ON THE CYCLOMATIC NUMBER OF LINEAR HYPERGRAPHS

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ABSTRACT. In this note it is shown that the number of cycles of a linear hypergraph is bounded below by its cyclomatic number.

Keywords: *linear hypergraph, hypertree, cyclomatic number.*

1. Notation and preliminary results

A *simple hypergraph* $H = (X, \mathbb{E})$, with *order* $|X|$ and size $m = |\mathbb{E}|$, consists of a *vertex-set* $V(H) = X$ and an *edge-set* $\mathbb{E}(H) = \mathbb{E}$, where $E \subseteq X$ and $|E| \geq 2$ for each E in $\mathbb{E}$. H is *h-uniform*, or is an h-hypergraph, if $|E| = h$ for each E in $\mathbb{E}$ and H is *linear* if no two edges intersect in more than one vertex. The number of edges containing a vertex x is its *degree* $d_H(x)$.

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A path of *length* k joining vertices u and v in H is a subhypergraph consisting of k distinct vertices $x_0 = u, x_1, ..., x_k = v$ and k distinct edges $E_1, ..., E_k$ of H such that $x_{i-1}, x_i \in E_i$ for each i $(1 \leq i \leq k)$.
Similarly, a cycle C of length k in H [3] is a subhypergraph comprising k distinct vertices $x_1, ..., x_k$ and k distinct edges $E_1, ..., E_k$ of H such that $x_i, x_{i+1} \in E_i$ for each i, $1 \leq i \leq k-1$ and $x_1, x_k \in E_1$. Two vertices u, v of H are in the same *component*, if there is a path joining them. If H has only one component then it is *connected*; otherwise it has $p \geq 2$ *connected components*. An *hypertree* is a connected linear hypergraph without cycles.

Lemma 1.1. *Let H be a linear hypergraph without cycles and with order n, size m and p components. Then*

$$n = \sum_{E \in \mathbb{E}(H)} |E| - m + p.$$

Proof. This property follows from the fact that an hypertree with m edges has order

$$\sum_{E \in \mathbb{E}(H)} |E| - m + 1.$$

□

If H is an h-uniform hypertree then $n = m(h-1) + p$ [6].
The cyclomatic number of a graph G(also called co-rank of G) [4] is denoted by $\mu(G)$ and it is equal to $m - n + p$, where m, n and p are the number of edges, vertices and components of G, respectively. If G is cycle-free, i.e., G is forest, then $\mu(G) = 0$.

Acharya and Las Vergnas defined a parameter denoted by $\mu(H)$ for an hypergraph H and also called the cyclomatic number of H so that if H is simple graph, $\mu(H)$ is a generalization of this concept. In order to define this notion (see [3]), let

H be an hypergraph such that $\mathbb{E}(H) = \{E_1, E_2, ..., E_m\}$ and $V(H) = \cup_{i=1}^m E_i$. The intersection multigraph $L(H)$ associated with the hypergraph H is valuated by associating to each edge $u = e_i e_j$ the weight $w(u) = \mid E_i \cap E_j \mid$. IF F is a spanning forest of $L(H)$, its weight is $w(F) = \sum_{u \in F} w(u)$. The cyclomatic number of an hypergraph H without isolated vertices (i.e., such that $V(H) = \cup_{i=1}^m E_i$) is defined as follows:

$$\mu(H) = \sum_{i=1}^{m} \mid E_i \mid - \mid X \mid - w_H,$$

where w_H is the maximum weight of a spanning forest $F \subset L(H)$.

If H has a single edge E_1, we have $\mu(H) = \mid E_1 \mid - \mid E_1 \mid = 0$; the same situation holds if H has only two edges E_1 and E_2 since this case $\mu(H) = \mid E_1 \mid + \mid E_2 \mid - \mid E_1 \cup E_2 \mid - \mid E_1 \cap E_2 \mid = 0$. In these two cases H cannot contain any cycle. In general we have $\mu(H) \geq 0$ [3].

Acharya [1] established sharp bounds for the cyclomatic number $\mu(H)$ and Acharya and Las Vergnas [2] showed that if $\mu(H) = 0$ then the maximal edges of H with respect to inclusion are the cliques of a traingulated graph. By computing in two different ways the sum of cardinalities of all the edges of a hypergraph, Lewin [5] proved that $\mu(H) = 0$ if and only if some maximum forest of the weighted intersection multigraph $L(H)$ has the property that for every vertex of H the subgraph of the forest induced by these edges containing that vertex is connected.

If H is a linear hypergraph without isolated vertices, having order n, size m and p components, a maximum spanning forest F of $L(H)$ has cyclomatic number $\mu(F) = 0$. It follows that $m(F) - n(F) + p(F) = 0$, where $m(F), n(F)$ and $p(F)$ denote the number of edges, vertices and components of F, respectively.

But $n(F) = m$ and $p(F) = p$, which implies $m(F) = m - p$. By the linearity of H each edge of $L(H)$ has weight 1, hence $w(F) = m(F) = m - p$. In this case

$$\mu(H) = \sum_{E \in \mathbb{E}(H)} |E| - n - m + p. \quad (1)$$

Note that for linear hypergraphs H the condition that H does not contain isolated vertices is immeterial. Indeed, let H_1 be a hypergraph deduced from the linear hypergraph H by adding new s isolated vertices. In this case

$$\sum_{E \in \mathbb{E}(H_1)} | E | = \sum_{E \in \mathbb{E}(H)} | E |,\ n_1 = n + s,\ m_1 = m \ and\ p_1 = p + s.$$

By (1) we deduce that $\mu(H_1 = \mu(H)$.

In particular, if H is a simple graph (with or without isolated vertices), then $\mu(H) = 2m - n - m + p = m - n + p$ is the cyclomatic number of H.

It is well known that the cyclomatic number of any graph G is a lower bound for the number of all cycles of G. In the next section we shahh prove this property also holds for linear hypergraphs.

2. Main Result

Theorem 2.1: *Let H be a linear hypergraph of order n, size m, having k cycles and p connected components. Then*

$$k \geq \mu(H) = \sum_{E \in \mathbb{E}(H)} |E| - n - m + p. \quad (2)$$

Proof: We shall apply induction on the number of edges. If $m = 0$, then H is cycle-free, $n = p$, $k = \mu(H) = 0$ and (2) is an equality. Let $m \geq 1$ and suppose (2) is valid for any linear hypergraph of order n and size at most $m-1$. If we delete an edge E from an hypergraph H the resulting hypergraph, denoted by $H - E$, has by definition $V(H-E) = V(H)$ and $\mathbb{E}(H-E) = \mathbb{E}(H) \backslash \{E\}$. By deleting one edge, say E_1 from H we get a linear hypergraph $H' = H - E_1$ having order n, size $m-1$, k' cycles and p' components for which (2) holds, i.e.

$$k' \geq \mu(H') = \sum_{E \in \mathbb{E}(H')} |E| - n - m + 1 + p'. \qquad (3)$$

Suppose that E_1 includes $s \geq 0$ vertices of degree one. Vertices of degree greater than or equal to two of E_1 can be partitioned into two groups:

One group consists of $r \geq 0$ vertices $z_1, ..., z_r$ which belong to r distinct components $C_1, ..., C_r$ in H' such that $z_i \in C_i$ for $1 \leq i \leq r$; the other group includes t disjoint subsets of vertices $X_1 = \{z_1^1, z_2^1, ..., z_{m_1}^1\}$, $X_2 = \{z_1^2, z_2^2, ..., z_{m_2}^2\}$,..., $X_t = \{z_1^t, z_2^t, ..., z_{m_t}^t\}$ such that all vertices of X_i belong to a component C_i^1 in H' for $1 \leq i \leq t$ and all components $C_1, ..., C_r$, $C_1^1, ..., C_t^1$ are distinct (see fig.1). It is clear that

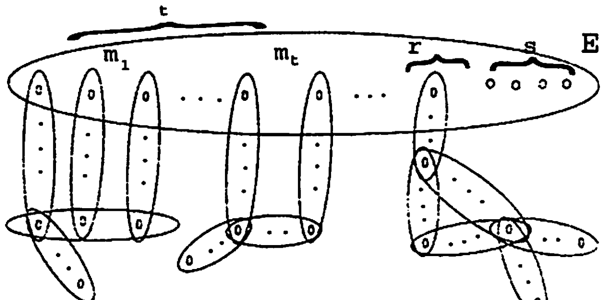

FIGURE 1

$$| E_1 | = \sum_{i=1}^{t} m_i + r + s \qquad (4)$$

and

$$p' = p + t + r + s - 1. \qquad (5)$$

On the other hand, for any two vertices z^i_p and z^i_q in the same subset X_i there exists a path $P^i_{p,q}$ joining them in the component C^1_i of H'. By the linearity of H' these paths must be distinct for $1 \leq p < q \leq m_i$. These paths in H' together the edge E_1 generate distinct cycles in H. It follows that

$$k \geq k' + \sum_{i=1}^{t} \binom{m_i}{2} \geq k' + \sum_{i=1}^{t}(m_i - 1) = k' + \sum_{i=1}^{t} m_i - t$$

$$\geq \sum_{E \in \mathbb{E}(H)} |E| - |E_1| - n - m + 1 + p' + \sum_{i=1}^{t} m_i - t \qquad \text{by (3).}$$

Now (2) follows since $|E_1|$ and p' are given by (4) and (5). □

The inequality (2) is an equality for example for acyclic linear hypergraphs by Lemma 1.1. Also (2) becomes an equality if linear hypergraph H has only three edges. By considering all the cases for these edges (pairwise disjoint, inducing a star, a path of length three, a path of length two and a path of length one or a cycle), it is easy to see that $k = \mu(H) \in \{0, 1\}$. But if H has $m = 4$ edges then the inequality (2) may be strict.

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