KERNELS BY MONOCHROMATIC DIRECTED PATHS IN m-COLORED DIGRAPHS WITH QUASI-TRANSITIVE CHROMATIC CLASSES

HORTENSIA GALEANA-SÁNCHEZ, BERNARDO LLANO, AND JUAN JOSÉ MONTELLANO-BALLESTEROS.

ABSTRACT. In this paper, we give sufficient conditions for the existence of kernels by monochromatic directed paths (m.d.p.) in digraphs with quasi-transitive colorings. Let D be an m-colored digraph. We prove that if every chromatic class of D is quasi-transitive, every cycle is quasi-transitive in the rim and D does not contain polychromatic triangles, then D has a kernel by m.d.p. The same result is valid if we preserve the first two conditions before and replace the last one by: there exists $k \geq 4$ such that every $\overrightarrow{C}_k$ is quasi-monochromatic and every $\overrightarrow{C}_l$ $(3 \leq l \leq k-1)$ is not polychromatic. Finally, we also show that if every chromatic class of D is quasi-transitive, every cycle in D induces a quasi-transitive digraph and D does not contain polychromatic $\overrightarrow{C}_3$, then D has a kernel by m.d.p. Some corollaries are obtained for the existence of kernels by m.d.p. in m-colored tournaments.

1. INTRODUCTION

In this paper we study the existence of kernels by monochromatic directed paths (m.d.p.) in digraphs with restricted colorings of its arcs. Specially, we focus to the so called quasi-transitive colorings (to be defined in the next section). The research on kernels by m.d.p. goes back to the classical paper of Sands, Sauer and Woodrow (see [10]) who proved that every 2-colored digraph has a kernel by m.d.p. (particularly, 2-colored tournaments have this property). They also posed the following problem.

Problem 1. *Let T be a 3-colored tournament not containing polychromatic directed triangles. Must T contain a kernel by m.d.p.?*

Date: December 17, 2007.

2000 *Mathematics Subject Classification.* Primary 05C20.

Key words and phrases. m-colored quasi-transitive digraphs, kernel by monochromatic paths, quasi-transitive chromatic class.

This paper is in final form and no version of it will be submitted for publication elsewhere.

In 1988, Shen proved that m-colored tournaments without polychromatic *triangles* (directed triangles and transitive subtournaments of order 3) have a kernel by m.d.p. It is also proved that if $m \geq 5$, the condition on T not containing polychromatic triangles cannot be dropped (see [11]). The case $m = 4$ was answered in [7]. In both papers, infinite families of tournaments of order n are constructed showing that the condition is best possible for $m \geq 4$. For $m = 3$, the problem is still open. Other results concerning Problem 1 can be found in [3], [8] and [9], and more recently in [6].

It is of particular interest to find sufficient conditions to the existence of kernels by monochromatic directed paths (m.d.p.) in general digraphs, or at least in broader classes of them given the well-known difficulty of the problem. For instance, in [4] and [5] it was considered the family of m-colored digraphs resulting from the deletion of a single arc of some m-colored tournament with n vertices. They give conditions on the chromaticity of short directed cycles and on the closure of the digraphs to prove the existence of a kernel by m.d.p and in some cases, to show that the digraphs in question are kernel-perfect.

As mentioned before, we will deal with digraphs with quasi-transitive colorings. Let D be an m-colored digraph. We prove that if every chromatic class of D is quasi-transitive, every cycle is quasi-transitive in the rim and D does not contain polychromatic triangles, then D has a kernel by m.d.p. The same result is valid if we preserve the first two conditions before and replace the last one by: there exists $k \geq 4$ such that every $\overrightarrow{C}_k$ is quasi-monochromatic and every $\overrightarrow{C}_l$ $(3 \leq l \leq k-1)$ is not polychromatic. Finally, we also show that if every chromatic class of D is quasi-transitive, every cycle in D induces a quasi-transitive digraph and D does not contain polychromatic $\overrightarrow{C}_3$, then D has a kernel by m.d.p. Some corollaries are obtained for the existence of kernels by m.d.p. in m-colored tournaments.

2. Preliminaries

Let $D = (V, A)$ be a finite digraph, where V and A denote the sets of vertices and arcs of D respectively. For $\varnothing \neq S \subseteq V(D)$ (resp. $\varnothing \neq S \subseteq A(D)$) we denote by $D[S]$ the induced (resp. arc-induced) subdigraph of D by the subset S. An arc $(u, v) \in A(D)$ is *asymmetrical* (resp. *symmetrical*) if $(v, u) \notin A(D)$ (resp. $(v, u) \in A(D)$). A symmetrical arc (u, v) is denoted by $[u, v]$. A digraph D is said to be *asymmetrical* (resp. *symmetrical*) if every arc of D is asymmetrical (resp. symmetrical). We define the *asymmetrical* (resp. *symmetrical*) *part of* D, denoted by $Asym(D)$ (resp. $Sym(D)$), as the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. A digraph D is *asymmetrical* if $Asym(D) = D$. A *semicomplete digraph* D has no pair of nonadjacent vertices.

A digraph D is said to be m-colored if the arcs of D are colored with m colors. Given $u, v \in V(D)$, a directed path from u to v of D is *monochromatic* if all its arcs have the same color. Directed paths and monochromatic directed paths from u to v are denoted by $u \rightsquigarrow v$ and $u \rightsquigarrow_m v$, respectively. A subdigraph D' of D is called *quasi-monochromatic* if with at most one exception all of its arcs are colored alike. D' is called *polychromatic* if its arcs are colored with at least three colors.

A nonempty set $S \subseteq V(D)$ is an *absorbent set by monochromatic directed paths* (m.d.p.) if for every vertex $u \in V(D) - S$ there exists $v \in S$ such that $u \rightsquigarrow_m v$. A *kernel* K of D is an independent set of vertices so that for every $u \in V(D) - K$ there exists $(u, v) \in A(D)$, where $v \in K$. A digraph D is *kernel-perfect* if every induced subdigraph of D has a kernel.

We will use the following well-known result.

Theorem 1 ([2], Théorème 4.2). *If every directed cycle of a digraph D has a symmetrical arc, then D is kernel-perfect.*

Let D be an m-colored digraph. A set $K \subseteq V(D)$ is called a *kernel by m.d.p.* if

(i) for every $u, v \in K$ there is no m.d.p between u and v, and
(ii) for every $x \in V(D) - K$ there exists $y \in K$ such that $x \rightsquigarrow_m y$.

If $D = (V, A)$ is an m-colored digraph, then the *closure of* D, denoted by $\mathfrak{C}(D)$, is the m-colored digraph defined by

$$V(\mathfrak{C}(D)) = V(D) \text{ and}$$

$$A(\mathfrak{C}(D)) = A(D) \cup \{(u, v) \text{ of color } i : \exists u \rightsquigarrow_m v \text{ of color } i \text{ in } D\}.$$

Remark 1. *(i) For every digraph D, $\mathfrak{C}(D)$ is isomorphic to $\mathfrak{C}(\mathfrak{C}(D))$.*
(ii) D has a kernel by m.d.p. if and only if $\mathfrak{C}(D)$ has a kernel.

A digraph D is called *quasi-transitive* if whenever distinct vertices $x, y, z \in V(D)$ such that $(x, y) \in A(D)$ and $(y, z) \in A(D)$, there exists at least $(x, z) \in A(D)$ or $(z, x) \in A(D)$. A directed cycle $\overrightarrow{C}_k = (u_0, u_1, ..., u_k, u_0)$ of D ($k \geq 2$) is *quasi-transitive in the rim* if for every $i = 0, 1, ..., k$, there exists $(u_i, u_{i+2}) \in A(D)$ or $(u_{i+2}, u_i) \in A(D)$, where indices are taken modulo $k + 1$.

A *tournament* with n vertices is an orientation of the complete graph K_n. Observe that a tournament T is an asymmetrical quasi-transitive digraph and every cycle contained in T is trivially quasi-transitive in the rim.

We denote by $\overrightarrow{C}_3$ and TT_3 the directed triangle and the transitive tournament with three vertices, respectively. We will simply call both of them *triangles*.

Let D be an m-colored digraph. A *chromatic class* of D is the set of arcs of a same color. We say that a chromatic class S is *quasi-transitive* if $D[S]$ is a quasi-transitive digraph. Abusing the notation, we will denote the arc-induced subdigraph by a chromatic class with its color.

Proposition 1 ([1], Proposition 3.1). *Let D be a quasi-transitive digraph. Suppose that $P = (x_1, x_2, ..., x_k)$ is a minimal directed path $x_1 \rightsquigarrow x_k$. Then $D[V(P)]$ is a semicomplete digraph and $(x_j, x_i) \in A(D)$ for every $j > i+1$, unless $k = 4$, in which case the arc between x_1 and x_k may be absent.*

Corollary 1 ([1], Corollary 3.2). *If a quasi-transitive digraph D has a directed path $x \rightsquigarrow y$ but $(x, y) \notin A(D)$, then either $(y, x) \in A(D)$, or there exist vertices $u, v \in V(D) - \{x, y\}$ such that (x, u, v, y) and (y, u, v, x) are directed paths in D.*

As immediate consequence of this corollary, we have the following

Corollary 2. *If a quasi-transitive digraph D has a directed path $x \rightsquigarrow y$ but there does not exist a directed path $y \rightsquigarrow x$ in D, then $(x, y) \in A(D)$.*

Using this result, the next lemma directly follows.

Lemma 1. *Let D be an m-colored digraph such that every chromatic class is quasi-transitive. If $\overrightarrow{C}_k$ is a a directed cycle in $Asym(\mathfrak{C}(D))$, then $\overrightarrow{C}_k$ is a directed cycle in $Asym(D)$.*

3. Results

We will need a previous lemma in order to prove the following theorems. It is useful to point out that if D is an m-colored digraph and $\overrightarrow{C}_k$ is an asymmetrical directed cycle in $\mathfrak{C}(D)$, then there is a vertex of $\overrightarrow{C}_k$ where the cycle changes the color of its arcs.

Lemma 2. *Let D be an m-colored digraph such that*

(i) every cycle is quasi-transitive in the rim and
(ii) D does not contain polychromatic triangles.

Suppose that $k \geq 3$, $\overrightarrow{C}_k = (u_0, u_1, \ldots, u_{k-1}, u_0)$ is an asymmetrical directed cycle in $\mathfrak{C}(D)$ of minimum length and for some $i \in \{0, 1, \ldots, k-1\}$, u_i is a vertex where $\overrightarrow{C}_k$ changes of color B to color R. Then

a) $(u_{i+1}, u_{i-1}) \notin A(D)$ and $(u_{i-1}, u_{i+1}) \in A(D)$ (the indices are taken modulo k) and
b) there exists $u_{i+1} \rightsquigarrow_m u_{i-1}$ of color G different from B and R.

Proof. Suppose that there exists an asymmetrical directed cycle $\overrightarrow{C}_k = (u_0, u_1, ..., u_{k-1}, u_0)$ in $\mathfrak{C}(D)$. By Lemma 1, $\overrightarrow{C}_k$ is a directed cycle of D. We fix $\overrightarrow{C}_k$, an asymmetrical directed cycle of minimum length. Since $\overrightarrow{C}_k$ is asymmetrical in $\mathfrak{C}(D)$, there exists a change of color of its arcs in one vertex, that is, there exists $i \in \{0, 1, ..., k-1\}$ such that (u_{i-1}, u_i) is colored B and

(u_i, u_{i+1}) is colored R. By condition (i), there exists $(u_{i-1}, u_{i+1}) \in A(D)$ or $(u_{i+1}, u_{i-1}) \in A(D)$.

Notice that $(u_{i+1}, u_{i-1}) \notin A(D)$. If it is not the case, $(u_{i-1}, u_i, u_{i+1}, u_{i-1})$ is a directed triangle and, by condition (ii), (u_{i+1}, u_{i-1}) is colored B or R. But then $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$ or $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, respectively, a contradiction to the asymmetry of $\overrightarrow{C}_k$ in $\mathfrak{C}(D)$. Therefore, $(u_{i-1}, u_{i+1}) \in A(D)$ and part a) is proven.

Since $\overrightarrow{C}_k$ is of minimum length, there exists $u_{i+1} \rightsquigarrow_m u_{i-1}$ of color G. This color is different from B and R, otherwise, if $G = B$ or $G = R$, then $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$ or $[u_i, u_{i-1}] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\overrightarrow{C}_k$ in $\mathfrak{C}(D)$. □

Theorem 2. *Let D be an m-colored digraph such that*

(i) every chromatic class is quasi-transitive,

(ii) every cycle is quasi-transitive in the rim and

(iii) D does not contain polychromatic triangles.

Then D has a kernel by m.d.p.

Proof. Let $\mathfrak{C}(D)$ be the closure of D. We will prove that every directed cycle of $\mathfrak{C}(D)$ has a symmetrical arc and thus D has a kernel by m.d.p. in virtue of Theorem 1 and Remark 1(ii). We proceed by contradiction.

Suppose that there exists an asymmetrical directed cycle in $\mathfrak{C}(D)$, denoted by $\overrightarrow{C}_k = (u_0, u_1, ..., u_{k-1}, u_0)$. We fix $\overrightarrow{C}_k$, an asymmetrical directed cycle of minimum length in $\mathfrak{C}(D)$. By Lemma 1, $\overrightarrow{C}_k$ is a directed cycle of D. Since D satisfies conditions (ii) and (iii), we can apply the results of Lemma 2. Let $u_i \in V(\overrightarrow{C}_k)$ be a vertex where $\overrightarrow{C}_k$ changes color, that is, (u_{i-1}, u_i) is colored B and (u_i, u_{i+1}) is colored R. Then $(u_{i+1}, u_{i-1}) \notin A(D)$ and $(u_{i-1}, u_{i+1}) \in A(D)$, and there exists $u_{i+1} \rightsquigarrow_m u_{i-1}$ of color G different from B and R. We observe that u_i does not belong to the path $u_{i+1} \rightsquigarrow u_{i-1}$ (if $u_i = u_j$, where u_j is a vertex of the path $u_{i+1} \rightsquigarrow u_{i-1}$, then we have that $u_j \rightsquigarrow u_{i-1}$ and therefore $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\overrightarrow{C}_k$ in $\mathfrak{C}(D)$).

The arc $(u_{i-1}, u_{i+1}) \in A(D)$ is not colored G, because otherwise, we have a 3-colored $TT_3 \cong D[u_{i-1}, u_i, u_{i+1}]$. Thus, (u_{i-1}, u_{i+1}) is colored B or R. The chromatic class G does not contain the arc (u_{i-1}, u_{i+1}). Since G is quasi-transitive, using Corollary 1, there exist vertices $v_k, v_l \in V(G) - \{u_{i-1}, u_{i+1}\}$ $(k \neq l)$ such that $(u_{i+1}, v_k, v_l, u_{i-1})$ and $(u_{i-1}, v_k, v_l, u_{i+1})$ are directed paths in G.

The cycle $(u_i, u_{i+1}, v_k, v_l, u_{i-1}, u_i)$ is quasi-transitive in the rim and then there exists $(u_i, v_k) \in A(D)$ or $(v_k, u_i) \in A(D)$. We have two cases:

(1) If $(u_i, v_k) \in A(D)$, then it is not colored G. Otherwise, there exists the directed path (u_i, v_k, v_l, u_{i-1}) colored G which implies that

$[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$. But then (u_i, v_k) is colored B and R at the same time, since $D[u_i, u_{i+1}, v_k] \cong TT_3$ and $D[u_i, v_k, u_{i-1}] \cong TT_3$ satisfy condition (iii), a contradiction.

(2) If $(v_k, u_i) \in A(D)$, then it is neither colored G, since otherwise $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$, once again a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$. But then (v_k, u_i) is colored B and R at the same time, since $(v_k, u_i, u_{i+1}, v_k) \cong \vec{C}_3$ and $D[u_i, v_k, u_{i-1}] \cong TT_3$ satisfy condition (iii), a contradiction.

□

Corollary 3. *Let T be an m-colored tournament such that*

(i) every chromatic class is quasi-transitive and

(ii) T does not contain polychromatic triangles.

Then T has a kernel by m.d.p.

Theorem 3. *Let D be an m-colored digraph such that*

(i) every chromatic class is quasi-transitive,

(ii) every cycle is quasi-transitive in the rim and

(iii) there exists $k \geq 4$ such that every $\vec{C}_k$ is quasi-monochromatic and every $\vec{C}_l$ $(3 \leq l \leq k-1)$ is not polychromatic.

Then D has a kernel by m.d.p.

Proof. We start in the same way as in the proof of Theorem 2. Suppose that there exists an asymmetrical directed cycle $\vec{C}_k = (u_0, u_1, ..., u_{k-1}, u_0)$ in $\mathfrak{C}(D)$. We fix $\vec{C}_k$, an asymmetrical directed cycle of minimum length in $\mathfrak{C}(D)$. By Lemma 1, $\vec{C}_k$ is a directed cycle of D. Let $u_i \in V(\vec{C}_k)$ be a vertex where the cycle changes from color B to color R, that is, (u_{i-1}, u_i) is colored B and (u_i, u_{i+1}) is colored R. Since D satisfies conditions (ii) and (iii) of the Theorem, we can apply the results of Lemma 2 and we have that $(u_{i-1}, u_{i+1}) \in A(D)$ and $(u_{i+1}, u_{i-1}) \notin A(D)$, and there exists $u_{i+1} \rightsquigarrow_m u_{i-1}$ of color G different from B and R. As in the proof before, u_i does not belong to the path $u_{i+1} \rightsquigarrow u_{i-1}$. We denote $u_{i+1} \rightsquigarrow_m u_{i-1}$ by the m.d.p. $P = (u_{i+1}, v_1, v_2, ..., v_t, u_{i-1})$ of color G $(t \geq 1)$ and suppose that P is of minimum length. Consider the following two cases:

Case 1. *$(u_{i-1}, u_{i+1}) \in A(D)$ is not colored G.*

By condition (ii), chromatic class G is quasi-transitive and since P is of minimum length, Corollary 1 can be applied, that is, $P = (u_{i+1}, v_1, v_2, u_{i-1})$, and $(u_{i+1}, v_1, v_2, u_{i-1})$ and $(u_{i-1}, v_1, v_2, u_{i+1})$ are directed paths in G. Then, $(u_i, u_{i+1}, v_1, v_2, u_{i-1}, u_i) \cong \vec{C}_5$ is polychromatic. If $k \geq 5$, we arrive to a contradiction with condition (iii). So, we only have to check the case when $k = 4$.

Claim 1.1. $(v_1, u_i) \in A(D)$ *and it is colored* R.

If $(u_i, v_1) \in A(D)$, then $(u_i, v_1, v_2, u_{i-1}, u_i) \cong \vec{C}_4$. By condition (iii), every $\vec{C}_4$ is quasi-monochromatic in D and therefore (u_i, v_1) is colored G. This implies that $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$.

Then $(v_1, u_i) \in A(D)$ and we have that $(v_1, u_i, u_{i+1}, v_1) \cong \vec{C}_3$. Observe that (v_1, u_i) cannot be colored with a color different from G and R (otherwise $(v_1, u_i, u_{i+1}, v_1) \cong \vec{C}_3$ is polychromatic, a contradiction to condition (iii) of the Theorem). The arc (v_1, u_i) is not colored G, otherwise $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$. Therefore (v_1, u_i) is colored R.

Claim 1 is proven.

Thus, we have to check whether $(u_i, v_2) \in A(D)$ or $(v_2, u_i) \in A(D)$.

Claim 1.2. $(u_i, v_2) \in A(D)$ *and it is colored* B.

If $(v_2, u_i) \in A(D)$, then $(u_i, u_{i+1}, v_1, v_2, u_i) \cong \vec{C}_4$. By condition (iii), every $\vec{C}_4$ is quasi-monochromatic in D and therefore (v_2, u_i) is colored G. This implies that $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$.

Then $(u_i, v_2) \in A(D)$ and we have that $(v_2, u_{i-1}, u_i, v_2) \cong \vec{C}_3$. Observe that (u_i, v_2) cannot be colored with a color different from G and B (otherwise $(v_2, u_{i-1}, u_i, v_2) \cong \vec{C}_3$ is polychromatic, a contradiction to condition (iii) of the Theorem). The arc (u_i, v_2) is not colored G, otherwise $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of $\vec{C}_k$ in $\mathfrak{C}(D)$. Therefore (u_i, v_2) is colored B.

Claim 2 is proven.

By Claims 1.1 and 1.2, $(u_i, v_2, u_{i+1}, v_1, u_i) \cong \vec{C}_4$ is polychromatic, a contradiction to condition (iii) of the Theorem.

Case 2. $(u_{i-1}, u_{i+1}) \in A(D)$ *is colored* G.

Recalling that P is of minimum length, we can apply Proposition 1 and then $D[V(P)]$ is a semicomplete digraph colored G, $(u_{i-1}, v_j) \in A(D)$ for $j = 1, 2, ..., t-1$, $(v_m, v_j) \in A(D)$, where $t \geq m > j+1$ and $j = 1, 2, ..., t-2$, and $(v_j, u_{i+1}) \in A(D)$ for every $j = 2, 3, ..., t-1$. We have the cycle

$$(u_{i-1}, u_{i+1}, v_1, v_2, ..., v_t, u_{i-1}) \cong \vec{C}_{t+2}$$

and by condition (iii), $t \geq k-2$.

Using condition (ii) of the Theorem, if there exists $\alpha \in \{1, ..., t-1\}$ such that $(u_i, v_\alpha) \in A(D)$, then $(u_i, v_{\alpha+1}) \in A(D)$ or $(v_{\alpha+1}, u_i) \in A(D)$. We analyze the following cases.

(2.1) If there is no $\beta \in \{1, 2, ..., t\}$ such that $(v_\beta, u_i) \in A(D)$, then for every $\alpha \in \{1, 2, ..., t\}$ we have that $(u_i, v_\alpha) \in A(D)$ and all of them are not colored G (note that if there exits one of them colored G, then $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$, a contradiction). Consider the cycle

$$\overrightarrow{C} = (u_i, u_{i+1}, v_1, v_2, ..., v_t, u_{i-1}, u_i).$$

This cycle is of length $l = t+3 > k$. Since $(u_i, v_\alpha) \in A(D)$ for every $\alpha \in \{1, 2, ..., t\}$, there exists the cycle

$$\overrightarrow{C}' = (u_i, v_{t-(k-3)}, v_{t-(k-2)}, ..., v_t, u_{i-1}, u_i)$$

of length k which is not quasi-monochromatic, a contradiction to condition (iii).

(2.2) If $(v_{\alpha+1}, u_i) \in A(D)$, then by condition (ii), $(v_\alpha, u_i) \in A(D)$ or $(u_i, v_\alpha) \in A(D)$. If there is no $\beta \in \{1, 2, ..., t\}$ such that $(u_i, v_\beta) \in A(D)$, then for every $\alpha \in \{1, 2, ..., t\}$ we have that $(v_\alpha, u_i) \in A(D)$. All these arcs are not colored G, otherwise we obtain the contradiction $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$. We proceed similarly as in case (2.1).

(2.3) Let

$$\alpha = \min\{j \in \{1, 2, ..., t\} : (v_j, u_i) \in A(D)\}.$$

Observe that in virtue of condition (ii), $(u_i, v_1) \in A(D)$ or $(v_1, u_i) \in A(D)$, and therefore α does exist. Moreover, for every $\beta < \alpha$, $(u_i, v_\beta) \in A(D)$. We claim that $\alpha < k-2$. If it is not the case, $\alpha \geq k-2$ and

$$(u_i, v_{\alpha-(k-2)}, v_{\alpha-(k-3)}, ..., v_\alpha, u_i) \cong \overrightarrow{C}_k,$$

(if $\alpha = k-2$, then $v_{\alpha-(k-2)} = u_{i+1}$) which is not quasi-monochromatic, a contradiction to condition (iii) (recall that $(u_i, v_{\alpha-(k-2)})$ and (v_α, u_i) are not colored G because otherwise, $[u_{i-1}, u_i] \in A(\mathfrak{C}(D))$ or $[u_i, u_{i+1}] \in A(\mathfrak{C}(D))$, a contradiction to the asymmetry of of $\overrightarrow{C}_k$ in $\mathfrak{C}(D)$).

Let

$$\beta = \max\{j \in \{1, 2, ..., t\} : (u_i, v_j) \in A(D)\}.$$

With a similar argument as before, for every $\alpha > \beta$, $(v_\alpha, u_i) \in A(D)$ and $\beta > t-(k-3)$. Therefore:

(2.3.1) If $\beta > \alpha$, then

a) if $\beta - \alpha \geq k-2$, then

$$(u_i, v_\beta, v_{\beta-(k-3)}, v_{\beta-(k-2)}, ..., v_{\beta-1}, v_\alpha, u_i) \cong \overrightarrow{C}_k$$

is not quasi-monochromatic, a contradiction to condition (iii) (once more recall that (u_i, v_β) and (v_α, u_i) are not colored G), and

b) if $\beta - \alpha \leq k - 3$, then

$$(u_i, v_\gamma, v_{\gamma+1}, ..., v_\alpha, ..., v_\beta, v_{\beta+1}, ..., v_\delta, u_i) \cong \overrightarrow{C}_k$$

is not quasi-monochromatic for $\delta - \gamma = k - 2$, a contradiction to condition (iii) (again recall that (u_i, v_γ) and (v_δ, u_i) are not colored G).

(2.3.2) If $\beta < \alpha$, then $\beta = \alpha - 1$ by the definitions of α and β. Thus

$$(u_i, v_\gamma, ..., v_\beta, v_\alpha, ..., v_\delta, u_i) \cong \overrightarrow{C}_k$$

is not quasi-monochromatic for $\delta - \gamma = k - 2$, a contradiction to condition (iii) of the Theorem.

□

In a sense, this theorem generalizes Theorem 8 of [9] for a broader class of m-colored digraphs, those with quasi-transitive chromatic classes and for which every cycle is quasi-transitive in the rim.

Proposition 2. *Let D be an asymmetrical m-colored digraph such that*

(i) every chromatic class is quasi-transitive and

(ii) D does not contain polychromatic $\overrightarrow{C}_3$.

If $(u,v) \in A(D)$ is asymmetrical in $\mathfrak{C}(D)$, then there is no $\overrightarrow{C}_3$ in D containing (u,v).

Proof. Suppose that $(u,v) \in A(D)$ is asymmetrical in $\mathfrak{C}(D)$ and there exists a $\overrightarrow{C}_3$ in D containing (u,v). Let $\overrightarrow{C}_3 = (u,v,w,u)$. By condition (ii), two of the arcs are colored B. Since B is quasi-transitive and D is asymmetrical, the remaining arc is colored B. Therefore $\overrightarrow{C}_3 = (u,v,w,u)$ is monochromatic and so, symmetrical in $\mathfrak{C}(D)$, which is a contradiction. □

Theorem 4. *Let D be an asymmetrical m-colored digraph such that*

(i) every chromatic class is quasi-transitive,

(ii) every cycle induces a quasi-transitive digraph and

(iii) D does not contain polychromatic $\overrightarrow{C}_3$.

Then D has a kernel by m.d.p.

Proof. By contradiction, suppose that there exists an asymmetrical directed cycle $\overrightarrow{C}_k = (u_0, u_1, ..., u_{k-1}, u_0)$ in $\mathfrak{C}(D)$. By Lemma 1, $\overrightarrow{C}_k$ is a directed cycle of D. Since $\overrightarrow{C}_k$ induces a quasi-transitive digraph in D, there exists $(u_0, u_2) \in A(D)$ or $(u_2, u_0) \in A(D)$. By Proposition 2, there does not exist $\overrightarrow{C}_3$ in D containing (u_i, u_{i+1}) for every $i = 0, 1, ..., k$ (indices are taken modulo $k+1$). Then $(u_2, u_0) \notin A(D)$, since otherwise $(u_0, u_1, u_2, u_0) \cong \overrightarrow{C}_3$

in D, a contradiction. Therefore, $(u_0, u_2) \in A(D)$. In the same manner, $(u_0, u_3) \in A(D)$. If $(u_3, u_0) \in A(D)$, then (u_0, u_2, u_3, u_0) would be a directed triangle, once again a contradiction. We continue this procedure to obtain that $(u_0, u_j) \in A(D)$ for every $j = 2, 3, ..., k-1$. But then $(u_0, u_{k-1}, u_k, u_0) \cong \overrightarrow{C}_3$ in D, a contradiction to Proposition 2. □

The following corollary solves Problem 1 in the special case when the m-colored tournaments have quasi-transitive chromatic classes for every color.

Corollary 4. *Let T be an m-colored tournament such that*

(i) every chromatic class is quasi-transitive,

(ii) D does not contain polychromatic $\overrightarrow{C}_3$.

Then T has a kernel by m.d.p.

We conclude with some remarks:

(1) Condition (i) in Theorems 2, 3 and 4 (every chromatic class is quasi-transitive) cannot be removed. The already mentioned counterexamples of [7] and [11] still remain valid.

(2) We believe that condition (ii) in Theorems 2 and 3 (every cycle is quasi-transitive in the rim) could be replaced by another one. In particular, if one can prove Theorem 3 omitting this condition, a direct generalization of Theorem 8 of [9] will be obtained.

References

[1] J. Bang-Jensen and J. Huang, *Quasi-transitive Digraphs*, J. Graph Theory **20**, No.2 (1985) 141-161.

[2] P. Duchet, *Graphes noyau-parfaits,* Annals of Discrete Math. **9** (1980) 93-101.

[3] H. Galeana-Sánchez, *On monochromatic paths and monochromatic cycles in edge colored tournaments,* Discrete Math. **156** (1996) 103-112.

[4] H. Galeana-Sánchez, *Kernels in edge-colored digraphs,* Discrete Math. **184** (1998) 87-99.

[5] H. Galeana-Sánchez and J. J. García-Ruvalcaba, *Kernels in the closure of colored digraphs,* Discuss. Math. Graph Theory **20** (2000) 243-254.

[6] H. Galeana-Sánchez, B. Llano and J. J. Montellano-Ballesteros, *Absorbent sets and kernels by monochromatic directed paths in m-colored tournaments* (to appear in Australasian Journal of Combinatorics)

[7] H. Galeana-Sánchez and R. Rojas-Monroy, *A counterexample to a conjecture on edge-colored tournaments,* Discrete Math. **282** (2004) 275-276.

[8] H. Galeana-Sánchez and R. Rojas-Monroy, *Monochromatic paths and at most 2-colored arc sets in edge-colored tournaments,* Graphs and Combin. **21** (2005) 307-317.

[9] G. Hahn, P. Ille and R. E. Woodrow, *Absorbing sets in arc-colored tournaments,* Discrete Math. **283** (2004) 93-99.

[10] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths in edge-colored digraphs,* J. Combin. Theory Ser. B **33** (1982) 271-275.

[11] M. G. Shen, *On monochromatic paths in m-colored tournaments*, J. Combin. Theory Ser. B **45** (1988) 108-111.

(H. Galeana-Sánchez and J. J. Montellano-Ballesteros) Instituto de Matemáticas, UNAM, Ciudad Universitaria, 04510, México, D. F.
E-mail address: hgaleana@matem.unam.mx and juancho@matem.unam.mx

(B. Llano) Departamento de Matemáticas, Universidad Autónoma Metropolitana, Iztapalapa, San Rafael Atlixco 186, Colonia Vicentina, 09340, México, D.F.
E-mail address: llano@xanum.uam.mx