A note on the independence number of strong products of odd cycles

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Abstract

The determination of the zero-capacity of a noisy channel have inspired researches of the independence number of the strong product of odd cycles. The independence number for two infinite families of the strong product of three odd cycles is considered in this paper. In particular, we present the independence number of $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ and an upper bound on the independence number of $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$. The results are partially obtained by a computer search.

Key words: independence number, strong product, Shannon capacity.

1 Introduction and preliminaries

The Shannon capacity is an important information theoretical parameter because it represents the effective size of an alphabet in a communication model represented by a graph. The study of this parameter was introduced by Shannon in [11]. It turns out that the solution of the problem requires the determination of the independence number of product of graphs which contain odd cycles [1, 2, 5, 6, 10, 12, 14].

It may be interesting to note that the capacity of C_5 was studied already by Shannon in 1956 and was determined only in 1979 by Lovász [8]. Furthermore, for C_7, or more generally, for the cycle graph C_n with n odd, the

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Shannon capacity is still unknown. Lovász in [8] defined the graph invariant known as Lovász number that is an upper bound on the Shannon capacity. In order to establish the lower bound on the Shannon capacity of a graph, one may search for the independence number of the strong product of its copies. This approach has been used in [4, 13] where large independent sets in the strong product of four 7-cycles where found. This results improved the best known lower bound for the Shannon capacity of C_7.

The *strong product* of graphs G and H is the graph $G \boxtimes H$ with vertex set $G \times H$ and $(x_1, x_2)(y_1, y_2) \in E(G \boxtimes H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$ or $x_2 y_2 \in E(H)$ and $x_1 = y_1$ or $x_1 y_1 \in E(G)$ and $x_2 y_2 \in E(H)$. The strong product is commutative and associative in an obvious way, having the trivial graph as a unit. Let $a = (a_1, a_2) \in G \boxtimes H$. The graph induced on the set $H_a = \{(a_1, y) \mid y \in H\}$ is the *H-layer* through a. The graph induced on the set $G_a = \{(x, a_2) \mid x \in G\}$ is the *G-layer* through a. If G and H are cycles, then a H-layer is called a *row* and a G-layer a *column*.

The independence numbers of the strong product of even cycles are well known. The problem is much more involved for cycles with odd length. Hales [5], as well as Sonnemann and Krafft [12], discovered the explicit formula for the independence numbers of the strong product of two odd cycles.

Theorem 1 [5, 12] *For $j, k \in \mathbb{N}$, $j \geq k$*

$$\alpha(C_{2j+1} \boxtimes C_{2k+1}) = jk + \lfloor \tfrac{k}{2} \rfloor.$$

The independence numbers of products of more then two odd cycles are still unknown in most cases. Even for the product of three cycles there are several products with unknown independence numbers.

The next lemma can be used to bound independent size growth in the strong products of cycles.

Lemma 2 *[5] For any graph G and $k \geq 2$,*

$$\alpha(G \boxtimes C_{2k+1}) \leq k\alpha(G) + \left\lfloor \frac{\alpha(G)}{2} \right\rfloor.$$

The next lemma shows that the largest independent set of $C_{2k+1} \boxtimes G$ can be extended to $C_{2k+3} \boxtimes G$ if the upper bound from Lemma 2 is reached.

Lemma 3 [12] *For any graph G and any $k \in \mathbb{N}$,*

(i) $\alpha(C_{2k+3} \boxtimes G) \geq \lceil \frac{2k+3}{2k+1} \alpha(C_{2k+1} \boxtimes G) \rceil$

(ii) $\alpha(C_{2k+3} \boxtimes G) = \lfloor \frac{2k+3}{2} \alpha(G) \rfloor$ *if* $\alpha(C_{2k+1} \boxtimes G) = \lfloor \frac{2k+1}{2} \alpha(G) \rfloor$.

The following result is also well-known.

Lemma 4 [5] *For any graph G and $n \in \mathbb{N}$,*

$$\alpha(G \boxtimes K_n) = \alpha(G).$$

This paper considers the independence numbers of two infinite families of graphs. Section 2 deals with the independence number of $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$, while Section 3 presents the upper bound on the independence number of $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$. The results are partially obtained by a computer search.

2 $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$

It is known that the independence number of $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ is between $13k + 5$ and $13k + 6$, c.f. [13], where it was also conjectured that the independence numbers for $C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ reach the upper bound if k is large enough. This is indeed true, as follows from the main result of this section:

Theorem 5 *Let $k \geq 4$. Then*

$$\alpha(C_7 \boxtimes C_9 \boxtimes C_{2k+1}) = 13k + 6.$$

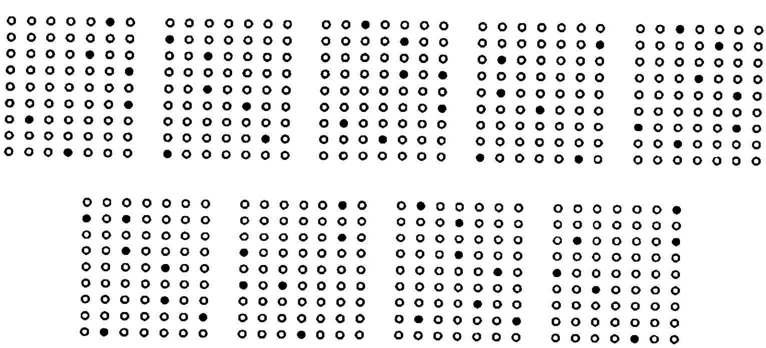

Figure 1: 58 independent vertices of $C_7 \boxtimes C_9 \boxtimes C_9$.

Proof.

We found an independent set of $C_7 \boxtimes C_9 \boxtimes C_9$ with cardinality 58 depicted in Fig. 1. Note that $(C_7 \boxtimes C_9)$-layers are presented in a clockwise order - the first row from left to right and then in the second row from right to left. Since $\alpha(C_7 \boxtimes C_9) = 13$ by Theorem 1, the assertion now follows from Lemma 3 (ii). □

In the rest of the section we report on the computer search that enabled the independent set from Fig. 1. The idea is introduced in [7] in a more general framework, but for our purposes the following description will be sufficient.

From Lemma 4 it follows that $\alpha(C_7 \boxtimes C_9 \boxtimes K_2) = \alpha(C_7 \boxtimes C_9) = 13$. In other words, two consecutive $(C_7 \boxtimes C_9)$-layers can have together at most 13 independent vertices.

We therefore define a directed graph D as follows.

In the set $S_{7,6}$ are all independent sets of $C_7 \boxtimes C_9 \boxtimes K_2$ with 13 vertices such that a $(C_7 \boxtimes C_9)$-layer admits either 6 or 7 independent vertices. A vertex $u = u_1u_2$ of $S_{7,6}$ is therefore composed of components u_1 and u_2, such that u_i póssesses 6 or 7 vertices and $|u_1| + |u_2| = |u| = 13$.

Analogously we define the set $S_{6,6}$ composed of all independent sets of $C_7 \boxtimes C_9 \boxtimes K_2$ with 12 vertices, such that both $(C_7 \boxtimes C_9)$-layers admits exactly 6 independent vertices. More formally, $w = w_1u_2 \in S_{6,6}$ if an only if $|w_i| = 6$ and $|w_1| + |w_2| = |w| = 12$.

We then set $V(D) := S_{7,6} \cup S_{6,6}$.

Let $u = u_1u_2$ and $v = v_1v_2$ be vertices of $S_{7,6}$.

We make an arc from u to v in D if and only if $u_2 = v_1$. Note that if uv is an arc in D, then $u_1u_2v_2$ induce an independent set in $C_7 \boxtimes C_9 \boxtimes P_3$.

Let $u = u_1u_2 \in S_{7,6}$ and $w = w_1w_2 \in S_{6,6}$. We make an arc from w to u in D if and only if $w_2 = u_1$ and we make an arc from u to w in D if and only if $u_2 = w_1$. Apparently, if uw (resp. wu) is an arc in D, then $u_1u_2w_2$ (resp. $w_1w_2u_2$) induce an independent set in $C_7 \boxtimes C_9 \boxtimes P_3$.

The next proposition was then the basis for our computer search.

Proposition 6 *$C_7 \boxtimes C_9 \boxtimes C_{2k+1}$ admits an independent set with $13k + 6$ independent vertices if and only if D admits a closed directed walk of length $2k + 1$.*

Since the cardinality of $S_{6,6}$ is too large, we first constructed (by using a computer program) the subgraph D' of D induced by $S_{7,6}$. The graph consists of 30702672 vertices with a maximum output degree of 96. Into the set of vertices of D' we then add a vertex $w = w_1w_2$ of $S_{6,6}$ if $w_1 = u_2$ and $w_2 = v_1$, for $u = u_1u_2 \in S_{7,6}$ and $v = v_1v_2 \in S_{7,6}$. For the obtained subgraph of D induced by $\{w\} \cup S_{7,6}$ we applied the breadth-first search algorithm in order to find a cycle of minimum length.

The computations were performed on various computers, mainly in Windows environment, but some also using Linux Ubuntu operating system. The hardware used for computations was also diverse: Intel i7 930 based personal computer, Intel Q9400 based machine and a computer cluster (with up to 24 processor cores). All computations were carried out during three months. The development environment and class libraries Lazarus (version of pascal language) were used to write all necessary programs.

3 $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$

Note first that $\alpha(C_{13} \boxtimes C_{13}) = 39$ by Theorem 1. The existence of an independent set with 247 vertices of $C_{13} \boxtimes C_{13} \boxtimes C_{13}$ was proven in [2]. Moreover, recently it was established that the independent set with more than 247 vertices cannot exist in this graph [3]. From Lemma 2 and Lemma 3 then it follows that the independence number of $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$, $k \geq 7$, is between $38k + 19$ and $39k + 19$. We will show that the upper bound can be substantially reduced.

All operations in the sequel are performed modulo 13. We also identify the vertices of C_{13} with the elements of $\mathbb{Z}_{13}$.

We first define the following sets for $i \in \mathbb{Z}_{13}$.

$$X_i = \{(4i + 2k + 2, k) \mid k \in \mathbb{Z}_{13}\},$$

$$Y_i = \{(4i + 11k + 6, k) \mid k \in \mathbb{Z}_{13}\},$$

$$Z_i = \{(2i + 6k + 9, k) \mid k \in \mathbb{Z}_{13}\},$$

$$W_i = \{(2i + 7k + 8, k) \mid k \in \mathbb{Z}_{13}\}.$$

Note that the the vertices in the above sets are independent in $C_{13} \boxtimes C_{13}$. We next define for $i \in \mathbb{Z}_{13}$:

$$\widehat{X_i} := X_{i-1} \cup X_i \cup X_{i+1},$$

$$\widehat{Y_i} := Y_{i-1} \cup Y_i \cup Y_{i+1},$$

$$\widehat{Z_i} := Z_{i-1} \cup Z_i \cup Z_{i+1},$$

$$\widehat{W_i} := W_{i-1} \cup W_i \cup W_{i+1}.$$

Proposition 7 *Let S be an independent set with 39 vertices of $C_{13} \boxtimes C_{13}$. Then there exists $i \in \mathbb{Z}_{13}$ such that S equals one of $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, or $\widehat{W_i}$.*

Proof. It is easy to establish that $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, and $\widehat{W_i}$ are independent sets of cardinality 39 in $C_{13} \boxtimes C_{13}$.

Let S be an independent set of $C_{13} \boxtimes C_{13}$ with cardinality 39. Note first that by Lemma 4 every row and column of $C_{13} \boxtimes C_{13}$ has to have exactly 3 vertices of S. Using this fact we wrote a simple computer program based on the exhaustive search which confirmed that $C_{13} \boxtimes C_{13}$ admits exactly 52 independent sets of cardinality 39, such that every of them equals one of $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, or $\widehat{W_i}$. □

For an independent set S of cardinality 39 we say that S is of *type* X, Y, Z, or W, if it equals $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, or $\widehat{W_i}$, respectively, for some $i \in \mathbb{Z}_{13}$.

Theorem 8 *Let $i, j \in \mathbb{Z}_{13}$. Let $\widehat{P_i}$ as well as $\widehat{Q_i}$ stand for one of $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, or $\widehat{W_i}$ and let P_i as well as Q_i stand for one of X_i, Y_i, Z_i, or W_i. Then*

(i) $|P_i \cap Q_j| = 1$ for $P \neq Q$.

(ii) If $u \in P_i$, then $\{u\} \cup P_j$ is an independent set if and only if $j \in \{i \pm 1, i \pm 2, i \pm 5\}$.

(iii) $|\widehat{P_i} \cap \widehat{P_{i+1}}| = 26$.

(iv) $|\widehat{P_{i-1}} \cap \widehat{P_{i+1}}| = 13$.

(v) $|\widehat{P_i} \cap \widehat{Q_j}| = 9$ for $P \neq Q$.

Proof.

(i) Suppose first that P_i and Q_j stand for X_i and Y_j, respectively. Let i and j be arbitrary but fixed elements from $\mathbb{Z}_{13}$. Then $(4i + 2k + 2, k)$ equals $(4j + 11k + 6, k)$ if and only if $4i + 2k + 2 \equiv 4j + 11k + 6 \pmod{13}$.

We then get

$$4k \equiv (4j + 9i + 4) \pmod{13}.$$

It is well known, e.g. see [9, Theorem 2.17], that

$$ax \equiv b \pmod{m}$$

has a solution if and only if $\gcd(a, m) = g$ divide b. If this condition is met, the total number of solutions is g (mod m). Since gcd(13,4)=1, the assertion follows.

The proofs for other pairs of P_i and Q_j are analogous (note that 13 is a prime number), we may therefore conclude that this part of the proof is complete.

(ii) For an arbitrary $u \in P_i$, the set $\{u\} \cup P_j$ is independent if for every $v \in P_i$, the difference of u and v is not in the set $\{-1, 0, 1\} \times \{-1, 0, 1\}$. The assertion can be checked very easily by hand or by a computer.

(iii) It follows from the definitions of $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, and $\widehat{W_i}$.

(iv) It follows from the definitions of $\widehat{X_i}$, $\widehat{Y_i}$, $\widehat{Z_i}$, and $\widehat{W_i}$.

(v) It follows from (i). □

If S is an independent set in $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$, let S_i denote the intersection of S with a $(C_{13} \boxtimes C_{13})$-layer through i. By slight abuse of notation, S_i will also stand for the projection of S_i to $C_{13} \boxtimes C_{13}$.

Theorem 9 *Let $k \geq 7$. Then*

$$38k + 19 \leq \alpha(C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}) \leq 38k + \lfloor \frac{k}{2} \rfloor + 19.$$

Proof. Since $\alpha(C_{13} \boxtimes C_{13} \boxtimes C_{13}) = 247$ [3], the lower bound follows from Lemma 3 (i). We therefore have to show that for $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$, $k \geq 7$, an independent set with more than $38k + \lfloor \frac{k}{2} \rfloor + 19$ vertices does not exist.

Suppose then to the contrary that there is an independent set of cardinality $38k + \lfloor \frac{k}{2} \rfloor + 20$ for some $k \geq 7$. Let then S be an independent set of cardinality $38k + \lfloor \frac{k}{2} \rfloor + 20$ in $C_{13} \boxtimes C_{13} \boxtimes C_{2k+1}$ such that k is minimal.

Since $\alpha(C_{13} \boxtimes C_{13} \boxtimes K_2) = \alpha(C_{13} \boxtimes C_{13}) = 39$, then there must be

(a) a sequence $S_{j-1}, S_j, S_{j+1}, S_{j+2}, S_{j+3}$ with cardinalities 19, 20, 19, 20, 19, respectively or

(b) a sequence $S_{j-2}, S_{j-1}, S_j, S_{j+1}, S_{j+2}, S_{j+3}$ with cardinalities 19, 20, 19, 19, 20, 19, respectively.

Suppose now that a sequence (a) exist. From Theorem 8 (v) it follows that $S_{j-1} \cup S_j$, $S_j \cup S_j$, $S_{j+1} \cup S_{j+2}$, and $S_{j+2} \cup S_{j+3}$ must be of the same type. Suppose w.l.o.g. that they are all of type X.

We first show the following

Fact. If $|S_{j-1}| = |S_{j+1}| = 19$ and $|S_j = 20|$ then $S_{j-1} \cup S_j$ and $S_j \cup S_{j+1}$ cannot both equal $\widehat{X_i}$.

Proof: Suppose $S_{j-1} \cup S_j = S_j \cup S_{j+1} = \widehat{X_i}$. Then $S \setminus (S_j \cup S_{j+1})$ is an independent set of cardinality $38(k-1) + \lfloor \frac{k-1}{2} \rfloor + 20$ in $C_{13} \boxtimes C_{13} \boxtimes C_{2k-1}$ which contradicts the minimality of k. □

Let then $S_j \cup S_{j+1} = \widehat{X_i}$ for some $i \in \mathbb{Z}_{13}$. Since by the above Fact $S_{j-1} \cup S_j$ cannot equal $\widehat{X_i}$, from Theorem 8 it follows that $S_{j-1} \cup S_j = \widehat{X_{i\pm1}}$. Suppose w.l.o.g. that $S_{j-1} \cup S_j = \widehat{X_{i-1}}$. Then $S_{j+1} \cup S_{j+2}$ and $S_{j+2} \cup S_{j+3}$ have to equal $\widehat{X_{i+1}}$ and $\widehat{X_{i+2}}$, respectively. Since S_j is a subgraph of both $\widehat{X_{i-1}}$ and $\widehat{X_i}$ and since S_{j+1} is a subgraph of both $\widehat{X_i}$ and $\widehat{X_{i+1}}$, we have $S_j = X_{i-1} \cup X_i'$ and $S_{j+1} = X_{i+1} \cup (X_i \setminus X_i')$ for some $X_i' \subset X_i$. The situation is depicted in Fig. 2. Furthermore, since $S_{j+1} \cup S_{j+2} = \widehat{X_{i+1}}$, then it follows that S_{j+2} has to contain the set X_i'. But since $S_{j+2} \cup S_{j+3} = \widehat{X_{i+2}}$, the set S_{j+2} cannot contain vertices from X_i and we obtain a contradiction.

Suppose now that a sequence (b) exist. We can conclude analogously as above that $S_{j-2} \cup S_{j-1}$, $S_{j-1} \cup S_j$, $S_j \cup S_{j+1}$, $S_{j+1} \cup S_{j+2}$ and $S_{j+2} \cup S_{j+3}$ must be of the same type. Suppose again that they are all of type X and since the Fact holds also for this case suppose w.l.o.g. that $S_{j-2} \cup S_{j-1} = \widehat{X_{i-1}}$ and $S_{j-1} \cup S_j = \widehat{X_i}$. Since the vertices of S_{j-1} compose $\widehat{X_{i-1}}$, S_j

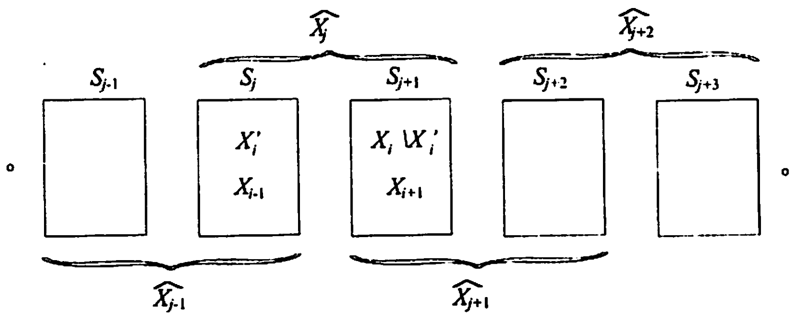

Figure 2: Distribution of independent vertices in consecutive $(C_{13} \boxtimes C_{13})$-layers for case (a).

has to contain all vertices of X_{i+1} and some vertices of $X_{i-1} \cup X_i$. We distinguish two cases:

Case 1. S_j contains at least one vertex from X_{i-1}. Then by Theorem 8 (ii) the set S_{j+1} has to be a subset of $X_{i+1} \cup X_{i-1} \cup X_i$. Moreover, since $|S_{j+1} \cup S_{j+2}| = 39$ and $|S_j \cup S_{j+1}| = 38$, for some $u \in \widehat{X_i}$ we get $S_j \cup S_{j+1} = \widehat{X_i} - \{u\}$. For the same reason we get $S_{j+1} \cup S_{j+2} = \widehat{X_i}$. Let then S' be obtained from S by moving u from S_{j+2} to S_{j+1}. Note that S' is an independent set of cardinality $|S|$. Moreover, $|S_j'| = |S_{j+2}'| = 19$ and $|S_{j+1}'| = 20$ such that $S_{j-1}' \cup S_j'$ and $S_j' \cup S_{j+1}'$ both equals $\widehat{X_i}$. Since we proved with the Fact that the latter leads to a contradiction, this part of the proof is settled.

Case 2. S_j does not contain vertices from X_{i-1}. Then $S_j = X_{i+1} \cup X_i'$, where X_i' is a subset of X_i. By Theorem 8 (iv) is then S_{j+1} a subset either of $\widehat{X_i}$ or $\widehat{X_{i+1}}$. If $S_{j+1} \subset \widehat{X_i}$, we obtain a contradiction analogous to Case 1. Let then $S_{j+1} \subset \widehat{X_{i+1}}$. From Theorem 8 (ii) it follows that $S_{j+1} = X_{i+2} \cup (X_i \setminus X_i') - \{u\}$, where u is some vertex of $X_i \setminus X_i'$. (Note that u cannot be a vertex of X_{i+2}, since then $S_{j+2} \cup S_{j+3}$ is also equal to $\widehat{X_{i+1}}$ which leads to a contradiction.) Let then S' be obtained from S by moving u from S_{j+2} to S_{j+1}. Again, S' is an independent set of cardinality $|S|$, with $|S_j'| = |S_{j+2}'| = 19$ and $|S_{j+1}'| = 20$ such that $S_j' \cup S_{j+1}'$ and $S_{j+1}' \cup S_{j+2}'$ both equals $\widehat{X_{i+1}}$. The resulting contradiction completes the proof of Case 2. □

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