(G) + A(G)$ and $L(G) = D(G) - A(G)$ are unitarily similar. In particular, they have the same spectrum.*

Lemma 5. *Let v be a vertex of a connected graph G with $d(v) \geq 2$, and $G_k (k \geq 1)$ be the graph obtained from G by attaching a new path $P : v(= v_0)v_1v_2 \cdots v_k$ of length k at v. Let x be a unit eigenvector of G corresponding to $\mu(G)$. Then for any fixed i $(i = 0, 1, \ldots, k-1)$, we have $|x_{v_{i+1}}| \leq |x_{v_i}|$ and $x_{v_i}x_{v_{i+1}} \leq 0$, with equalities if only if $x_{v_0} = 0$.*

The proof of Lemma 5 is similar to the proof of Lemma 3.3 in [6].

Lemma 6 [6]. *Let v be a vertex of a connected graph G and suppose that $v_1, v_2, \ldots, v_s$ are pendant vertices of G which are adjacent to v. Let G^* be the graph obtained from G by adding any t $(1 \leq t \leq \frac{s(s-1)}{2})$ edges among $v_1, v_2, \ldots, v_s$. Then we have $\mu(G) = \mu(G^*)$.*

3. All the unicyclic graphs in $\mathcal{U}_{(n,d)}$ with $\mu(G) > n - d + 2$

Theorem 1. *Let $G \in \mathcal{U}_{(n,d)}$. If $d \geq 3$ and $n - d \geq 5$, then $\mu(G) > n - d + 2$ if and only if G is one of the following unicyclic graphs:*

(1) $U^{0}_{n-d-2}(i)$, $3 \leq i \leq d$; (2) $\Delta^{0}_{0,n-d-2}(i)$, $2 \leq i \leq d$;
(3) $\nabla^{0}_{n-d-3}(i)$, $2 \leq i \leq d$.

Proof. For $3 \leq i \leq d$, by Lemma 1, we have

$$\mu(U^{0}_{n-d-2}(i)) > d(v_i) + 1 = n - d + 2,$$

Similarly, for $2 \leq i \leq d$, we have

$$\mu(\Delta^{0}_{0,n-d-2}(i)) > n - d + 2, \qquad \mu(\nabla^{0}_{n-d-3}(i)) > n - d + 2.$$

Let $G \in \mathcal{U}_{(n,d)}$. Suppose that $G \neq U_{n-d-2}^{0}(i)$, $3 \leq i \leq d$; $G \neq \triangle_{0,n-d-2}^{0}(i)$, $\nabla_{n-d-3}^{0}(i)$, $2 \leq i \leq d$. Now our aim is to show that $\mu(G) \leq n-d+2$.

In the case when $\max\{\, d(u)+d(v) \,|\, uv \in E(G)\,\} \leq n-d+2$, by Lemma 2, we have

$$\mu(G) \leq \max\{\, d(u)+d(v) \,|\, uv \in E(G)\,\} \leq n-d+2.$$

So we may assume that there exists an edge $e = uv$ of G such that $d(u) + d(v) \geq n-d+3$. Let $P_{d+1} : v_1v_2\cdots v_{d+1}$ be a path of G. We claim that at least one of u and v belong to P_{d+1}. Assume, on the contrary, that neither u nor v belongs to P_{d+1}. Since G is a unicyclic graph, there exist at most two edges of G between P_{d+1} and edge uv. This implies that $d(u) + d(v) \leq n-d+1$, a contradiction. Thus at least one of u and v belongs to P_{d+1}. We distinguish the following two cases.

Case 1. One of u and v belongs to P_{d+1}. Without loss of generality, we may assume that $u = v_i$. If $N(u) \cap N(v) = \emptyset$, since $d(u)+d(v) \geq n-d+3$, it follows that G must belong to $\mathcal{U}_1$, where

$$\mathcal{U}_1 = \{\, U_s^t(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d,\ s \geq 0,\ t \geq 1\,\}.$$

If $N(u) \cap N(v) \neq \emptyset$, since G is a unicyclic graph, it follows that $|N(u) \cap N(v)| = 1$. We obtain similarly that G must belong to $\mathcal{U}_2 \cup \mathcal{U}_3$, where

$$\begin{aligned} \mathcal{U}_2 &= \{\, \triangle_{0,s}^{t}(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d,\ s \geq 0,\ t \geq 1\,\}, \\ \mathcal{U}_3 &= \{\, \nabla_s^t(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d-1,\ s \geq 0,\ t \geq 1\,\}. \end{aligned}$$

Case 2. Both u and v belong to P_{d+1}. Without loss of generality, we may assume that $u = v_{i-1}$ and $v = v_i$. If $N(u) \cap N(v) = \emptyset$, since $d(u) + d(v) \geq n-d+3$, it follows that G must belong to $\cup_{i=4}^{8}\mathcal{U}_i$, where

$$\begin{aligned} \mathcal{U}_4 &= \{\, \square_{s,t}(i) \in \mathcal{U}_{(n,d)} \,|\, 2 \leq i \leq d,\ s \geq 0,\ t \geq 0\,\}, \\ \mathcal{U}_5 &= \{\, \nabla_{s,t}(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d,\ s \geq 1,\ t \geq 0\,\}, \\ \mathcal{U}_6 &= \{\, U_{s,t}^{+}(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d-1,\ s \geq 1,\ t \geq 0\,\}, \\ \mathcal{U}_7 &= \{\, U_{s,t}^{-}(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d,\ s \geq 1,\ t \geq 0\,\}, \\ \mathcal{U}_8 &= \{\, U_{s,t}(i) \in \mathcal{U}_{(n,d)} \,|\, 3 \leq i \leq d,\ s \geq 1,\ t \geq 0\,\}, \end{aligned}$$

$\square_{s,t}(i)$, $U_{s,t}^{+}(i)$, $U_{s,t}^{-}(i)$ and $U_{s,t}(i)$ are shown in Figs. 4-7 respectively, $\nabla_{s,t}(i) \in \mathcal{U}_{(n,d)}$ is the graph obtained from $\nabla_t^0(i)$ by attaching $s(\geq 1)$ pendant edges to the vertex v_{i-1}.

If $N(u) \cap N(v) \neq \emptyset$, since G is a unicyclic graph, it follows that $|N(u) \cap N(v)| = 1$. Since $d(u)+d(v) \geq n-d+3$, it follows that G must belong to

$\mathcal{U}_9 \cup \mathcal{U}_{10} \cup \mathcal{U}_{11}$, where

$$\begin{aligned} \mathcal{U}_9 &= \{ \Delta^1_{r,s}(i) \in \mathcal{U}_{(n,d)} \mid 3 \le i \le d,\ r \ge 0,\ s \ge 0 \}, \\ \mathcal{U}_{10} &= \{ \Delta^{0,*}_{r,s}(i) \in \mathcal{U}_{(n,d)} \mid 3 \le i \le d-1,\ s \ge 0,\ r \ge 0 \}, \\ \mathcal{U}_{11} &= \{ \Delta^0_{r,s}(i,j) \in \mathcal{U}_{(n,d)} \mid 3 \le i \le d,\ 2 \le j \le d,\ s \ge 0,\ r \ge 0 \} \backslash \mathcal{A}, \end{aligned}$$

$\mathcal{A} = \{ \Delta^0_{0,n-d-2}(i) \mid 3 \le i \le d \}$, $\Delta^{0,*}_{r,s}(i)$ and $\Delta^0_{r,s}(i,j)$ are shown in Fig. 9 and Fig. 8 respectively.

For any $G \in \mathcal{U}_1$, from the definition of $\mathcal{U}_1$, we have $s + t = n - d - 2$, $1 \le t \le n-d-2$, $0 \le s \le n-d-3$. By properties of the function $f(x) = x + ax^{-1}$ $(a > 0)$, we have

$$\begin{aligned} d(u) + m(u) &\le s + 3 + \frac{s+2+t+2+2}{s+3} = s + 3 + \frac{n-d+4}{s+3} \\ &\le \max\{ 3 + \frac{n-d+4}{3},\ n - d + \frac{n-d+4}{n-d} \} \le n - d + 2 \end{aligned}$$

for $n - d \ge 5$. For any other vertex w of G, we can similarly verify that

$$d(w) + m(w) \le n - d + 2$$

for $n - d \ge 5$. By Lemma 3, we have

$$\mu(G) \le \max\{ d(w) + m(w) \mid w \in V(G) \} \le n - d + 2.$$

Similarly, by Lemma 3, we can verify that when $n-d \ge 5$, $\mu(G) \le n-d+2$ holds for any $G \in \cup_{i=2}^{11} \mathcal{U}_i$.

Combining the above arguments, we obtain a proof of Theorem 1.

4. The Laplacian spectral radius of unicyclic graphs in $\mathcal{U}_{(n,d)}$

Theorem 2. Suppose that $d \ge 3$ and $n - d \ge 6$. Then for any $G \in \mathcal{U}_{(n,d)}$

$$\mu(G) \le \max\{ \mu(U^{\,0}_{n-d-2}(i)) \mid 3 \le i \le d \}.$$

The proof of Theorem 2 follows immediately from Theorem 1 and the following lemmas 7 and 8.

Lemma 7. *Suppose that $d \ge 3$, $n - d \ge 5$. Then*

$$\mu(\Delta^0_{0,n-d-2}(i)) < \mu(U^{\,0}_{n-d-2}(i)), \ \textit{for } 3 \le i \le d;$$
$$\mu(\Delta^0_{0,n-d-2}(2)) < \mu(U^{\,0}_{n-d-2}(d)).$$

Proof. For $3 \le i \le d$, let $\mu = \mu(\Delta^0_{0,n-d-2}(i))$, and x be a unit eigenvector of $\Delta^0_{0,n-d-2}(i)$ corresponding to $\mu(\Delta^0_{0,n-d-2}(i))$. Then

$$L(\Delta^0_{0,n-d-2}(i))x = \mu x, \quad \mu = x^T L(\Delta^0_{0,n-d-2}(i))x.$$

We associate with x a labeling of $\Delta_{0,n-d-2}^{0}(i)$ in which vertex v_j is labeled x_j for $j = 1, 2, \ldots, d+1$ and u is labeled x_u, where u is shown in Fig. 2. So we have

$$-x_i + 2x_u - x_{i-1} = \mu x_u, \qquad -x_i - x_u + 3x_{i-1} - x_{i-2} = \mu x_{i-1}.$$

From the two Equations, we have

$$x_{i-1} - x_{i-2} = (3-\mu)(x_u - x_{i-1}).$$

If $x_u - x_{i-1} = 0$, then $x_{i-1} = x_{i-2}$. By Lemma 5, we have $x_{i-1} = 0$. From above we get further $x_u = x_{i-1} = x_{i-2} = x_i = 0$. Again by Lemma 5, we have $x = 0$, a contradiction. Hence $x_u - x_{i-1} \neq 0$. Since $n - d \geq 5$, by Lemma 1, we have $\mu > 7$.

By Courant-Fisher Theorem for Hermite matrices, we further have

$$\begin{aligned} & \mu(U_{n-d-2}^{0}(i)) - \mu(\Delta_{0,n-d-2}^{0}(i)) \\ \geq\ & x^T \left(L(U_{n-d-2}^{0}(i)) - L(\Delta_{0,n-d-2}^{0}(i))\right) x \\ =\ & 2x_u x_{i-1} - 2x_u x_{i-2} - x_{i-1}^2 + x_{i-2}^2 \\ =\ & 2(x_{i-1} - x_{i-2})(x_u - x_{i-1}) + (x_{i-1} - x_{i-2})^2 \\ =\ & (\mu - 3)(\mu - 5)(x_u - x_{i-1})^2 > 0. \end{aligned}$$

For $i = 2$, since both $\Delta_{0,n-d-2}^{0}(2) - uv_1$ and $U_{n-d-2}^{0}(d)$ are bipartite graphs, by Lemma 6, Lemma 4 and the Perron-Frobenius theory of non-negative matrices, we have

$$\mu(\Delta_{0,n-d-2}^{0}(2)) = \mu(\Delta_{0,n-d-2}^{0}(2) - uv_1) < \mu(U_{n-d-2}^{0}(d)).$$

This completes the proof.

Lemma 8. *Suppose that $d \geq 3$, $n - d \geq 5$ and $2 \leq i \leq d$. Then for $3 \leq i \leq d$,*

$$\mu(\nabla_{n-d-3}^{0}(i)) < \mu(U_{n-d-2}^{0}(i))$$

and

$$\mu(\nabla_{n-d-3}^{0}(2)) < \mu(U_{n-d-2}^{0}(d)).$$

Proof. For $3 \leq i \leq d$, since $\nabla_{n-d-3}^{0}(i) - uw$ is bipartite graphs (u and w are shown in Fig. 3), by similar arguments to that in the proof of Lemma 7, we have

$$\mu(\nabla_{n-d-3}^{0}(i)) = \mu(\nabla_{n-d-3}^{0}(i) - uw) < \mu(U_{n-d-2}^{0}(i)).$$

For $i = 2$, we obtain similarly $\mu(\nabla_{n-d-3}^{0}(2)) < \mu(U_{n-d-2}^{0}(d))$. This completes the proof.

5. Two conjectures

Concerning Theorem 2, we have further the following conjecture.

Conjecture 1. Suppose that $d \geq 3$ and $n-d \geq 6$. Then for any $G \in \mathcal{U}_{(n,d)}$,

$$\mu(G) \leq \mu(U^{0}_{n-d-2}(\lceil \frac{d}{2} \rceil + 1)),$$

and the equality holds if and only if $G = U^{0}_{n-d-2}(\lceil \frac{d}{2} \rceil + 1)$.

Denote by $U_t(a,b)$ the unicyclic graph shown in Fig. 10, where a, b, t are all nonnegative integers. We abbreviate $U_{n-d-2}(a,b) = U(a,b)$. It is easy to see that $U_t(a,0) = U_{t-1}(a,1)$ for $t \geq 1$, and $U^{0}_{n-d-2}(i) = U(i-3, d+1-i)$ for $3 \leq i \leq d$.

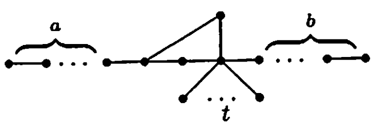

Fig. 10 $U_t(a,b)$

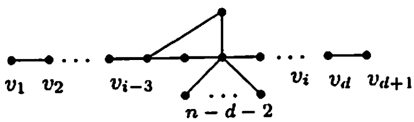

Fig. 11 $U(i-3, d+1-i)$

Obviously, Conjecture 1 follows from Theorem 2 and the following Conjecture 2.

Conjecture 2. Suppose that $n - d \geq 6$. If $d \geq 3$ is odd, then

$$\mu(U(a,b)) < \mu(U(a+1, b-1)) \quad \text{for} \quad b - a \geq 3,$$

$$\mu(U(a,b)) < \mu(U(a-1, b+1)) \quad \text{for} \quad a - b \geq 1.$$

And if $d \geq 4$ is even, then

$$\mu(U(a,b)) < \mu(U(a+1, b-1)) \quad \text{for} \quad b - a \geq 4,$$

$$\mu(U(a,b)) < \mu(U(a-1, b+1)) \quad \text{for} \quad a - b \geq 0.$$

Next we show that some particular cases of Conjecture 2 are true, and from these particular cases and Theorem 2 we prove that Conjecture 1 is true for $3 \leq d \leq 6$.

Lemma 9. *Let v be a vertex of a connected graph G, and suppose that a path of length k is attached to G by its end vertex at v to form G_k. In particular, $G_0 = G$. Then*

$$\phi(G_k) = (x-2)\phi(G_{k-1}) - \phi(G_{k-2}), \ (k \geq 2).$$

The proof of lemma 8 can be obtained easily by expansion of $\det(xI - L(G_k))$.

Lemma 10. *If $d \geq 5$ is odd, $a = \lceil \frac{d}{2} \rceil - 3$, and $n - d \geq 5$, then*

$$\mu(U(a, a+3)) < \mu(U(a+1, a+2)),$$

$$\mu(U(a+3, a)) < \mu(U(a+2, a+1)) < \mu(U(a+1, a+2)).$$

Proof. By the direct calculation, we have

$$\begin{aligned}\phi(U_{n-5}(0,1)) &= x(x-2)(x-1)^{n-5}[x^3 - (n+3)x^2 + (4n-2)x - 2n] \\ &= x(x-1)^{n-5}[x^4 - (n+5)x^3 + (6n+4)x^2 - (10n-4)x + 4n],\end{aligned}$$

$$\phi(U_{n-6}(1,1)) = x(x-2)(x-1)^{n-6}[x^4 - (n+4)x^3 + (6n-4)x^2 - (8n-12)x + 2n].$$

In the following, let $t = n - d - 2$. Then $t \geq 3$. By Lemma 9, we have

$$\begin{aligned}\phi(U(a, a+3)) &= (x-2)\phi(U(a, a+2)) - \phi(U(a, a+1)), \\ \phi(U(a+1, a+2)) &= (x-2)\phi(U(a, a+2)) - \phi(U(a-1, a+2)).\end{aligned}$$

Thus we have

$$\phi(U(a, a+3)) - \phi(U(a+1, a+2)) = \phi(U(a-1, a+2)) - \phi(U(a, a+1)).$$

Repeating above procedure, we have

$$\begin{aligned} & \phi(U(a, a+3)) - \phi(U(a+1, a+2)) \\ = \; & \phi(U(a-1, a+2)) - \phi(U(a, a+1)) = \cdots = \phi(U(0,3)) - \phi(U(1,2)) \\ = \; & (x-2)\phi(U(0,2)) - \phi(U(0,1)) - [(x-2)\phi(U(1,1)) - \phi(U(1,0))] \\ = \; & (x-2)[(x-2)\phi(U(0,1)) - \phi(U(0,0))] - \phi(U(0,1)) \\ & -(x-2)\phi(U(1,1)) + \phi(U(1,0)) \\ = \; & x^2(x-2)(x-1)^{t-1}[x^3 - (t+7)x^2 + (3t+14)x - 2t + 12] \\ = \; & x^2(x-2)(x-1)^{t-1}[x(x-t-4)(x-3) + 2x - 2t + 12]. \end{aligned}$$

It follows that when $x \geq \mu(U(a, a+3)) > t + 4$,

$$\phi(U(a, a+3)) > \phi(U(a+1, a+2)).$$

Therefore $\mu(U(a, a+3)) < \mu(U(a+1, a+2))$.

By similar reasoning, we have when $x \geq \mu(U(a+3, a)) > t + 4$,

$$\begin{aligned} & \phi(U(a+3, a)) - \phi(U(a+2, a+1)) = \phi(U(3,0)) - \phi(U(2,1)) \\ = \; & x^2(x-2)(x-1)^{t-1}[(t+1)x^3 - 7(t+1)x^2 + (3t+14)x - 6t - 8] \\ = \; & x^2(x-2)(x-1)^{t-1}[(t+1)x(x-1)(x-6) + (7t+8)x - 6t - 8] > 0, \end{aligned}$$

and when $x \geq \mu(U(a+2,a+1)) > t+4$

$$\phi(U(a+2,a+1)) - \phi(U(a+1,a+2)) = \phi(U(1,0)) - \phi(U(0,1))$$
$$= tx^2(x-2)^2(x-1)^{t-1} > 0.$$

Therefore

$$\mu(U(a+3,a)) < \mu(U(a+2,a+1)) < \mu(U(a+1,a+2)).$$

This completes the proof.

Lemma 11. *If $d \geq 6$ is even, $a = \lceil \frac{d}{2} \rceil - 3$ and $n-d \geq 6$, then*

$$\mu(U(a,a+4)) < \mu(U(a+1,a+3)),$$
$$\mu(U(a+4,a)) < \mu(U(a+3,a+1)) < \mu(U(a+2,a+2)) < \mu(U(a+1,a+3)).$$

Proof. Let $t = n-d-2$. Then $t \geq 4$. By similar arguments to that in the proof of Lemma 10, we have when $x \geq \mu(U(a,a+4)) > t+4$,

$$\phi(U(a,a+4)) - \phi(U(a+1,a+3)) = \phi(U(0,4)) - \phi(U(1,3))$$
$$= x^2(x-2)(x-1)^{t-1}[x^4 - (t+9)x^3 + (5t+27)x^2 - (7t+31)x + 4t + 12]$$
$$= x^2(x-2)(x-1)^{t-1}[x^2(x-5)(x-t-4) + 7x^2 - (7t+31)x + 4t + 12] > 0,$$

when $x \geq \mu(U(a+4,a)) > t+4$,

$$\phi(U(a+4,a)) - \phi(U(a+3,a+1)) = \phi(U(4,0)) - \phi(U(3,1))$$
$$= x^2(x-2)(x-1)^{t-1}[(t+1)x^4 - 9(t+1)x^3 + (26t+27)x^2 - (27t-31)x + 8t - 12]$$
$$= x^2(x-2)(x-1)^{t-1}[(t+1)x^2(x-3)(x-6) + (8t+9)x^2 - (27t-31)x + 8t - 12]$$
$$> 0,$$

and when $x \geq \mu(U(a+3,a+1)) > t+4$,

$$\phi(U(a+3,a+1)) - \phi(U(a+2,a+2)) = \phi(U(2,0)) - \phi(U(1,1))$$
$$= (t+1)x^2(x-2)(x-4)(x-1)^t > 0.$$

Therefore

$$\mu(U(a,a+4)) < \mu(U(a+1,a+3)),$$
$$\mu(U(a+4,a)) < \mu(U(a+3,a+1)) < \mu(U(a+2,a+2)).$$

Similarly, we have

$$\phi(U(a+2,a+2)) - \phi(U(a+1,a+3)) = \phi(U(1,1)) - \phi(U(0,2))$$

$= -x^2(x-2)(x-1)^{t-1}(x^2-(t+5)x+4).$

Let $F(x) = -(x^2 - (t+5)x + 4)$. It is easy to see that $F(x)$ is decreasing strictly on the interval $[t+4,\ t+4+\frac{3}{t+3}]$, and $F(t+4+\frac{3}{t+3}) > 0$ for $t \geq 4$. So we have

$$\phi(U(a+2, a+2)) - \phi(U(a+1, a+3)) > 0$$

on the interval $[t+4,\ t+4+\frac{3}{t+3}]$. Moreover, by Lemma 1 and 3, we have

$$t+4 < \mu(U(a+2, a+2)), \mu(U(a+1, a+3)) \leq t+4+\frac{3}{t+3}.$$

Thus

$$\mu(U(a+2, a+2)) < \mu(U(a+1, a+3)).$$

This completes the proof.

Theorem 3. Suppose that $3 \leq d \leq 6$, $n-d \geq 6$. Then for any $G \in \mathcal{U}_{(n,d)}$,

$$\mu(G) \leq \mu(U_{n-d-2}^{0}(\lceil \frac{d}{2} \rceil + 1)),$$

and the equality holds if and only if $G = U_{n-d-2}^{0}(\lceil \frac{d}{2} \rceil + 1)$.

Proof. For $d = 3$, by Theorem 2 we have nothing to prove. For $d = 4$, by similar arguments to that in the proof of Lemma 11, we have

$$\mu(U_{n-d-2}^{0}(4)) < \mu(U_{n-d-2}^{0}(3)).$$

Then Theorem 3 follows from Theorem 2. For $d = 5, 6$, Theorem 3 follows from Theorem 2, Lemma 10 and Lemma 11 immediately. This completes the proof.

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