On the diameter of integral circulant graphs

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Abstract

Integral circulant graphs have been proposed as potential candidates for modelling quantum spin networks with perfect state transfer between antipodal sites in the network. We show that the diameter of these graphs is at most $O(\ln \ln n)$, and further improve the recent result of Saxena, Severini and Shparlinski.

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1 Introduction

It has been shown in [2, 3] that for a quantum spin network with a mirror symmetry to allow the perfect state transfer it is necessary that the ratio of differences of any two pairs of adjacency matrix eigenvalues be rational. Integral graphs, which have integers only as the eigenvalues of their adjacency matrix [1], naturally satisfy this condition. Actually, it can be easily shown that a graph G satisfies the above condition if and only if there exists a square-free positive integer b such that all eigenvalues of G are in the form $a\sqrt{b}$, with a being an integer. Setting $b = 1$ then yields the class of integral graphs.

Motivated by this, Saxena, Severini and Shparlinski [9] studied the properties, notably the vertex degree and the diameter, of integral circulant

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graphs. They obtained an upper bound on the number of vertices of an integral circulant graph in terms of its degree, and sharp bounds on the diameter of integral circulant graphs.

Let us recall that, for $n \in \mathbb{N}$ and $S \subseteq \{1, 2, \ldots, n-1\}$, the *circulant graph* $G(n, S)$ is the graph with n vertices, labelled with integers modulo n, such that each vertex i is adjacent to $\#S$ other vertices $i + s \pmod n$ for each $s \in S$. The set S is called the *symbol* of $G(n, S)$. As we will consider undirected graphs only, we assume that $s \in S$ if and only if $n - s \in S$, so that the vertex i is adjacent to both $i \pm s \pmod n$ for each $s \in S$. Circulant graphs have already found important applications as a class of interconnection networks in parallel and distributed computing [7].

The *diameter* of a graph G, denoted by $\operatorname{diam} G$, is the longest among the shortest paths between any two vertices of G. Other graph theoretical notions may be found, for example, in [10].

Recently, So [8] has characterized the integral circulant graphs. Let

$$G_n(d) = \{k : gcd(n, k) = d, \quad 1 \le k < n\}$$

be a set of all integers less than n having the same greatest common divisor d with n. In particular, $\#G_n(d) = \varphi(n/d)$, where

$$\varphi(m) = \#\{s : gcd(m, s) = 1, \quad 1 \le s < m\}$$

denotes the Euler phi-function [6]. Let D_n be a set of positive divisors d of n, with $d \le n/2$. So has proved

Theorem 1 ([8]) *A circulant graph $G(n, S)$ is integral if and only if*

$$S = \bigcup_{d \in D} G_n(d) \tag{1}$$

for some set of divisors $D \subseteq D_n$.

We continue to study the diameter of integral circulants, showing that it is at most $O(\ln \ln n)$. In addition, we improve the upper bound from [9] on the diameter of integral circulants.

2 An upper bound on the diameter

Throughout this section, let $G(n; S)$ be an integral circulant graph whose symbol is, via (1), defined by a corresponding set D of divisors of n. It is easy to see that G is connected if and only if $gcd(n, D) = 1$ [7, Corollary 4.2].

Notice that $G(n, S)$ is a Cayley graph of the additive group of $\mathbb{Z}_n$ with respect to the Cayley set S, where $\mathbb{Z}_n$ is the residue ring modulo n. Saxena, Severini & Shparlinski [9] have shown the following theorem.

Theorem 2 ([9]) *Let t be the size of the smallest set of additive generators of $\mathbb{Z}_n$ contained in D. Then $t \leq \operatorname{diam} G(n,S) \leq 2t+1$.*

Let $\{d_1, d_2, \ldots, d_t\} \subseteq D$ be the smallest set of additive generators of $\mathbb{Z}_n$. Thus, $1 \in \mathbb{Z}_n$ is a linear combination of these elements and therefore,

$$\gcd(n, d_1, d_2, \ldots, d_t) = 1.$$

Next, let $\omega(n)$ be the number of distinct prime factors of n. The minimality of t implies that

$$gcd(n, d_1, \ldots, d_{s-1}, d_{s+1}, \ldots, d_t) > 1$$

for every $s = 1, \ldots, t$. Thus, for each s there exists a prime divisor p_s of n such that $p_s \nmid d_s$ and $p_s \mid d_i$ for all $i \neq s$. We may therefore define the mapping $\alpha : \{d_1, \ldots, d_t\} \mapsto \{p_1, \ldots, p_{\omega(n)}\}$ by $\alpha(d_s) = p_s$. Evidently, if $d_{s_1} \neq d_{s_2}$ then $p_{s_1} \neq p_{s_2}$, i.e., the mapping α is injective and thus, $t \leq \omega(n)$. Therefore, we have proved

Theorem 3 *In an integral circulant graph $G(n,S)$,* $\operatorname{diam} G(n,S) \leq 2\omega(n)+1$.

As

$$\omega(n) \approx \ln\ln n + B_1 + \sum_{k=1}^{\infty}\left(-1+\sum_{j=0}^{k-1}\frac{\gamma_j}{j!}\right)\frac{(k-1)!}{(\ln n)^k},$$

where B_1 is the *Mertens constant* and γ_j are the *Stieltjes constants*, it follows that

Corollary 4 *In an integral circulant graph $G(n,S)$,* $\operatorname{diam} G(n,S) = O(\ln\ln n)$.

We now go on to improve the upper bound from Theorem 2.

Theorem 5 *Let $\{d_1, d_2, \ldots, d_t\} \subseteq D$ be the smallest set of additive generators of $\mathbb{Z}_n$. If n/d_i is odd for each $i = 1, 2, \ldots, t$, then* $\operatorname{diam} G(n,S) \leq 2t$.

Proof For $i = 1, 2, \ldots, t$ and $j = 0, 1, \ldots, n-1$, let

$$H_n(d_i) = \{h :\ d_i | h\} \subseteq \{0, 1, \ldots, n-1\},$$

and let $j + H_n(d_i)$ denote a subgraph of $G(n,S)$, whose vertices are of the form $j+h$, $h \in H_n(d_i)$, with two vertices $j+h'$ and $j+h''$ being adjacent if $h''-h' \in G_n(d_i)$. We claim that the subgraph $j + H_n(d_i)$ has diameter at most two.

Let $j + h_1$ and $j + h_2$ be two distinct vertices that are not adjacent in $j + H_n(d_i)$. As $h_1, h_2 \in H_n(d_i)$, we can write $h_1 = d_i g_1$, $h_2 = d_i g_2$. Next, we want to write $g_2 - g_1$ as a sum $f' + f''$, where f', f'' are both relatively prime to n/d_i.

Let $n/d_i = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where $p_1, \ldots, p_m$ are distinct prime divisors of n/d_i. As n/d_i is odd, we have $p_k > 2$ for all $k = 1, \ldots, m$. Now, the requirement that f', f'' be both relatively prime to n/d_i can be written as a system of congruences

$$f' \equiv q_k \pmod{p_k}, \qquad \text{for } k = 1, \ldots, m, \tag{2}$$

such that $q_k \not\equiv 0, g_2 - g_1 \pmod{p_k}$. The assumption $p_k > 2$ guarantees that we can choose numbers q_k such that $q_k \not\equiv 0, g_2 - g_1 \pmod{p_k}$. The existence of a solution to (2) is then guaranteed by the Chinese Remainder Theorem [5].

Therefore, both $j + h_1$ and $j + h_2$ are adjacent to $j + d_i f'$ in $j + H_n(d_i)$, and, thus, the diameter of $j + H_n(d_i)$ is at most two.

Now we are ready to prove the upper bound on the diameter of $G(n, S)$. Let u and v be two arbitrary vertices of $G(n, S)$. As the set $\{d_1, d_2, \ldots, d_t\}$ generates $\mathbb{Z}_n$, there exist coefficients $\alpha_1, \alpha_2, \ldots, \alpha_t \in \mathbb{Z}_n$ such that

$$v - u \equiv \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_t d_t \pmod{n}. \tag{3}$$

A walk W from u to v may then be constructed from the following pieces:

- a path from u to $u + \alpha_1 d_1$ within $u + H_n(d_1)$,
- a path from $u + \alpha_1 d_1$ to $u + \alpha_1 d_1 + \alpha_2 d_2$ within $(u + \alpha_1 d_1) + H_n(d_2)$,
- ...
- a path from $u + \sum_{i=1}^{t-1} \alpha_i d_i$ to $u + \sum_{i=1}^{t} \alpha_i d_i = v$ within $(u + \sum_{i=1}^{t-1} \alpha_i d_i) + H_n(d_t)$.

As the length of each piece of W is at most two, we conclude that the distance between u and v is at most $2t$. □

3 Concluding remarks

Integral circulant graphs turn out to have a very small diatemer compared to the number of vertices. Thus, even when they do allow perfect state transfer in quantum spin networks, the communication distance they provide will be considerably lower than the one established by a hypercube on 3^m vertices [3].

This shows that possible candidate graphs for quantum spin networks with a perfect state transfer should better be searched for among integral graphs with a large diameter. However, all known classes of integral graphs [1] have a diameter that is either fixed or of order $\log n$. This poses the following

Open Problem *What is the order of the largest diameter of an integral graph with a given number of vertices or edges?*

As a possible solution, we briefly mention a class of integral graphs that may yield the correct answer to the above problem. A *superpath* SP_m is a graph whose vertices are divided into columns $C_1, C_2, \ldots, C_{2m}$, whose sizes are

$$\begin{aligned} \#C_{2i-1} &= n-i+1, \qquad i=1,2,\ldots,m, \\ \#C_{2i} &= i, \qquad i=1,2,\ldots,m. \end{aligned}$$

Then for each $j=1,2,\ldots,2m-1$, each two vertices from the neighboring columns C_j and C_{j+1} are joined by an edge. The superpath SP_m has $n=m^2+m$ vertices and the diameter $2m-1$. It can be shown that the spectrum of SP_m consists of simple eigenvalues $\pm m$, $\pm(m-1)$, $\ldots$, ± 2, ± 1, and the eigenvalue 0 of multiplicity m^2-m (details will be given elsewhere).

Thus, superpaths are integral graphs whose diameter is of order $O(\sqrt{n})$. We do not know whether a higher order for the diameter of integral graphs can be achieved.

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