

ON THE CYCLOMATIC NUMBER OF LINEAR HYPERGRAPHS

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ABSTRACT. In this note it is shown that the number of cycles of a linear hypergraph is bounded below by its cyclomatic number.

Keywords: *linear hypergraph, hypertree, cyclomatic number.*

1. Notation and preliminary results

A *simple hypergraph* $H = (X, \mathbb{E})$, with *order* $|X|$ and *size* $m = |\mathbb{E}|$, consists of a *vertex-set* $V(H) = X$ and an *edge-set* $\mathbb{E}(H) = \mathbb{E}$, where $E \subseteq X$ and $|E| \geq 2$ for each E in \mathbb{E} . H is *h -uniform*, or is an *h -hypergraph*, if $|E| = h$ for each E in \mathbb{E} and H is *linear* if no two edges intersect in more than one vertex. The number of edges containing a vertex x is its *degree* $d_H(x)$.

*The research was partially supported by Abdus Salam School of Mathematical Sciences and Higher Education Commission of Pakistan.

A path of length k joining vertices u and v in H is a subhypergraph consisting of k distinct vertices $x_0 = u, x_1, \dots, x_k = v$ and k distinct edges E_1, \dots, E_k of H such that $x_{i-1}, x_i \in E_i$ for each i ($1 \leq i \leq k$).

Similarly, a cycle C of length k in H [3] is a subhypergraph comprising k distinct vertices x_1, \dots, x_k and k distinct edges E_1, \dots, E_k of H such that $x_i, x_{i+1} \in E_i$ for each $i, 1 \leq i \leq k - 1$ and $x_1, x_k \in E_1$. Two vertices u, v of H are in the same *component*, if there is a path joining them. If H has only one component then it is *connected*; otherwise it has $p \geq 2$ *connected components*. An *hypertree* is a connected linear hypergraph without cycles.

Lemma 1.1. *Let H be a linear hypergraph without cycles and with order n , size m and p components. Then*

$$n = \sum_{E \in \mathbb{E}(H)} |E| - m + p.$$

Proof. This property follows from the fact that an hypertree with m edges has order

$$\sum_{E \in \mathbb{E}(H)} |E| - m + 1.$$

□

If H is an h -uniform hypertree then $n = m(h - 1) + p$ [6].

The cyclomatic number of a graph G (also called co-rank of G) [4] is denoted by $\mu(G)$ and it is equal to $m - n + p$, where m, n and p are the number of edges, vertices and components of G , respectively. If G is cycle-free, i.e., G is forest, then $\mu(G) = 0$.

Acharya and Las Vergnas defined a parameter denoted by $\mu(H)$ for an hypergraph H and also called the cyclomatic number of H so that if H is simple graph, $\mu(H)$ is a generalization of this concept. In order to define this notion (see [3]), let

H be an hypergraph such that $\mathbb{E}(H) = \{E_1, E_2, \dots, E_m\}$ and $V(H) = \cup_{i=1}^m E_i$. The intersection multigraph $L(H)$ associated with the hypergraph H is valued by associating to each edge $u = e_i e_j$ the weight $w(u) = |E_i \cap E_j|$. If F is a spanning forest of $L(H)$, its weight is $w(F) = \sum_{u \in F} w(u)$. The cyclomatic number of an hypergraph H without isolated vertices (i.e., such that $V(H) = \cup_{i=1}^m E_i$) is defined as follows:

$$\mu(H) = \sum_{i=1}^m |E_i| - |X| - w_H,$$

where w_H is the maximum weight of a spanning forest $F \subset L(H)$.

If H has a single edge E_1 , we have $\mu(H) = |E_1| - |E_1| = 0$; the same situation holds if H has only two edges E_1 and E_2 since this case $\mu(H) = |E_1| + |E_2| - |E_1 \cup E_2| - |E_1 \cap E_2| = 0$. In these two cases H cannot contain any cycle. In general we have $\mu(H) \geq 0$ [3].

Acharya [1] established sharp bounds for the cyclomatic number $\mu(H)$ and Acharya and Las Vergnas [2] showed that if $\mu(H) = 0$ then the maximal edges of H with respect to inclusion are the cliques of a triangulated graph. By computing in two different ways the sum of cardinalities of all the edges of a hypergraph, Lewin [5] proved that $\mu(H) = 0$ if and only if some maximum forest of the weighted intersection multigraph $L(H)$ has the property that for every vertex of H the subgraph of the forest induced by these edges containing that vertex is connected.

If H is a linear hypergraph without isolated vertices, having order n , size m and p components, a maximum spanning forest F of $L(H)$ has cyclomatic number $\mu(F) = 0$. It follows that $m(F) - n(F) + p(F) = 0$, where $m(F)$, $n(F)$ and $p(F)$ denote the number of edges, vertices and components of F , respectively.

But $n(F) = m$ and $p(F) = p$, which implies $m(F) = m - p$. By the linearity of H each edge of $L(H)$ has weight 1, hence $w(F) = m(F) = m - p$. In this case

$$\mu(H) = \sum_{E \in \mathbb{E}(H)} |E| - n - m + p. \quad (1)$$

Note that for linear hypergraphs H the condition that H does not contain isolated vertices is immaterial. Indeed, let H_1 be a hypergraph deduced from the linear hypergraph H by adding new s isolated vertices. In this case

$$\sum_{E \in \mathbb{E}(H_1)} |E| = \sum_{E \in \mathbb{E}(H)} |E|, \quad n_1 = n + s, \quad m_1 = m \quad \text{and} \quad p_1 = p + s.$$

By (1) we deduce that $\mu(H_1) = \mu(H)$.

In particular, if H is a simple graph (with or without isolated vertices), then $\mu(H) = 2m - n - m + p = m - n + p$ is the cyclomatic number of H .

It is well known that the cyclomatic number of any graph G is a lower bound for the number of all cycles of G . In the next section we shall prove this property also holds for linear hypergraphs.

2. Main Result

Theorem 2.1: *Let H be a linear hypergraph of order n , size m , having k cycles and p connected components. Then*

$$k \geq \mu(H) = \sum_{E \in \mathbb{E}(H)} |E| - n - m + p. \quad (2)$$

Proof: We shall apply induction on the number of edges. If $m = 0$, then H is cycle-free, $n = p$, $k = \mu(H) = 0$ and (2) is

an equality. Let $m \geq 1$ and suppose (2) is valid for any linear hypergraph of order n and size at most $m - 1$. If we delete an edge E from an hypergraph H the resulting hypergraph, denoted by $H - E$, has by definition $V(H - E) = V(H)$ and $\mathbb{E}(H - E) = \mathbb{E}(H) \setminus \{E\}$. By deleting one edge, say E_1 from H we get a linear hypergraph $H' = H - E_1$ having order n , size $m - 1$, k' cycles and p' components for which (2) holds, i.e.

$$k' \geq \mu(H') = \sum_{E \in \mathbb{E}(H')} |E| - n - m + 1 + p'. \quad (3)$$

Suppose that E_1 includes $s \geq 0$ vertices of degree one. Vertices of degree greater than or equal to two of E_1 can be partitioned into two groups:

One group consists of $r \geq 0$ vertices z_1, \dots, z_r which belong to r distinct components C_1, \dots, C_r in H' such that $z_i \in C_i$ for $1 \leq i \leq r$; the other group includes t disjoint subsets of vertices $X_1 = \{z_1^1, z_2^1, \dots, z_{m_1}^1\}$, $X_2 = \{z_1^2, z_2^2, \dots, z_{m_2}^2\}, \dots, X_t = \{z_1^t, z_2^t, \dots, z_{m_t}^t\}$ such that all vertices of X_i belong to a component C_i^1 in H' for $1 \leq i \leq t$ and all components $C_1, \dots, C_r, C_1^1, \dots, C_t^1$ are distinct (see fig.1). It is clear that

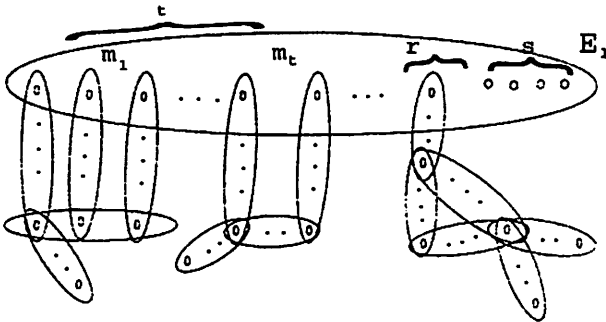


FIGURE 1

$$|E_1| = \sum_{i=1}^t m_i + r + s \quad (4)$$

and

$$p' = p + t + r + s - 1. \tag{5}$$

On the other hand, for any two vertices z_p^i and z_q^i in the same subset X_i there exists a path $P_{p,q}^i$ joining them in the component C_i^1 of H' . By the linearity of H' these paths must be distinct for $1 \leq p < q \leq m_i$. These paths in H' together the edge E_1 generate distinct cycles in H . It follows that

$$\begin{aligned} k &\geq k' + \sum_{i=1}^t \binom{m_i}{2} \geq k' + \sum_{i=1}^t (m_i - 1) = k' + \sum_{i=1}^t m_i - t \\ &\geq \sum_{E \in \mathbb{E}(H)} |E| - |E_1| - n - m + 1 + p' + \sum_{i=1}^t m_i - t \quad \text{by (3)}. \end{aligned}$$

Now (2) follows since $|E_1|$ and p' are given by (4) and (5). \square

The inequality (2) is an equality for example for acyclic linear hypergraphs by Lemma 1.1. Also (2) becomes an equality if linear hypergraph H has only three edges. By considering all the cases for these edges (pairwise disjoint, inducing a star, a path of length three, a path of length two and a path of length one or a cycle), it is easy to see that $k = \mu(H) \in \{0, 1\}$. But if H has $m = 4$ edges then the inequality (2) may be strict.

Acknowledgement

The authors are indebted to the referee for the valuable comments that improved the original version of this paper.

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