New results on sum graph theory

Haiying Wang[1]* Chuantao Li[2,3]†
1. The School of Science,
China University of Geosciences(Beijing), Beijing 100083, P.R.China
2. Shandong Institute of Physical Education and Sports,
Jinan, Shandong, 250014, P.R.China
3. School of Geophysics and Information Technology,
China University of Geosciences(Beijing),Beijing 100083,P.R.China

Abstract The concept of the sum graph and integral sum graph were introduced by F.Harary. In this paper, we gain some upper and lower bounds on the sum number and the integral sum number of a graph and these bounds are sharp, and some new properties on the integral sum graph. Using these results, we could directly investigate and determine the exclusive integral sum numbers, the exclusive sum numbers, the sum numbers and the integral sum numbers of the graphs $K_n\backslash E(2P_3)$, $K_n\backslash E(P_3)$ and any graph H with minimum degree $\delta(H) = n-2$ respectively as n is more than a given number. Then they will be the beginning of a new thought of research on the (exclusive) sum graph and the (exclusive) integral sum graph.
MSC: 05C78
Keywords Sum Graph; Integral Sum Graph; Exclusive Sum Graph; Exclusive Integral Sum Graph.

Section 1. Introduction

For a simple graph $H = (V(H), E(H))$, let $V(H)$ denote its vertex set and $E(H)$ its edge set. If a vertex of a graph is adjacent to every other of the graph then it is called a saturated vertex. All other notation and terminology are referred to [1].

The concept of the sum graph and the integral sum graph were introduced by F.Harary ([2][3]). Let N denote the set of all positive integers. The sum graph $G^+(S)$ of a finite subset $S \subset N$ is the graph $(S, E(G))$ with $uv \in E(G)$ if and only if $u + v \in S$.

A simple graph G is said to be a sum graph if it is isomorphic to the sum graph of some $S \subset N$. We say that S gives a sum labeling for G.

*E-mail: whycht@126.com.
†This research is supported by the Fundamental Research Funds for the Central Universities

In a labeling of a graph, vertices whose label corresponds to an edge are said to be working vertices. It has been realized that certain graphs can only be labeled in such way that all the working vertices are also isolated; such graphs are called exclusive.

For a simple graph $H = (V(H), E(H))$, the sum number $\sigma(H)$ of a graph H is the smallest number of isolated vertices which when added to H result in a sum graph. The exclusive sum number, $\varepsilon(H)$, of the graph H is the smallest integer r such that $H \cup \overline{K_r}$ has an exclusive sum labeling.

The integral sum graph, the exclusive integral sum graph, the integral sum number $\zeta(H)$ and the exclusive integral sum number $\epsilon(H)$ of a connected graph H are also defined when S is extended from the positive integers set N to the integer set Z. It is obvious that $\varepsilon(H) \geq \sigma(H) \geq \zeta(H)$ and $\varepsilon(H) \geq \epsilon(H) \geq \zeta(H)$ for a connected graph H.

Exclusive (integral) sum graphs are of interest for two reasons: they may be easier to label optimally, and they may be more likely to have a large sum number. It turns out that K_n and W_n are exclusive, while F_n and $K_{m,n}$ are not. Some useful results have been gained ([2-14]), but the characterization of (exclusive) integral sum graphs remains an open problem.

In this paper, we gain some upper and lower bounds on the sum number, the exclusive sum number, and the integral sum number, the exclusive integral sum number of a graph and these bounds are sharp, and some new results on the (exclusive) integral sum graph. Using these results, we could directly investigate and determine the exclusive integral sum numbers, the exclusive sum numbers, the sum numbers and the integral sum numbers of the graphes $K_n \backslash E(2P_3)$, $K_n \backslash E(P_3)$ and the connected graph H with minimum degree $\delta(H) = n - 2$ respectively as n is more than a given number. Then they will be the beginning of a new thought of research on the (exclusive) sum graph and the (exclusive) integral sum graph.

To simplify notations, we may assume that the vertices of a (exclusive) sum graph and an (exclusive) integral sum graph are identified with their labeling throughout this paper.

Section 2. New lower bounds on the (integral) sum number

In this section, we will give new lower bounds on the (integral) sum number of a connected graph H. Let an integral sum graph $G = H \cup \overline{K_r}$, which implies that some isolated vertices are added to a connected graph H and results in an integral sum graph G. Let S is the sum labeling set or the integral sum labeling of G and C is the set of its isolated vertices.

Theorem 2.1 Let $H = (V(H), E(H))$ be a simple graph and $\delta(H) > \frac{n+1}{2}$, where n be the vertices number of H and $n \geq 6$. Then H is not an integral sum graph.

Proof: We argue by contradiction. Assume that H is an integral sum graph. Then $u + u' \in V(H)$ for any edge $uu' \in E(H)$. Let $V(H) = \{u_1, u_2, \cdots, u_n\}$, where $u_1 < u_2 < \cdots < u_n$. Then we obtain at least $(2n-3)$ numbers $u_1 + u_2 < u_1 + u_3 < \cdots < u_1 + u_n < u_2 + u_n < u_3 + u_n < \cdots < u_{n-1} + u_n$. Let $U = \{u_1 + u_2, u_1 + u_3, \cdots, u_1 + u_n, u_2 + u_n, u_3 + u_n, \cdots, u_{n-1} + u_n\}$.

Since two vertices u_1 and u_n have $(n-1)-d_H(u_1)$ and $(n-1)-d_H(u_n)$ non-adjacent vertices respectively, in the set U there are at least $(2n-3)-[(n-1)-d_H(u_1)]-[(n-1)-d_H(u_n)]$ numbers which must be edge sums of the graph H. Since H is an integral sum graph, $(2n-3)-[(n-1)-d_H(u_1)]-[(n-1)-d_H(u_n)] \leq n$. that is, $d_H(u_1)+d_H(u_n) \leq n+1$. Combined $d_H(u_1) \geq \delta(H)$, $d_H(u_n) \geq \delta(H)$ and $\delta(H) > \frac{n+1}{2}$, we have $2\delta(H) \leq d_H(u_1) + d_H(u_n) \leq n+1$, which implies $\delta(H) \leq \frac{n+1}{2}$, but $\delta(H) > \frac{n+1}{2}$, a contradiction.

Thus, Theorem 2.1 holds. □

Theorem 2.2 Let H be a connected graph with the vertices number n and the minimum degree $\delta(H)$. If each vertex of H is non-working, then the sum number $\sigma(H)$, the integral sum number $\zeta(H)$, the exclusive sum number $\varepsilon(H)$, the exclusive integral sum number $\epsilon(H)$ of the graph H satisfy $\varepsilon(H) \geq \sigma(H) \geq \zeta(H) \geq 2\delta(H) - 1$ and $\epsilon(H) \geq \zeta(H) \geq 2\delta(H) - 1$.

Proof: Let $n \geq 3$ and $H = (V(H), E(H))$. Without loss of generality, we may assume that $V(H) = \{x_1, x_2, x_3, ..., x_n\}$, where $x_1 < x_2 < ... < x_n$ and $x_n > 0$ (otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

Since $x_1 + x_2 < x_1 + x_3 < ... < x_1 + x_n < x_2 + x_n < ... < x_{n-1} + x_n$, these $2n-3$ numbers are distinct. Since two vertices x_1 and x_n have $(n-1)-d_{(H)}(x_1)$ and $(n-1)-d_H(x_n)$ non-adjacent vertices respectively, in the set U there are at least $(2n-3)-[(n-1)-d_H(x_1)]-[(n-1)-d_H(x_n)]$, that is, $d_H(x_1)+d_H(x_n)-1$ numbers which must be edge sums of the graph H. Since $d_H(x_1) \geq \delta(H)$ and $d_H(x_n) \geq \delta(H)$, $d_H(x_1) + d_H(x_n) - 1 \geq 2\delta(H) - 1$.

On the other hand, since each vertex of H is non-working, $x_i + x_j \in C$ for all edges $x_i x_j \in E(H)$. Then $\varepsilon(H) \geq \sigma(H) \geq \zeta(H) \geq 2\delta(H) - 1$ and $\epsilon(H) \geq \zeta(H) \geq 2\delta(H) - 1$.

Thus, Theorem 2.4 holds. □

Corollary 2.3 Let H be a simple graph with vertices number $n \geq 4$. If $\delta(H) = n-1$, then $\varepsilon(H) = \epsilon(H) = \sigma(H) = \zeta(H) = 2\delta(H) - 1 = 2n - 3$, that is, the lower bound of Theorem 2.2 is best possible.

Proof: If $\delta(H) = n-1$, then H is the complete graph, denoted K_n, which each vertex's degree is $n-1$. Reference [4] has proved that each vertex of K_n is non-working and $\sigma(H) = \zeta(H) = \delta(H) - 1 = 2n - 3$ for any $n \geq 4$. Combined Theorem 2.2, Corollary 2.3 holds. □

Up to now, we automatically think about the following questions.

Question 1. Let H be a simple graph with the vertices number n and the minimum degree $\delta(H)$. The lower bound on $\varepsilon(H)$, $\epsilon(H)$, $\zeta(H)$ and $\sigma(H)$ in Theorem 2.2 is best possible for all $1 \leq \delta(H) \leq n-2$?

To be surprise, the numbers $\varepsilon(H)$, $\epsilon(H)$, $\zeta(H)$ and $\sigma(H)$ of almost of the graphs which have been investigated and determined by some researchers are best possible of the bounds in Theorem 2.2 as n is more than a given number. Then we are considering the following question and try to answer it.

Question 2. Which graphs are exclusive?

First we will consider some graphs H with higher degree. such that $\Delta(H) = n-1$, that is, H has at least one saturated vertex.

Section 3. Properties on the integral sum graph

In this section, give some Properties on the integral sum graph G. Let $G = H \cup \overline{K_r}$ be an integral sum graph and shows that the least isolated vertices are added to a connected graph H and results in an integral sum graph G. Let S is the sum labeling set or the integral sum labeling of G and C is the set of its isolated vertices.

Theorem 3.1 Let $H = (V(H), E(H))$ be a connected graph with n vertices and one saturated vertex v_s. If there exists $u \in V(H)$ such that $u + v_s \in C$ and $d_H(u) \geq 3$, then $u + u_i \in C$ for any edge $uu_i \in E(H)$.

Proof: Let $H = (V(H), E(H))$ be a connected graph with n vertices and one saturated vertex v_s. For any edge $uu_i \in E(H)$, since $u + v_s \in C$ and $d_G(v_s) = n-1$, $(u+u_i)+v_s = (u+v_s)+u_i \notin S$, which implies $u+u_i \in \{v_s\} \cup C$. Then there is at most one edge adjacent to the vertex u such that its sum equals v_s and the others belong to the isolated set C. Since $d_H(u) \geq 3$, there exists at least two distinct vertices in the subset $N(u) \backslash \{v_s\}$, denoted u_{i_1}, u_{i_2}.

If there is one edge adjacent to the vertex u such that its sum is v_s. Then there is the unique edge and denote it uu_{i_1}. Then $u + u_{i_1} = v_s$ and $u + u_{i_2} \in C$. Since $u+u_{i_2} \in C$, $v_s+u_{i_2} = (u+u_{i_1})+u_{i_2} = (u+u_{i_2})+u_{i_1} \notin S$, but $v_s+u_{i_2} \in S$, a contradiction. Thus, $u + u_i \in C$ for any $uu_i \in E(H)$. □

Theorem 3.2 Let $H = (V(H), E(H))$ be a connected graph with n vertices and $n \geq 4$. If H has two distinct saturated vertices such that their sums belong to the isolated set, then all of the edges sums adjacent to them also belong to the isolated set.

Proof: Let v_{s_1}, v_{s_2} be two distinct saturated vertices of H. Then v_{s_1}, v_{s_2} have edges with all other vertices of H. For any edge $v_i v_{s_1} \in E(H)$, since $v_{s_1} + v_{s_2} \in C$, $(v_i + v_{s_1}) + v_{s_2} = v_i + (v_{s_1} + v_{s_2}) \notin S$, which implies $v_i + v_{s_1} \in \{v_{s_2}\} \cup C$. Thus, there is at most one edge adjacent v_{s_1} such that its sum is v_{s_2} and others belong to the set of isolated vertices C. Since $d(v_{s_1}) = n - 1 \geq 3$, there exists at least two distinct vertices in the set $N(s_1) \backslash \{v_{s_2}\}$, denoted v_{i_1}, v_{i_2}.

If there is one edge adjacent to the vertex v_{s_1} such that its sum is v_{s_2}. Then there is the unique edge and denote it $v_{i_1} v_{s_1}$. So $v_{i_1} + v_{s_1} = v_{s_2}$ and $v_{i_2} + v_{s_1} \in C$. Then $v_{s_2} + v_{i_2} = (v_{i_1} + v_{s_1}) + v_{i_2} = (v_{i_2} + v_{s_1}) + v_{s_1} \notin S$, but $v_{s_2} + v_{i_2} \in S$, a contradiction.

Thus, $v_i + v_{s_1} \in C$ for $v_i v_{s_1} \in E(H)$. Similarly to v_{s_2}. □

Theorem 3.3 Let $H = (V(H), E(H))$ be not an integral sum graph with one saturated vertex v_s and n vertices. Then there exists one edge adjacent to v_s such that its edge sum belongs to the isolated set.

Proof: Let H be not an integral sum graph with one saturated vertex v_s and n vertices. Then $0 \notin S$ and $d(v_s) = n - 1$.

Let $N(v_s) = \{v_1, v_2, \cdots, v_{n-1}\}$, where $v_1 < v_2 < \cdots < v_{n-1}$. Then $V(H) = \{v_1, v_2, \cdots, v_{n-1}, v_s\}$ and $v_s+v_1, v_s+v_2, \cdots, v_s+v_1 \in S$ with $v_s+v_1 < v_s+v_2 < \cdots < v_s+v_{n-1}$. If $v_s+v_i \in V(H)$, then $v_s+v_i = v_i$ for any $i \in \{1, 2, \cdots, n-1\}$, which implies $v_s = 0$, a contradiction.

Thus, there exists one edge adjacent to v_s such that its edge sum belongs to the isolated set. □

Theorem 3.4 Let H be not an integral sum graph with one saturated vertex v_s. If $\delta(H) \geq 3$, then any edge sum belongs to the isolated set.

Proof: Let u be any vertex of H except for the saturated vertex v_s. Since v_s is one saturated vertex of G, there is an edge between u and v_s. By Theorem 3.3 and 3.1, $u + u_i \in C$ for any edge $uu_i \in E$. □

Section 4. The (exclusive) sum number and the (exclusive) integral sum number of the graph $K_n \backslash E(2P_3)$

In this section, consider the connected graph $H = K_n \backslash E(2P_3)$. Let $V(K_n \backslash E(2P_3)) = \{v_1, v_2, \cdots, v_n\}$ and the two deleted 3−paths $2P_3$ are denoted $v_1v_2v_3$, and $v_4v_5v_6$ respectively, n is the vertices number of $K_n \backslash E(2P_3)$ and then $d_G(v_2) = d_G(v_5) = n-3$, $d_G(v_1) = d(v_3) = d_G(v_4) = d_G(v_6) = n-2$ and $d(v_i) = n-1$ for any $i \in \{7, 8, \cdots, n\}$ as n is more than a given number ($n \geq 8$).

Let $G = [K_n \backslash E(2P_3)] \cup \overline{K_r}$ be an integral sum graph, which shows that the least isolated vertices are added to $K_n \backslash E(2P_3)$ and results in an integral sum graph G and S is an integral sum labeling of G and C is the set of isolated vertices.

Lemma 4.1 $K_n \backslash E(2P_3)$ is not an integral sum graph for $n \geq 8$.

Proof: Since $\delta(K_n \backslash E(2P_3)) = n-3$ and $n \geq 8$, $\delta(K_n \backslash E(2P_3) > \frac{n+1}{2}$. By Theorem 2.1, $K_n \backslash E(2P_3)$ is not an integral sum graph for $n \geq 8$. □

Lemma 4.2. G is exclusive.

Proof: Let S is the sum labeling set or the integral sum labeling of $G = [K_n \backslash E(2P_3)] \cup \overline{K_r}$ and C is the set of its isolated vertices. Since $\delta(G) \geq 3$, by Theorem 3.4, $u + v \in C$ for any edge $uv \in E(G)$.

Lemma 4.3. $\varepsilon(K_n \backslash E(2P_3)) \geq \sigma(K_n \backslash E(2P_3)) \geq \zeta(K_n \backslash E(2P_3)) \geq 2n-7$ and $\varepsilon(K_n \backslash E(2P_3)) \geq \epsilon(K_n \backslash E(2P_3)) \geq 2n-7$ for $n \geq 8$.

Proof: By Theorem 2.2, Lemma 4.3 holds. □

Lemma 4.4. $\epsilon(K_n \backslash E(2P_3)) \leq \varepsilon(K_n \backslash E(2P_3)) \leq 2n-7$ for $n \geq 8$.

Proof: Let $K_n \backslash E(2P_3) = (V, E)$ and $V = \{x_1, x_2, x_3, \cdots, x_n\}$ and $S = V \cup C$, where C is the set of its isolated vertices and $n \geq 8$.

First, let $x_i = (i-1) \times 10 + 1$ and $c_j = j \times 10 + 2$. Then $V = \{(i-1) \times 10 + 1 : i = 1, 2, ..., n\}$ and the isolated set $C = \{c_j : j = 1, 2, ..., 2n-3\} - \{c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}\}$. This key is to find $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$ and two deleted 3-paths $v_1v_2v_3$, and $v_4v_5v_6$.

Second, let us look for them and verify that this is an exclusive sum labeling in detail. Let $\{c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}\} = \{c_1, c_2, c_{2n-4}, c_{2n-3}\}$. Then the isolated set $C = \{c_j : j = 1, 2, ..., 2n-3\} - \{c_1, c_2, c_{2n-4}, c_{2n-3}\}$ and $E(2P_3) = \{x_1x_2, x_1x_3; x_nx_{n-1}, x_nx_{n-2}\}$. In fact, we have

(1) The vertices in S are distinct.

(2) For any vertices $x_i \in \{x_1, x_2, x_3, \cdots, x_n\}$ and $c_k \in C$, since $x_i + c_k \equiv 3(\text{mod}10)$, $x_i + c_k \notin S$.

(3) For any distinct vertices $c_i, c_k \in C$, since $c_i + c_k \equiv 4(\text{mod}10)$, $c_i + c_k \notin S$.

(4) Let $1 \leq i \neq j \leq n$. For any distinct vertices $x_i, x_j \in \{x_1, x_2, x_3, \cdots, x_n\}$, $x_i + x_j = (i+j-2) \times 10 + 2 = c_{i+j-2}$.

Since $x_i + x_j = c_1 \Longleftrightarrow i+j-2 = 1 \Longleftrightarrow i+j = 3 \Longleftrightarrow (i,j) = (1,2)$.

Since $x_i + x_j = c_2 \Longleftrightarrow i+j-2 = 2 \Longleftrightarrow i+j = 4 \Longleftrightarrow (i,j) = (1,3)$.

Since $x_i + x_j = c_{2n-4} \iff i + j - 2 = 2n - 4 \iff i + j = 2n - 2 \iff (i, j) = (n - 2, n)$.

Since $x_i + x_j = c_{2n-3} \iff i + j - 2 = 2n - 3 \iff i + j = 2n - 4 \iff (i, j) = (n - 1, n)$.

Then it is an exclusive sum labeling of $(K_n \backslash E(2P_3)) \cup (2n-7)K_1$, where the two deleted 3−paths $2P_3$ are denoted $x_2x_1x_3$, and $x_{n-2}x_nx_{n-1}$ respectively. □

Theorem 4.1. $\zeta(K_n \backslash E(2P_3)) = \sigma(K_n \backslash E(2P_3)) = \epsilon(K_n \backslash E(2P_3)) = \varepsilon(K_n \backslash E(2P_3)) = 2n - 7$ for $n \geq 8$.

Proof: By Lemma 4.3 and 4.4, Theorem 4.1 holds. □

Section 5. The (exclusive) sum number and the (exclusive) integral sum number of the graph $K_n \backslash E(P_3)$

In this section, consider the connected graph $H = K_n \backslash E(P_3)$. Let $V(K_n \backslash E(P_3)) = \{v_1, v_2, \cdots, v_n\}$, the deleted 3−path P_3 is denoted $v_1v_2v_3$, n is the vertices number of the graph $K_n \backslash E(P_3)$. Then $d(v_2) = n - 3$, $d(v_1) = d(v_3) = n - 2$ and $d(v_i) = n - 1$ for any $i \in \{4, 5, \cdots, n\}$ as n is more than a given number ($n \geq 8$).

Let $G = [K_n \backslash E(P_3)] \cup \overline{K_r}$ be an integral sum graph, which shows that the least r isolated vertices are added to $K_n \backslash E(P_3)$ with n vertices number and results in an integral sum graph G and S is an integral sum labeling of G and C is the set of isolated vertices.

Lemma 5.1 $K_n \backslash E(P_3)$ is not an integral sum graph for $n \geq 8$.

Proof: Since $\delta(K_n \backslash E(P_3)) = n - 3$ and $n \geq 8$, $\delta(K_n \backslash E(P_3) > \frac{n+1}{2}$. By Theorem 2.1, $K_n \backslash E(P_3)$ is not an integral sum graph for $n \geq 8$. □

Lemma 5.2 The integral sum graph G is exclusive.

Proof Let S is the sum labeling set or the integral sum labeling of $G = [K_n \backslash E(P_3)] \cup \overline{K_r}$ and C is the set of its isolated vertices. Since $\delta(K_n \backslash E(P_3)) \geq 3$, by Theorem 3.4, $u + v \in C$ for any edge $uv \in E(G)$. □

Lemma 5.3 $\varepsilon(K_n \backslash E(P_3)) \geq \sigma(K_n \backslash E(P_3)) \geq \zeta(K_n \backslash E(P_3)) \geq 2n - 5$ and $\varepsilon(K_n \backslash E(P_3)) \geq \epsilon(K_n \backslash E(P_3)) \geq 2n - 5$ for $n \geq 8$.

Proof Without loss of generality, we may assume that $V(K_n \backslash E(P_3)) = \{x_1, x_2, x_3, ..., x_n\}$, where $x_1 < x_2 < ... < x_n$ and $x_n > 0$ (otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

Since $x_1 + x_2 < x_1 + x_3 < ... < x_1 + x_n < x_2 + x_n < ... < x_{n-1} + x_n$, these $2n - 3$ numbers are distinct. Since two vertices x_1 and x_n have $(n - 1) - d(x_1)$ and $(n - 1) - d(x_n)$ non-adjacent vertices respectively, in the set U there are at least $\min\{(2n - 3) - [(n - 1) - d(x_1)]], (2n - 3) - [(n - 1) - d_H(x_n)]\}$ numbers which must be edge sums of the graph $K_n \backslash E(P_3)$.

Since $\delta(K_n \backslash E(P_3)) = n - 3$, $d(x_1) \geq n - 3$ and $d(x_n) \geq n - 3$. Then $\min\{(2n - 3) - [(n - 1) - d(x_1)]], (2n - 3) - [(n - 1) - d_H(x_n)]\} = 2n - 5$.

Since each vertex of $K_n \backslash E(P_3)$ is non-working, $x_i + x_j \in C$ for all edges $x_ix_j \in E(K_n \backslash E(P_3))$. Then $\varepsilon(K_n \backslash E(P_3)) \geq \sigma(K_n \backslash E(P_3)) \geq \zeta(K_n \backslash E(P_3)) \geq 2n - 5$ and $\varepsilon(K_n \backslash E(P_3)) \geq \epsilon(K_n \backslash E(P_3)) \geq 2n - 5$ for $n \geq 8$. □

Lemma 5.4 $\epsilon(K_n \backslash E(P_3)) \leq \varepsilon(K_n \backslash E(P_3)) \leq 2n - 5$ for $n \geq 8$.

Proof Let $K_n \backslash E(P_3) = (V, E)$ and $V = \{x_1, x_2, x_3, \cdots, x_n\}$ and $S = V \cup C$, where C is the set of isolated vertices and $n \geq 8$.

First, let $x_i = (i-1)\times 10+1$ and $c_j = j\times 10+2$. Then $V = \{(i-1)\times 10+1 : i = 1,2,...,n\}$ and the isolated set $C = \{c_j : j = 1,2,...,2n-3\} - \{c_{i_1}, c_{i_2}\}$. This key is to find c_{i_1}, c_{i_2} and the deleted 3-path $v_1v_2v_3$.

Second, let us look for them and verify that this is an exclusive sum labeling in detail. Let $\{c_{i_1}, c_{i_2}\} = \{c_1, c_2\}$. Then the isolated set $C=\{c_j : j = 1,2,...,2n-3\} - \{c_1, c_2\}$ and $E(P_3) = \{x_1x_2, x_2x_3\}$, which implies that the deleted path is $x_1x_2x_3$. In fact, we have

(1) The vertices in S are distinct.

(2) For any vertices $x_i \in \{x_1, x_2, x_3, \cdots, x_n\}$ and $c_k \in C$, since $x_i + c_k \equiv 3(\text{mod}10)$, $x_i + c_k \notin S$.

(3) For any distinct vertices $c_i, c_k \in C$, since $c_i+c_k \equiv 4(\text{mod}10)$, $c_i+c_k \notin S$.

(4) Let $1 \le i \neq j \le n$. For any distinct vertices $x_i, x_j \in \{x_1, x_2, x_3, \cdots, x_n\}$, $x_i + x_j = (i+j-2)\times 10 + 2 = c_{i+j-2}$.

Since $x_i + x_j = c_1 \Longleftrightarrow i+j-2 = 1 \Longleftrightarrow i+j = 3 \Longleftrightarrow (i,j) = (1,2)$.

Since $x_i + x_j = c_2 \Longleftrightarrow i+j-2 = 2 \Longleftrightarrow i+j = 4 \Longleftrightarrow (i,j) = (1,3)$.

Thus, it is an exclusive sum labeling of $(K_n\backslash E(P_3)) \cup (2n-5)K_1$, where the deleted 3−path P_3 is denoted $x_2x_1x_3$. □

Theorem 5.1 $\zeta(K_n\backslash E(P_3)) = \sigma(K_n\backslash E(P_3)) = \epsilon(K_n\backslash E(P_3))=\varepsilon(K_n\backslash E(P_3)) =2n-5$ for $n \ge 8$.

Proof By Lemma 5.3 and 5.4, Theorem 5.1 holds. □

Section 6. The (exclusive) sum number and the (exclusive) integral sum number of the connected graph with its minimum degree $\delta(G) = n-2$

In this section, we will consider the (exclusive) sum number and the (exclusive) integral sum number of the connected graph H with its minimum degree $\delta(H) = n-2$.

Let H be a simple graph with n vertices. If $\delta(H) = n-2$, then either $d_H(v) = n-2$ or $d_H(v) = n-1$ for any vertex v. Thus, if and only if H must be the connected graph, denoted $K_n\backslash E(rK_2)$, deleted r matchings from the complete graph K_n. Then if n is even then $1 \le r \le \frac{n}{2}$, otherwise, $1 \le r \le \frac{n-1}{2}$. Thus, $1 \le r \le \lceil\frac{n}{2}\rceil$.

Let $H = K_n\backslash E(rK_2)$, where the r deleted matchings $E\{rK_2\} = \{v_1v_1', v_2v_2', \cdots v_rv_r'\}$, n is the vertices number of H. Then either $d_H(v) = n-1$ or $d_H(v) = n-2$ for any vertex $v \in V(H)$. Besides, if r is odd, then $1 \le r \le \frac{n-1}{2}$; if r is even, then $1 \le r \le \frac{n}{2}$. Thus, $1 \le r \le \lceil\frac{n}{2}\rceil$.

Let $G = [K_n\backslash E(rK_2)] \cup \overline{K_r}$ be an integral sum graph, which shows that the least isolated vertices are added to $K_n\backslash E(rK_2)$ and results in an integral sum graph G and S is an integral sum labeling of G and C is the set of isolated vertices.

In this section, one special sign is used. For a vertex $v_i \in V(H)$, let v_i' denote the vertex such that $v_iv_i' \notin E(H)$, that is, v_iv_i' is one deleted matching and $v_iv_i' \in E(rK_2)$. According to the structure of the graph $H = K_n\backslash E(rK_2)$, if $d_H(v_i) = n-2$ then the vertex v_i' exists; if $d_H(v_i) = n-1$ then v_i' does not exist at all.

Lemma 6.1 For $n \geq 7$ and $r = 1$, $\sigma(K_n\backslash E(rK_2)) = \zeta(K_n\backslash E(rK_2)) = 2n - 4$.

Proof: For $n \geq 7$ and $r = 1$, $K_n\backslash E(rK_2) = K_n - e$. Reference [6][7] has determined that $\sigma(K_n - e) = \zeta(K_n - e) = 2n - 4$. □

Lemma 6.2 For $n \geq 7$ and $2 \leq r \leq \lceil\frac{n}{2}\rceil$, $K_n\backslash E(rK_2)$ is not an integral sum graph.

Proof: For $n \geq 7$ and $2 \leq r \leq \lceil\frac{n}{2}\rceil$, $\delta(K_n\backslash E(rK_2)) = n - 2 > \frac{n+1}{2}$. By Theorem 2.1, $K_n\backslash E(rK_2)$ is not an integral sum graph. □

Lemma 6.3 Let $n \geq 7$, $2 \leq r \leq \lceil\frac{n}{2}\rceil$ and an integral sum graph $G = [K_n\backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n\backslash E(rK_2) = (V, E)$. Assume $v_{\max}$ be a vertex which absolute value is maximum in V. Then there exists one edge adjacent to $v_{\max}$ such that their label sum belongs to the set of isolated vertices C.

Proof: Let $n \geq 7, 2 \leq r \leq \lceil\frac{n}{2}\rceil$ and an integral sum graph $G = [K_n\backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n\backslash E(rK_2) = (V, E)$. Assume $v_{\max}$ be a vertex which absolute value is maximum in the set V. Without loss of generality, we may assume $v_{\max} \in V$ and $v_{\max} > 0$ (Otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

We argue by contradiction. Suppose that $v + v_{\max} \in V$ for all edges $vv_{\max} \in E$. According to the choice of $v_{\max}$, we have $v < 0$ and $v + v_{\max} > 0$. Notice that $d_G(v_{\max}) \in \{n-2, n-1\}$. Then assume $v_1, v_2, \cdots, v_{n-3}, v_{n-2}$ are its distinct adjacent vertices, where $v_1 < v_2 < \cdots < v_{n-3} < v_{n-2} < 0$. Then $0 < v_1 + v_{\max} < v_2 + v_{\max} < \cdots < v_{n-3} + v_{\max} < v_{n-2} + v_{\max} < v_{\max}$ and they belong to the vertices set V.

Up to now, there are at most one vertex v_x, which signs may be positive. Since $0 \notin V$, $n - 2 \leq 1$, that is $n \leq 3$, but $n \geq 7$, which is a contradiction.

Thus, Lemma 6.3 holds. □

Lemma 6.4 Let $n \geq 7$, $2 \leq r \leq \lceil\frac{n}{2}\rceil$ and an integral sum graph $G = [K_n\backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n\backslash E(rK_2) = (V, E)$. Assume $v_{\max}$ be a vertex which absolute value is maximum in the set V. Then $v + v_{\max} \in C$ for all edges $vv_{\max} \in E$.

Proof: Let $n \geq 7, 2 \leq r \leq \lceil\frac{n}{2}\rceil$ and an integral sum graph $G = [K_n\backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n\backslash E(rK_2) = (V, E)$. Assume that $v_{\max}$ is the vertex which absolute value is maximum in the set V. Without loss of generality, we may assume that and $v_{\max} \in V$ and $v_{\max} > 0$ (Otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

According to the structure of the graph $K_n\backslash E(rK_2)$, $d_G(v_{\max}) \in \{n-2, n-1\}$. Since $n \geq 7$, there are at least 4 edges adjacent to the vertex $v_{\max}$. By Lemma 6.3, there exists one edge $v_{i_0}v_{\max} \in E$ such that $v_{i_0} + v_{\max} \in C$. Then for any edge $vv_{\max} \in E\backslash\{v_{i_0}v_{\max}\}$, $v_{i_0} + (v + v_{\max}) = (v_{i_0} + v_{\max}) + v \notin S$, which implies there is no edge between $(v + v_{\max})$ and v_{i_0}, that is, $v + v_{\max} \in (\{v_{i_0}\} \cup \{v'_{i_0}\}) \cup C$, denoted (1).

Thus, there are at most 2 such edges such that their sums belong to the vertices subset $\{v_{i_0}\} \cup \{v'_{i_0}\}$. Since $n \geq 7$, there must exist at least $(n-2) - 1 - 2 \geq 2$ edges adjacent to $v_{\max}$, denoted $v_{i_1}v_{\max}, v_{i_2}v_{\max} \in E$, such that their sums belong to the isolated set C, that is, $v_{i_1} + v_{\max}, v_{i_2} + v_{\max} \in C$.

Similarly (1), the sums of other edges adjacent to $v_{\max}$ must belong to $[(\{v_{i_0}\} \cup \{v'_{i_0}\}) \cap (\{v_{i_1}\} \cup \{v'_{i_1}\}) \cap (\{v_{i_2}\} \cup \{v'_{i_2}\})] \cup C$. Since v_{i_0}, v_{i_1} and v_{i_2} are distinct, $(\{v_{i_0}\} \cup \{v'_{i_0}\}) \cap (\{v_{i_1}\} \cup \{v'_{i_1}\}) \cap (\{v_{i_2}\} \cup \{v'_{i_2}\}) = \emptyset$. Then the sums of all edges adjacent to $v_{\max}$ must belong to the isolated set C. □

Lemma 6.5 Let $n \geq 7$, $2 \leq r \leq \lceil \frac{n}{2} \rceil$ and an integral sum graph $G = [K_n \backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n \backslash E(rK_2) = (V, E)$. Then $v_i + v_j \in C$ for any edge $v_i v_j \in E$.

Proof: Let $n \geq 7, 2 \leq r \leq \lceil \frac{n}{2} \rceil$ and an integral sum graph $G = [K_n \backslash E(rK_2)] \cup \overline{K_r}$ with $H = K_n \backslash E(rK_2) = (V, E)$. Assume that $v_{\max}$ is the vertex which absolute value is maximum in the set V. Without loss of generality, we may assume that and $v_{\max} \in V$ and $v_{\max} > 0$ (Otherwise, we just consider another integral sum labeling by using $(-1) \cdot S$ instead of S).

For any edge $v_i v_j \in E$, since $\delta(v_{\max}) \in \{n-1, n-2\}$, there is at least one vertex in the subset $\{v_i, v_j\}$ such that it is adjacent to the vertex $v_{\max}$. Without loss of generality, we may assume $v_i v_{\max} \in E$.

By Lemma 6.4, $v_i + v_{\max} \in C$. Thus, for any edge $v_i v_j \in E \backslash \{v_i v_{\max}\}$, $(v_i + v_{\max}) + v_j = (v_i + v_j) + v_{\max} \notin S$. So $v_i + v_j \in (\{v_{\max}\} \cup \{v'_{\max}\}) \cup C$, which implies that there are at most two distinct edges adjacent to v_i which sums are in $(\{v_{\max}\} \cup \{v'_{\max}\})$, denoted (2).

For $n \geq 7$, since $d(v_i) \in \{n-1, n-2\}$, At the same time, there are at least two distinct edges adjacent to v_i, denoted $v_i v_{j_1}$ and $v_i v_{j_2}$, such that their edge sums are in the set of isolated vertices C. Similarly (2), for any edge $v_i v_j \in E \backslash \{v_i v_{\max}, v_i v_{j_1}, v_i v_{j_2}\}$, $v_i + v_j \in [(\{v_{\max}\} \cup \{v'_{\max}\}) \cap (\{v_{j_1}\} \cup \{v'_{j_1}\}) \cap (\{v_{j_2}\} \cup \{v'_{j_2}\})] \cup C$. Since $v_{\max}, v_{j_1}$ and v_{j_2} are distinct, according to the structure of $K_n \backslash E(rK_2)$, $[(\{v_{\max}\} \cup \{v'_{\max}\}) \cap (\{v_{j_1}\} \cup \{v'_{j_1}\}) \cap (\{v_{j_2}\} \cup \{v'_{j_2}\})] = \emptyset$. Thus, $v_i + v_j \in C$.

Thus, Lemma 6.5 holds. □

Corollary 6.6 Let $n \geq 7$, $2 \leq r \leq \lceil \frac{n}{2} \rceil$ and an integral sum graph $G = [K_n \backslash E(rK_2)] \cup \overline{K_r}$, where $K_n \backslash E(rK_2)$ is a connected graph with $\delta(H) = n-2$. Then G is exclusive as n is more than a given number, for example, $n \geq 7$.

Proof: By Lemma 6.5, Corollary 6.6 holds. □

Lemma 6.7 For $n \geq 7$ and $2 \leq r \leq \lceil \frac{n}{2} \rceil$, $\varepsilon(K_n \backslash E(rK_2)) \geq \sigma(K_n \backslash E(rK_2)) \geq \zeta(K_n \backslash E(rK_2)) \geq 2n-5$ and $\varepsilon(K_n \backslash E(rK_2)) \geq \epsilon(K_n \backslash E(rK_2)) \geq 2n-5$.

Proof: Let $n \geq 7$ and $2 \leq r \leq \lceil \frac{n}{2} \rceil$. By Lemma 6.5, $G = [K_n \backslash E(rK_2)] \cup \overline{K_r}$ is an exclusive integral sum graph. By Theorem 2.2 and $\delta(K_n \backslash E(rK_2)) = n-2$, $\varepsilon(K_n \backslash E(rK_2)) \geq \sigma(K_n \backslash E(rK_2)) \geq \zeta(K_n \backslash E(rK_2)) \geq 2\delta(G) - 1 = 2(n-2) - 1 = 2n-5$ and $\varepsilon(K_n \backslash E(rK_2)) \geq \epsilon(K_n \backslash E(rK_2)) \geq 2\delta(G) - 1 = 2(n-2) - 1 = 2n-5$.
□

Lemma 6.8 For $n \geq 7$ and $2 \leq r \leq \lceil \frac{n}{2} \rceil$, $\epsilon(K_n \backslash E(rK_2)) \leq \varepsilon(K_n \backslash E(rK_2)) \leq 2n-5$.

Proof: For $n \geq 7$ and $2 \leq r \leq \lceil \frac{n}{2} \rceil$, let $K_n \backslash E(rK_2) = (V, E)$ and $V = \{x_1, x_2, x_3, \cdots, x_n\}$ and $S = V \cup C$, where C is the set of isolated vertices.

Let $x_i = (i-1) \times 10 + 1$ and $c_j = j \times 10 + 2$. Then $V = \{(i-1) \times 10 + 1 : i = 1, 2, \cdots, n\}$ and the isolated set $C = \{c_j : j = 1, 2, \cdots, 2n-3\} - \{c_{i_1}, c_{i_2}\}$. This key is to find c_{i_1}, c_{i_2} and the deleted matchings $E(rK_2)$ in the following cases respectively.

Case 1. n is odd and r is odd. Let $c_{i_1} = c_{r-1}$ and $c_{i_2} = c_{2n-r-2}$. The deleted matchings rK_2 are $x_1x_r, x_2x_{r-1}, \cdots, x_{\frac{r-1}{2}}x_{\frac{r+3}{2}}$ and $x_{n-r}x_n, x_{n-r-1}x_{n-1}, \cdots, x_{\frac{2n-r-1}{2}}x_{\frac{2n-r+1}{2}}$. Then the set of isolated vertices $C=\{c_j : j = 1, 2, \cdots, 2n-3\} - \{c_{r-1}, c_{2n-r-2}\}$. Verify that this is an exclusive sum labeling in detail.

In fact, we have

(1) The vertices in S are distinct.

(2) For any vertices $x_i \in \{x_1, x_2, x_3, \cdots, x_n\}$ and $c_k \in C$, since $x_i + c_k \equiv 3(\text{mod}10)$, $x_i + c_k \notin S$.

(3) For any distinct vertices $c_i, c_k \in C$, since $c_i + c_k \equiv 4(\text{mod}10)$, $c_i + c_k \notin S$.

(4) For $1 \le i \ne j \le n$ and any distinct vertices $x_i, x_j \in \{x_1, x_2, x_3, \cdots, x_n\}$, $x_i + x_j = (i + j - 2) \times 10 + 2 = c_{i+j-2}$.

Since $r \ge 2$ and $x_i + x_j = c_{r-1} \Longleftrightarrow i + j - 2 = r - 1 \Longleftrightarrow i + j = r + 1 \Longleftrightarrow (i, j) \in \{(1, r), (2, r-1), (3, r-2), \cdots, (\frac{r-1}{2}, \frac{r+3}{2})\}$.

Since $r \ge 2$ and $x_i + x_j = c_{2n-r-2} \Longleftrightarrow i + j - 2 = 2n - r - 2 \Longleftrightarrow i + j = 2n - r \Longleftrightarrow (i, j) \in \{(n-r, n), (n-r-1, n-1), (n-r-2, n-2), \cdots, (\frac{2n-r-1}{2}, \frac{2n-r+1}{2})\}$.

Then the deleted matchings are $x_1x_r, x_2x_{r-1}, \cdots, x_{\frac{r-1}{2}}x_{\frac{r+3}{2}}$ and $x_{n-r}x_n$, $x_{n-r-1}x_{n-1}, \cdots, x_{\frac{2n-r-1}{2}}x_{\frac{2n-r+1}{2}}$. Thus, it is one exclusive sum labeling of $(K_n \backslash E(rK_2)) \cup (2n-5)K_1$ when n and r are odd.

Case 2. n is odd and r is even. Let $c_{i_1} = c_r$ and $c_{i_2} = c_{2n-r-1}$. The deleted matchings rK_2 are $x_1x_{r+1}, x_2x_r, \cdots, x_{\frac{r}{2}}x_{\frac{r+4}{2}}$ and $x_{n-r+1}x_n, x_{n-r}x_{n-1}, \cdots, x_{\frac{2n-r}{2}}x_{\frac{2n-r+2}{2}}$. Then the set of isolated vertices $C=\{c_j : j = 1, 2, \cdots, 2n-3\} - \{c_r, c_{2n-r-1}\}$. Similarly, this is one exclusive sum labeling of $(K_n \backslash E(rK_2)) \cup (2n-5)K_1$ when n is odd and r is even.

Case 3. n is even and r is odd. Let $c_{i_1} = c_{r-2}$ and $c_{i_2} = c_{2n-r-2}$. The deleted matchings rK_2 are $x_1x_{r-1}, x_2x_{r-2}, \cdots, x_{\frac{r-1}{2}}x_{\frac{r+1}{2}}$ and $x_{n-r}x_n, x_{n-r+1}x_{n-1}, \cdots, x_{\frac{2n-r-2}{2}}x_{\frac{2n-r+2}{2}}$. Then the isolated set $C=\{c_j : j = 1, 2, \cdots, 2n-3\} - \{c_{r-2}, c_{2n-r-2}\}$. Similarly, this is one exclusive sum labeling of $(K_n \backslash E(rK_2)) \cup (2n-5)K_1$ when n is even and r is odd.

Case 4. n is even and r is even. Let $c_{i_1} = c_{r-1}$ and $c_{i_2} = c_{2n-r-2}$. The deleted matchings rK_2 are $x_1x_r, x_2x_{r-1}, \cdots, x_{\frac{r}{2}}x_{\frac{r+2}{2}}$ and $x_{n-r}x_n, x_{n-r+1}x_{n-1}, \cdots, x_{\frac{2n-r-2}{2}}x_{\frac{2n-r+2}{2}}$. Then the isolated set $C=\{c_j : j = 1, 2, \cdots, 2n-3\} - \{c_{r-1}, c_{2n-r-2}\}$. Similarly, this is one exclusive sum labeling of $(K_n \backslash E(rK_2)) \cup (2n-5)K_1$ when n and r are even.

Thus, $\sigma(K_n \backslash E(rK_2)) \le 2n-5$ for $n \ge 7$ and $2 \le r \le \lceil \frac{n}{2} \rceil$. □

Theorem 6.1 For any $n \ge 7$ and $1 \le r \le \lceil \frac{n}{2} \rceil$,

$$\zeta(K_n \backslash E(rK_2)) = \sigma(K_n \backslash E(rK_2)) = \begin{cases} 2n-4, & r = 1 \\ 2n-5, & 2 \le r \le \lceil \frac{n}{2} \rceil. \end{cases}$$

Proof: For $n \ge 7$, by Lemma 6.1, 6.7 and 6.8, Theorem 6.1 holds. □

Section 7. Conclusion

Similarly, using these results of Section 2 and 3, we could directly investigate and determine the (exclusive) sum number and the (exclusive) integral sum number of some classes of graphs H with $\delta(H) \geq 3$ or one saturated vertex v_s as the vertex number is more than a given number. This paper has proved that the method in this paper is much easier that the former, applied to determine the exclusive integral sum numbers, the exclusive sum numbers, the sum numbers and the integral sum numbers of the graphes. Then the method will be the beginning of a new thought of research on the (exclusive) sum graph and the (exclusive) integral sum graph.

Acknowledgements The authors is very grateful to the referee for his great comments and suggestions.

References

[1] J.A.Bondy, U.S.R.Murty, Graph Theory with Applications, MacMillan, New York, NY, 1976.

[2] F.Harary, Sum Graph and difference graphs, Congr. Numer., 72(1990): 101-108.

[3] F.Harary, Sum Graph over all the integers, Discrete Math., 124(1994): 99-105.

[4] Zhibo Chen, Note Harary's conjectures on integral sum graph, Discrete Mathematics,160(1996):241-244.

[5] M.Miller, Slamin, J.Ryan & W.F.Smyth, Labeling wheels for minimum sum number, JCMCC 28 (1998):289-297.

[6] Yan Wang & Bolian Liu, Some results on integral sum graphs, Discrete Mathematics, 240(2001): 219-229.

[7] W.He, X.Yu, H.Mi,Y.Xu, Y.Sheng & L.Wang, The (integral) sum number of $K_n - E(K_r)$, Discrete Math., 243(2002): 241-252.

[8] T.Nicholas, On integral sum labeling of dense graphs, Tamkang Journal of Mathematics, 41(2010):317-323.

[9] HaiyingWang, The sum numbers and the integral sum numbers of $K_{n+1} \backslash E(K_{1,r})$, Discrete Mathematics, 309(2009):4137-4143.

[10] HaiyingWang and JingzhenGao, The sum numbers and the integral sum numbers of $\overline{C_n}$ and $\overline{W_n}$, Ars Combinatoria, 96(2010): 479-488.

[11] HaiyingWang and JingzhenGao, The sum numbers and the integral sum numbers of $\overline{P_n}$ and $\overline{F_n}$, Ars Combinatoria, 96(2010): 9-31.

[12] HaiyingWang et al., The sum numbers and the integral sum numbers of the graph $K_n \backslash E(C_{n-1})$, Ars Combinatoria, 101(2011): 15-26.

[13] HaiyingWang and C. Li. The sum number and the integral sum number of a generalization of the dancing version of the cocktail party graph, Utilitas Mathematica, 86(2011), In press

[14] HaiyingWang et al. The sum numbers and the integral sum numbers of the graph $K_n \backslash E(P_{n-1})$, Utilitas Mathematica, 86(2011), In press