Bijective proofs of the hook formula for rooted trees

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Abstract

We present a bijective proof of the hook length formula for rooted trees based on the ideas of the bijective proof of the hook length formula for standard tableaux by Novelli, Pak and Stoyanovskii [10]. In section 4 we present another bijection for the formula.

MR Subject Classification: 05A05,05A15
Keywords: Hook length formula, bijective proof.

1 Introduction

Classically, there are three hook length formulas for the number of standard tableaux associated to posets [16] the Frame-Robinson-Thrall hook formula for the number of standard tableaux of a given Ferrers shape, the hook formula for the number of tableaux of a shifted shape, and the hook formula for the number of rooted trees with a standard labelling. (In the present paper we do not address the recent hook formulas for d-complete posets due to Peterson and Proctor (cf.[14]), which include the aforementioned as special cases.)
The problem of finding a bijective proof for these surprisingly compact formulas has a long history for the first case, the case of standard tableaux of an (ordinary) Ferrers shape (cf. the Introduction of [7]), culminating in what is now regarded as "the " bijective proof of the (ordinary) hook

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formula: the bijective proof of Novelli, Pak and Stoyanovskii [10].
Less attention had been paid to the problem of finding a bijective proof of the shifted hook formula (cf. [7]). However, recently Fischer [1] succeeded in finding such a bijective proof in the spirit of Novelli, Pak and Stoyanovskii. The purpose of this paper is to complete this program for the case of rooted trees as well. In fact, we do not only provide a bijective proof of the hook formula for rooted trees in the spirit of Novelli, Pak and Stoyanovskii (see Section 3.), we also provide a second, conceptually different, bijective proof (see Section 4.).
The plan of this paper is as follows. In Section 2 we recall some definitions and notations and state the main result. In Section 3 we define the first bijection (in the spirit of Novelli, Pak and Stoyanovskii). After defining a map and its inverse, we show, that these maps indeed define a bijection. The example we give, helps to understand how the maps work. In Section 4 we present the second bijection, by defining another map and its inverse, proving that there are inverse to each other and illustrating how these maps work by an example.

2 Notations and the main theorem

A poset $\mathcal{T}= (V, \leq)$ is a *rooted tree* if it has a unique minimal element. The Hasse diagram of $\mathcal{T}$ is a tree T in the graphic-theoretic sense of the term. The set of the nodes of the tree T is $V = V(T)$. If n is the number of elements in the poset then a bijection $S : V \to [n] = \{1, 2, \ldots, n\}$ is a *labelling* on $\mathcal{T}$. An order preserving bijection $S : V \to [n] = \{1, 2, \ldots, n\}$ ($v_1 \leq v_2$ implies $S(v_1) \leq S(v_2)$) is a *standard labelling* on $\mathcal{T}$. If v is a node of T, then the *hook of* v is

$$H_v = \{w \in \mathcal{T} \mid w \geq v\},$$

with corresponding *hook length* $h_v = |H_v|$. In fact the hook length is the number of successors of the node including the node itself. $f_{\mathcal{T}}$ denotes the number of standard labellings on the tree $\mathcal{T}$. The following theorem gives a formula for $f_{\mathcal{T}}$.

Theorem 1

$$f_{\mathcal{T}} = \frac{n!}{\prod_{v\in V(T)} h_v}, \tag{1}$$

In order to construct a bijective proof of Theorem 1, we multiply both sides of (1) by the denominator $\prod_{v\in V(T)} h_v$:

$$f_{\mathcal{T}} \times \prod_{v\in V(T)} h_v = n!. \tag{2}$$

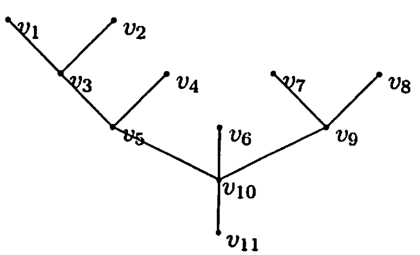
The right side of this equation can be interpreted as the number of labellings (permutations of $[n]$), and the left side as the number of pairs of a standard labelling and a hook function on $\mathcal{T}$. A *hook function* on $\mathcal{T}$ is a map $H : V \to Z$, such that $H(v) \in \{1, 2, \ldots, h_v\}$. We describe bijections between these two sets in Section 3 and Section 4, which proves Theorem 1.

3 The first bijection

3.1 The total order on the tree

We fix a total order on the nodes of the tree with the property that the order of a node is always greater than the order of its successors. We describe this total order by a map $V \to [n]$. First we define *the left most leaf of the tree.* This is the endnode of the unique path $P = \{t_1, t_2, \ldots, t_l\}$, where t_1 is the root and t_{i+1} is the first node (moving from left to the right) among the successors of t_i for all $1 \le i \le l-1$. We construct our map which gives the total order the following way: Consider the left most leaf of the tree and assign the least number to it. Delete this node from the set of the nodes and delete this number from the set of the numbers. A node is denoted by v_j if the number j has been associated to it. We define first a map from

Figure 1: The total order



the set of labellings (L) of the tree T to the set of a pair (S, H) where S is a standard labelling and H is a hook function of $\mathcal{T}$. We will see that this is a bijective map.

3.2 The map I: $L \to (S, H)$

The map I transforms a labelling to a pair (S, H) using a sequence of pairs of a labelling and a hook function (S_j, H_j), $1 \le j \le (n+1)$:

$$L \longrightarrow (S_1, H_1) \overset{Move(1)}{\longrightarrow} (S_2, H_2) \overset{Move(2)}{\longrightarrow} \cdots \overset{Move(n)}{\longrightarrow} (S_{n+1}, H_{n+1})$$

The sequence starts with (S_1, H_1), where S_1 is the labelling L and H_1 is the hook function with $h(v) = 1$ for all $v \in V(T)$.

Move(j) transforms the pair (S_j, H_j) into a pair (S_{j+1}, H_{j+1}) the following way: Start the process with considering the node v_j. We denote the label of a node v_j in S_j by l.

Step 0 Let v_i be the actual node with label l. Consider the set of the direct successors $(D(v_i))$ in S_j. Let v_{min} be the node with the minimal label (l_{min}) in $D(v_i)$.

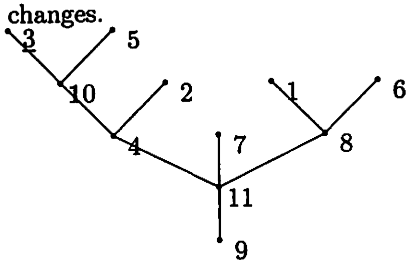
Step 1
- if $l_{min} < l$ then interchange the label of v_{min} and v_i. The actual node with l is v_{min}. Go to Step 0.
- if $l_{min} > l$ Go to Step 2.

Step 2 Let v_k be the node with the label l in S_{j+1}. We point to this node with the hook number which we associate to the node v_j and set $h(v_j) = j - k + 1$.

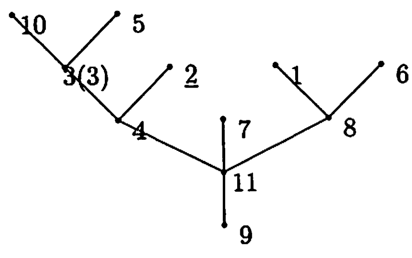
The label l slides actually from v_j to another node v_k along a unique path. The labels of the nodes of this path slide one node down and the endnode of the path receive the label l.

3.3 Example

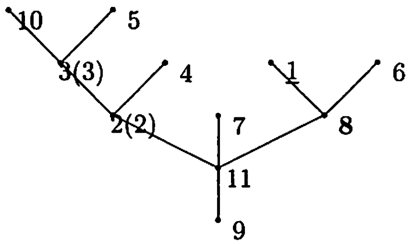
Lets consider an example. We describe only moves when the labelling changes.



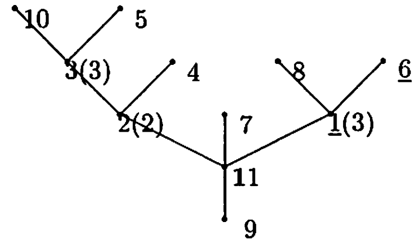
The first exchange will be necessary at Move(3) concerning the node v_3: $l = 10$ $l_{min} = 3$. And we set: $h(v_3) = 3 - 1 + 1 = 3$.



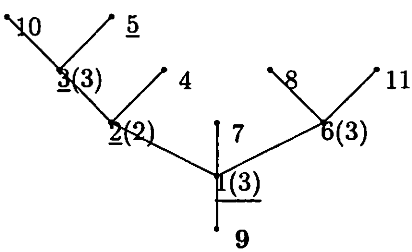
The next exchange is at Move(5): $l = 4$ and $l_{min} = 2$. After the exchange we set: $h(v_5) = 5 - 4 + 1 = 2$



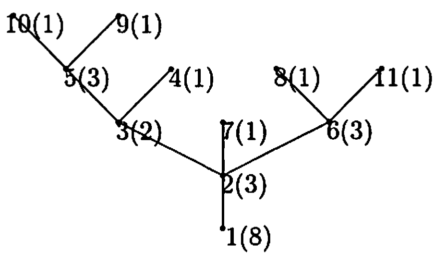
Move(9): .
$l = 8$, $l_{min} = 1$, so we exchange and set $h(v_9) = 9 - 7 + 1 = 3$.



Move(10):
$l = 11$, $l_{min} = 1$, we exchange. We consider now the successors of v_9: $l_{min} = 6 < 11$ so we do another exchange and Move(10) ends with setting $h(v_{10}) = 10 - 8 + 1 = 3$.



Move(11): $l = 9$. We have $l > l(v_{10}) > l(v_5) > l(v_3) > l(v_2)$, so Move(11) ends after 4 exchanges and $l = 9$ slides up to the node v_2. We have to set $h(v_{11}) = 11 - 2 + 1 = 8$.



This is the pair (S_{12}, H_{12}). The standard labelling of the tree and the hook function.

3.4 The map II: $(S, H) \to L$

We define map II which transforms a pair of a standard labelling and a hook function (S, H) into a labelling L of the tree T using a sequence of pairs (S'_j, H'_j), $1 \le j \le n$:

$$(S, H) \overset{Move'(n)}{\longrightarrow} (S'_n, H'_n) \overset{Move'(n-1)}{\longrightarrow} \cdots \overset{Move'(1)}{\longrightarrow} (S'_1, H'_1) \longrightarrow L$$

In the pair (S'_1, H'_1) S'_1 is the labelling L and H'_1 is the hook function with $h(v_j) = 1$ for all $1 \le j \le n$.
Move'(j): Consider v_j with its hook number $h(v_j)$. Set $k = j - (h(v_j) - 1)$.

($k < j$, so v_k is a successor of v_j.) Interchange the labels of the nodes of the unique walk from v_k to v_j step by step and set $h(v_j) = 1$.

3.5 The proof

Theorem 2 *The map I and map II are inverse to each other.*

Proof From the definition of Move(j) follows that S_j is a standard labelling of the subtree on $\{v_1, \dots, v_j\}$. It is obvious that Move(j) and Move'(j) are inverse to each other and the theorem follows.

4 A second bijection

4.1 Notations

There are other possibilities to define a bijection between the set of pairs (S, H) and the set of labellings L. We describe one in this section. In the first bijection we moved the labellings of the nodes. In this bijection in some sense we fix the labellings and move the nodes. We consider a labelling of the tree as a linear arrangement of the nodes and a label simple as the position of the node in this arrangement. A standard labelling is a special labelling which keeps the structure of the tree, the partial order of the nodes so it holds: if $v_i \le v_j$ in the total order for some i and j then v_i stands before v_j in the linear arrangement.
We fix the total order of the nodes. The *distance of two nodes v_i and v_j* is the number of the nodes of the unique path in the tree from v_i to v_j (involving v_i and v_j). We say that a node v_i is on the level k when the distance of the root and v_i is k. We fix the total order of the nodes according the following simple rule: we denote the node on the first level, the root by v_1. Let the right most node on the level $(k-1)$ be the node $v_{(k-1)^*}$. We denote the nodes on the level k from left to the right by $v_{(k-1)^*+1}, v_{(k-1)^*+2}, \dots, v_{k^*}$. $k^* - (k-1)^*$ is the number of the nodes on the level k. (See Figure 2.)
We write the pair (S, H) as a sequence of $(v_j, h(v_j))$, where the order of $(v_j, h(v_j))$ is given by the standard labelling S. For instance

The associated sequence to in Figure 3:

$$\begin{aligned}(S,H) &= (v_1,3)(v_3,4)(v_7,3)(v_2,1)(v_{11},1)(v_6,3)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1)\end{aligned}$$

Figure 2: The total order

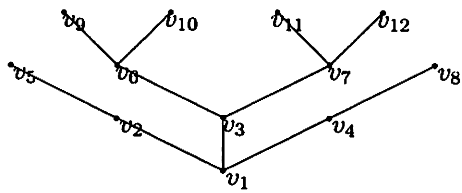
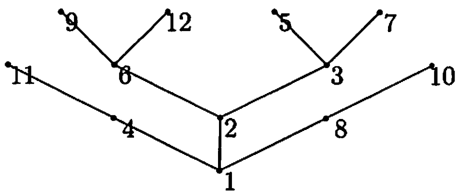
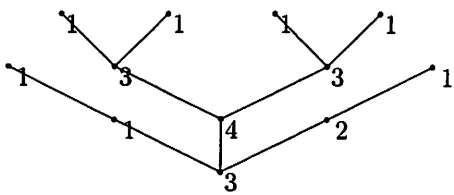


Figure 3: The standard labelling and hook function (S,H)



The standard labelling S



The hook function H

4.2 The map φ

This map transforms a pair (S,H) to a pair (S_{n+1}, H_{n+1}), a pair of a labelling and to a hook function with $h(v_j) = 1$ for all $1 \leq j \leq n$ step by step. We associate to S_{n+1} the labelling L.

$$(S,H) = (S_1, H_1) \overset{Step\ 1}{\longrightarrow} (S_2, H_2) \overset{Step\ 2}{\longrightarrow} \cdots \overset{Step\ n}{\longrightarrow} (S_{n+1}, H_{n+1}) \rightarrow L$$

We describe Step j: Consider the node v_j and the subsequence of its successors $A(v_j)$. Move the node v_j to the position signed by its hook number $h(v_j)$ among the members of $A(v_j)$ and set $h(v_j) = 1$. We arrange the other members of $A(v_j)$ in the remaining positions, occupying by $A(v_j)$ keeping their relative relations. The nodes outside of $A(v_j)$ keep their previous positions.

4.3 Example

We consider the tree which is shown in Figure 2 with the standard labelling and hook function given in Figure 3. We apply the map φ:

Step 1 Consider v_1. All the nodes are successors of v_1 and $h(v_1) = 3$. So v_1 moves to the third position of the whole sequence. We set $h(v_1) = 1$. The result of the first transformation is:

$$\begin{aligned}(S_2, H_2) &= (v_3,4)(v_7,3)(v_1,1)(v_2,1)(v_{11},1)(v_6,3)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 2 Consider v_2. The subsequence of the successors is $A(v_2) = v_2, v_5$ and $h(v_2) = 1$. So v_2 keeps its position and $h(v_2) = 1$. The result of this step:

$$\begin{aligned}(S_3, \dot{H}_3) &= (S_2, H_2) = (v_3,4)(v_7,3)(v_1,1)(v_2,1)(v_{11},1)\\ &\quad (v_6,3)(v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 3 Consider v_3. The subsequence of the successors is $A(v_3) = v_3, v_7, v_{11}, v_6, v_{12}, v_9, v_{10}$ and $h(v_3) = 4$. v_3 moves to the fourth position of the seven positions of the nodes from $A(v_3)$ and we set $h(v_3) = 1$. The result of this step is:

$$\begin{aligned}(S_4, H_4) &= (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)\\ &\quad (v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 4 Consider v_4. $A(v_4) = v_4, v_8$ and $h(v_4) = 2$. So

$$\begin{aligned}(S_5, H_5) &= (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)\\ &\quad (v_{12},1)(v_8,1)(v_9,1)(v_4,1)(v_5,1)(v_{10},1).\end{aligned}$$

Step 5 $(S_6, H_6) = (S_5, H_5)$ since $h(v_5) = 1$.

Step 6 Consider the node v_6. $A(v_6) = v_6, v_9, v_{10}$ and $h(v_6) = 3$. So

$$\begin{aligned}(S_7, H_7) &= (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_9,1)(v_3,1)\\ &\quad (v_{12},1)(v_8,1)(v_{10},1)(v_4,1)(v_5,1)(v_6,1).\end{aligned}$$

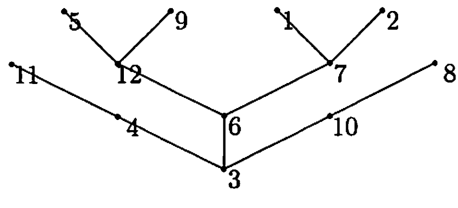
Step 7 Consider the node v_7. $A(v_7) = v_7, v_{11}, v_{12}$ and $h(v_7) = 3$. So

$$\begin{aligned}(S_8, H_8) &= (v_{11},1)(v_{12},1)(v_1,1)(v_2,1)(v_9,1)(v_3,1)\\ &\quad (v_7,1)(v_8,1)(v_{10},1)(v_4,1)(v_5,1)(v_6,1).\end{aligned}$$

Step 8-12 The other hook numbers $h(v_8), h(v_9), \ldots, h(v_{12})$ are all 1, so we have

$$(S_8, H_8) = (S_9, H_9) = \cdots = (S_{13}, H_{13}) \rightarrow L.$$

Figure 4: The labelling L



4.4 The map ψ

The map ψ transforms a labelling L of the nodes into a pair of (S, H), a pair of a standard labelling and a hook function. The n steps of the map are:

$$L \rightarrow (S^*_{n+1}, H^*_{n+1}) \overset{Step\ n^*}{\longrightarrow} (S^*_n, H^*_n) \overset{Step\ (n-1)^*}{\longrightarrow} \cdots \overset{Step\ 1^*}{\longrightarrow} (S^*_1, H^*_1) = (S, H)$$

We consider the nodes in the reverse order. During the generic step Step j^* we replace the node v_j and change $h(v_j)$ when its necessary. The successors of v_j were already investigated. We denote this subsequence by $A^*(v_j)$. We set $h(v_j)$ according the relative relation of v_j among the members of $A^*(v_j)$ and move v_j to the first position among $A^*(v_j)$. We arrange the other members of $A^*(v_j)$ in the remaining positions occupying by $A^*(v_j)$ keeping their relative relations. The nodes outside of $A^*(v_j)$ keep their previous positions.

4.5 Example

We apply now map ψ to

$$\begin{aligned} L = (S^*_{13}, H^*_{13}) &= (v_{11}, 1)(v_{12}, 1)(v_1, 1)(v_2, 1)(v_9, 1)(v_3, 1) \\ &\quad (v_7, 1)(v_8, 1)(v_{10}, 1)(v_4, 1)(v_5, 1)(v_6, 1). \end{aligned}$$

Step 12* Consider the last node $v_n = v_{12}$. The subsequence of its successors is $A^*(v_{12}) = v_{12}$. v_{12} is the first node in $A^*(v_{12})$. So $(S^*_{13}, H^*_{13}) = (S^*_{12}, H^*_{12})$.

Step 11* − 8* $(S^*_{12}, H^*_{12}) = (S^*_{11}, H^*_{11}) = \cdots = (S^*_8, H^*_8)$

Step 7* Consider v_7. $A^*(v_7) = v_{11}, v_{12}, v_7$. v_7 is in the third position. We put it to the first position in $A^*(v_7)$, set $h(v_7) = 3$ and shift the other nodes in $A^*(v_7)$:

$$(S_7^*, H_7^*) = (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_9,1)(v_3,1)(v_{12},1)(v_8,1)(v_{10},1)(v_4,1)(v_5,1)(v_6,1).$$

Step 6* Consider v_6. $A^*(v_6) = v_9, v_{10}, v_6$. v_6 is in the third position. We put it to the first position in $A^*(v_6)$, set $h(v_6) = 3$ and shift the other nodes in $A^*(v_6)$:

$$(S_6^*, H_6^*) = (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)(v_{12},1)(v_8,1)(v_9,1)(v_4,1)(v_5,1)(v_{10},1).$$

Step 5* $(S_5^*, H_5^*) = (S_6^*, H_6^*)$

Step 4* Consider v_4. $A^*(v_4) = v_8, v_4$. We put v_4 to the first position in $A^*(v_4)$ and set $h(v_4) = 2$.

$$(S_4^*, H_4^*) = (v_7,3)(v_{11},1)(v_1,1)(v_2,1)(v_6,1)(v_3,1)(v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).$$

Step 3* Consider v_3. $A^*(v_3) = v_7, v_{11}, v_6, v_3, v_{12}, v_9, v_{10}$. v_3 is in the third position. We put it to the first position and set $h(v_3) = 4$.

$$(S_3^*, H_3^*) = (v_3,4)(v_7,3)(v_1,1)(v_2,1)(v_{11},1)(v_6,1)(v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).$$

Step 2* $(S_2^*, H_2^*) = (S_3^*, H_3^*)$.

Step 1* We consider v_1. $A^*(v_1)$ is the whole sequence. v_1 is in the third position. We put it to the first position and set $h(v_1) = 3$.

$$(S, H) = (S_1^*, H^*) = (v_1,3)(v_3,4)(v_7,3)(v_2,1)(v_{11},1)(v_6,3)(v_{12},1)(v_4,2)(v_9,1)(v_8,1)(v_5,1)(v_{10},1).$$

4.6 The proof

Theorem 3 *The map φ and the map ψ are inverse to each other.*

Proof: First we give several important properties of the maps:

1. φ: The *Step* $j(S_j, H_j)$ change the position of v_j according to the hook number $h(v_j)$ in the subsequence $A(v_j)$.

   ψ: The *Step* $j^*(S_{j+1}, H_{j+1})$ change the hook number $h(v_j)$ according to the position of v_j in $A^*(v_j)$ and moves v_j to the first position in $A^*(v_j)$.

2. φ: After *Step* j of the map φ (in $\{S_i\}_{i>j}$) the node v_j keeps its position.

ψ: The node v_j keeps its position until *Step* j^* (in $\{S_i^*\}_{i>j}$).

It is obvious that given a set (S_j, H_j):

$$Step\, j^*(Step\, j(S_j, H_j)) = (S_j, H_j)$$

and given a set (S_{j^*}, H_{j^*}) :

$$Step\, j(Step\, j^*(S^*_{(j+1)}, H^*_{(j+1)})) = (S^*_{(j+1)}, H^*_{(j+1)}).$$

This means that *Step* j and *Step* j^* are inverse to each other and the theorem follows.

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