Orthogonal (g, f)-Factorizations in Graphs *†

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) \le f(x)$ for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(F)$. A (g, f)-factorization of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors. Let $F = \{F_1, F_2, \cdots, F_m\}$ be a factorization of G and H be a subgraph of G with m edges. If F_i, $1 \le i \le m$, has exactly one edge in common with H, we say that F is orthogonal to H. In this paper it is proved that every $(mg+k-1, mf-k+1)$-graph contains a subgraph R such that R has a (g, f)-factorization orthogonal to any given subgraph with k edges of G if $f(x) > g(x) \ge 0$ for each $x \in V(G)$ and $1 \le k \le m$, where m and k are two positive integers.

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1 Introduction

The factors, factorizations and orthogonal factorizations in graphs are very useful in combinatorial design, circuit layout, optimization and network

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design and so on [1]. The file transfer problem can be modeled as $(0, f)$-factorizations (or f-colorings) of a graph [2]. The designs of Latin squares and Room squares are related to orthogonal factorizations in graphs which were firstly presented by Alspach et al. [1].

The graphs considered in this paper will be finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x \in V(G)$, we denote the degree of x in G by $d_G(x)$. Let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) \le f(x)$ for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G satisfying that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has m edges in total. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. Let H be an m-subgraph of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of G is orthogonal to H if $|E(H) \cap E(F_i)| = 1$ for $1 \le i \le m$. Undefined notations and definitions in this paper can be found in [3].

Recently, many authors studied the factors [4-12], factorizations [13]. The interested readers may find many relevant results about factors and factorizations in [1,14]. Alspach et al. [1] presented the following problem: Given a subgraph H of G, does there exist a factorization F of G with some properties orthogonal to H? Liu proved that every $(mg+m-1, mf-m+1)$-graph has a (g, f)-factorization orthogonal to a star or a matching [15,16]. Li and Liu showed that every $(mg + m - 1, mf - m + 1)$-graph has a (g, f)-factorization orthogonal to any given subgraph with m edges [17]. Feng and Liu studied the orthogonal $(0, f)$-factorizations [18,19]. Zhou also studied the orthogonal $(0, f)$-factorizations [20,21]. Li, Chen and Yu showed that every $(mg+k, mf-k)$-graph contains a subgraph R such that R has a (g, f)-factorization orthogonal to a given subgraph with k edges. The purpose of this paper is to prove that for any k-subgraph H of an $(mg + k - 1, mf - k + 1)$-graph G, there exists a subgraph R of G which has a (g, f)-factorization orthogonal to H, where m and k are two positive integers with $1 \le k \le m$ and $f(x) > g(x) \ge 0$ for each $x \in V(G)$. Our result is an improvement of the results in [15,16,17,22].

2 Preliminary results

Let G be a graph and $S \subseteq V(G)$. For any function f defined on $V(G)$, we put $f(S) = \sum_{x \in S} f(x)$ and $f(\emptyset) = 0$. For $S \subseteq V(G)$ and $A \subset E(G)$, we denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident with vertices in S, and by $G - A$ the subgraph obtained from G by deleting the edges in A, and by $G[S]$ (rep. $G[A]$) the subgraph of G induced by S (rep. A). Let S and T be two disjoint subsets of $V(G)$. We write $E_G(S,T) = \{xy : xy \in E(G), x \in S \text{ and } y \in T\}$ and $e_G(S,T) = |E_G(S,T)|$. Let g and f be two integer-valued functions defined on $V(G)$. If C is a component of $G-(S\cup T)$ such that $g(x) = f(x)$ for each $x \in V(C)$, then we say that C is odd or even according to $e_G(T, V(C)) + f(V(C))$ being odd or even, respectively. We denote by $h_G(S,T)$ the number of the odd components of $G-(S\cup T)$. Set

$$\delta_G(S,T) = d_{G-S}(T) - g(T) - h_G(S,T) + f(S).$$

Note that when $f(x) \neq g(x)$ for each $x \in V(G)$, $h_G(S,T) = 0$.

In [16] Guizhen Liu got a necessary and sufficient condition for a graph to have a (g, f)-factor containing a given edge.

Lemma 2.1 [16] *Let G be a graph and $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $0 \leq g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor containing any given edge e of G if and only if*

$$\delta_G(S,T) \geq f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S,T)$$

for any two disjoint subsets S and T of $V(G)$, where $\varepsilon(S,T)$ is defined as follows:
(1) $\varepsilon(S,T) = 2$, if $e = uv$, $u, v \in S$;
(2) $\varepsilon(S,T) = 1$, if there exists a neutral component C of $G-(S\cup T)$ such that $e \in E_G(S, V(C))$;
(3) $\varepsilon(S,T) = 0$, otherwise.

The following result was proved by Guojun Li et al. [22].

Lemma 2.2 [22] *Let G be an $(mg+k, mf-k)$-graph, and H a k-subgraph of G, where $1 \leq k < m$ and $f(x) > g(x) \geq 0$ for each $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g, f)-factorization $F = \{F_1, F_2, \cdots, F_k\}$ orthogonal to H, and $G - F_1 - F_2 - \cdots - F_k$ is an $((m-k)g, (m-k)f)$-graph.*

3 The proofs of Main results

Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \leq g(x) < f(x)$ for each $x \in V(G)$. In order to prove the main theorem, we first prove the following lemma which plays a crucial role in the proof of our theorem.

Lemma 3.1 *Let G be an (mg, mf)-graph with $m \geq 1$ and $m \neq 2$, where $0 \leq g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor containing any given edge e of G.*

Proof. Obviously, the result holds for $m = 1$. In the following we may assume $m \geq 3$. According to Lemma 2.1 it suffices to show that for any two disjoint subsets S and T, we have

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S,T).$$

Claim 1. $\delta_G(S,T) \geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S)$.
Proof. Since G is an (mg, mf)-graph, we have

$$\begin{aligned}
\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\
&= f(S) + d_G(T) - e_G(S,T) - g(T) \\
&= \frac{1}{m} d_G(T) - g(T) + f(S) - \frac{1}{m} d_G(S) \\
&\quad + \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\
&\geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S).
\end{aligned}$$

Now, we divide this proof into three cases.
Case 1. If $e = uv$, $u, v \in S$, then $\varepsilon(S,T) = 2$.

Clearly, $d_{G-T}(S) \geq 2$. In the following we prove $\delta_G(S,T) \geq \varepsilon(S,T)$.
Case 1.1. $d_{G-S}(T) \neq 0$.

In view of Claim 1 and $d_{G-T}(S) \geq 2$, we obtain

$$\begin{aligned}
\delta_G(S,T) &\geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\
&\geq \frac{m-1}{m} + \frac{2}{m} = 1 + \frac{1}{m} > 1.
\end{aligned}$$

By the integrity of $\delta_G(S,T)$, we get

$$\delta_G(S,T) \geq 2 = \varepsilon(S,T).$$

Case 1.2. $d_{G-S}(T) = 0$.

If $g(x) = 0$ for each $x \in T$, then $\delta_G(S,T) = f(S) \geq |S| \geq 2 = \varepsilon(S,T)$. In the following we may assume there exists $x_1 \in T$ such that $g(x_1) > 0$. Since G is a simple graph, then we have

$$|S| \geq mg(T).$$

Thus, we obtain

$$f(S) \geq |S| \geq mg(T).$$

Hence, we have

$$\begin{aligned}\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\geq mg(T) - g(T) = (m-1)g(T) \\ &\geq m-1 \geq 2 = \varepsilon(S,T).\end{aligned}$$

Case 2. If there exists a neutral component C of $G-(S\cup T)$ such that $e \in E_G(S, V(C))$, then $\varepsilon(S,T) = 1$.

Obviously, $d_{G-T}(S) = |E_G(S, V(G)\setminus T)| \geq |E_G(S, V(G)\setminus (S\cup T))| \geq |E_G(S, V(C))| \geq 1$. In view of Claim 1, we obtain

$$\begin{aligned}\delta_G(S,T) &\geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \\ &\geq \frac{1}{m} d_{G-T}(S) \geq \frac{1}{m} > 0.\end{aligned}$$

According to the integrity of $\delta_G(S,T)$, we have

$$\delta_G(S,T) \geq 1 = \varepsilon(S,T).$$

Case 3. If neither case 1 nor case 2 holds, then $\varepsilon(S,T) = 0$.

According to Claim 1, we get that

$$\delta_G(S,T) \geq \frac{m-1}{m} d_{G-S}(T) + \frac{1}{m} d_{G-T}(S) \geq 0 = \varepsilon(S,T).$$

The proof is completed.

Now we are in a position to prove the main theorem.

Theorem 1 *Let G be an $(mg+k-1, mf-k+1)$-graph, and H a k-subgraph of G, where $1 \leq k \leq m$ and $m-k \neq 1$ and $f(x) > g(x) \geq 0$ for each $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g,f)-factorization orthogonal to H.*

Proof. In view of Lemma 3.1, the theorem holds for $k=1$. In the following we may assume $k \geq 2$. For any edge e of H, set $H' = H - e$. Then H' is a $(k-1)$-subgraph of G. According to Lemma 2.2, there exists a subgraph R' of G such that R' has a (g,f)-factorization $F' = \{F_1, F_2, \cdots, F_{k-1}\}$ orthogonal to H' and $G - F_1 - F_2 - \cdots - F_{k-1}$ is an $((m-k+1)g, (m-k+1)f)$-graph. Since $m-k \neq 1$, then $m-k+1 \neq 2$. By Lemma

3.1, $G - F_1 - F_2 - \cdots - F_{k-1}$ has a (g, f)-factor F_k containing e. Put $R = R' \cup F_k$. Clearly, R is a subgraph of G and R has a (g, f)-factorization $F = \{F_1, F_2, \cdots, F_{k-1}, F_k\}$ orthogonal to H.

Completing the proof.

In Theorem 1, if $k = m$, then we get the following corollary.

Corollary 1 [23] *Let G be an $(mg+m-1, mf-m+1)$-graph, and let g and f be two integer-valued functions defined on $V(G)$ such that $0 \le g(x) < f(x)$. If H is an m-subgraph of G, then G has a (g, f)-factorization orthogonal to H.*

By Theorem 1, the following result holds.

Corollary 2 [22] *Let G be an $(mg+k, mf-k)$-graph, and H a k-subgraph of G, where $1 \le k < m$ and $f(x) > g(x) \ge 0$ for each $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.*

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