The Geo-number of a Graph

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Abstract

Let G be a connected graph of order $p \geq 2$. The closed interval $I[x,y]$ consists of all vertices lying on some x-y geodesic of G. If S is a set of vertices of G, then $I[S]$ is the union of all sets $I[x,y]$ for $x,y \in S$. The geodetic number $g(G)$ is the minimum cardinality among the subsets S of $V(G)$ with $I[S] = V$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. For any vertex x in G, a set $S_x \subseteq V$ is an x-geodominating set of G if each vertex $v \in V$ lies on an x-y geodesic for some element y in S_x. The minimum cardinality of an x-geodominating set of G is defined as the x-geodomination number of G, denoted by $g_x(G)$ or simply g_x. An x-geodominating set S_x of cardinality $g_x(G)$ is called a g_x-set of G. If $S_x \cup \{x\}$ is a g-set of G, then x is called a geo-vertex of G. The set of all geo-vertices of G is called the geo-set of G and the number of geo-vertices of G is called the geo-number of G and it is denoted by $gn(G)$. For positive integers r, d and $n \geq 2$ with $r < d \leq 2r$, there exists a connected graph G of radius r, diameter d and $gn(G) = n$. Also, for each triple p, d and n with $3 \leq d \leq p-1$, $2 \leq n \leq p-2$ and $p-d-n+1 \geq 0$, there exists a graph G of order p, diameter d and $gn(G) = n$. If the x-geodomination number $g_x(G)$ is same for every vertex x in G, then G is called a vertex geodomination regular graph or for short VGR-graph. If $S_x \cup \{x\}$ is same for every vertex x in G, then G is called a perfect vertex geodomination graph or for short PVG-graph. We characterize a PVG-graph.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [5]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an *x-y geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. The *diameter diam* G of a connected graph G is the length of any longest geodesic. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is a *simplicial* or *extreme vertex* if the subgraph induced by its neighbors is complete. A *nonseparable* graph is connected, nontrivial, and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. A *connected block graph* is a connected graph in which each of its blocks is complete.

The *closed interval* $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a geodetic set if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. The geodetic number of a graph was introduced in [1, 6] and further studied in [3]. It was shown in [6] that determining the geodetic number of a graph is an NP-hard problem. Geodetic concepts were first studied from the point of view of domination by Chartrand, Harary, Swart, and Zhang in [2], where a pair x, y of vertices in a nontrivial connected graph G is said to *geodominate a vertex* v of G if $v \in I[x, y]$, that is, v lies on an x-y geodesic of G. In [2], geodetic sets and the geodetic number were referred to as *geodominating sets* and *geodomination number*. For a connected graph G and a set $W \subseteq V(G)$, a tree T contained in G is a *Steiner tree* with respect to W if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all vertices in G that lie on some Steiner tree with respect to W. The set W is a *Steiner set* for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the *Steiner number* $s(G)$.

The concept of vertex geodomination number was introduced by Santhakumaran and Titus [8] and further studied in [9]. A vertex y in a connected graph G is said to *x-geodominate* a vertex u if u lies on an x-y

geodesic. A set S of vertices of G is an *x-geodominating set* if each vertex $v \in V(G)$ is x-geodominated by some element of S. The minimum cardinality of an x-geodominating set of G is defined as the *x-geodomination number* of G, denoted by $g_x(G)$ or simply g_x. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set.

Every vertex of an x-y geodesic is x-geodominated by the vertex y. Since, by definition, a g_x-set is minimum, the vertex x and also the internal vertices of an x-y geodesic do not belong to a g_x-set. For the graph G given in Figure 1.1, $g_u(G) = 1$, $g_v(G) = 2$, $g_w(G) = 2$, $g_x(G) = 2$ and $g_y(G) = 1$ with minimum vertex geodominating sets $\{y\}$, $\{u, y\}$, $\{u, x\}$, $\{u, w\}$ and $\{u\}$ respectively.

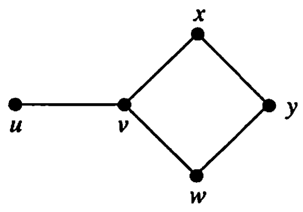

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Figure 1.1

It is proved in [8] that for any vertex x in G, g_x-set is unique and $1 \leq g_x(G) \leq p - 1$ for every vertex x in G. We characterized graphs which realize the bounds. It is also proved that $g(G) \leq g_x(G) + 1$ for every vertex x in G. An elaborate study of results in vertex geodomination with several interesting applications is given in [8, 9]. The following theorems will be used in the sequel.

Theorem 1.1. [1] *Let G be a connected graph. Then $g(G) = p$ if and only if $G = K_p$.*

Theorem 1.2. [1] *No cut vertex of G belongs to any minimum geodetic set of G.*

Theorem 1.3. [2] *Every geodominating set of a graph G contains every simplicial vertex of G. In particular, if the set W of simplicial vertices is a geodominating set of G, then W is the unique g-set of G and so $g(G) = |W|$.*

Theorem 1.4. [3] *For integers $m, n \geq 2$, $g(K_{m,n}) = min\{m, n, 4\}$.*

Theorem 1.5. [6] *For the wheel $W_{1,n}$,* $g(W_{1,n}) = \begin{cases} 4 & \text{for } n = 3, \\ \lceil \frac{n}{2} \rceil & \text{for } n \geq 4. \end{cases}$

Result 1.6. [8] *Every vertex of an x-y geodesic is x-geodominated by the vertex y. Since, by definition, a g_x-set is minimum, the vertex x and also the internal vertices of an x-y geodesic do not belong to the g_x-set.*

Theorem 1.7. [8] *Let G be a connected graph.*

(i) *Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to the g_x-set for any vertex x in G.*

(ii) *For any vertex x, eccentric vertices of x belong to the g_x-set.*

(iii) *No cut vertex of G belongs to any g_x-set.*

Theorem 1.8. [8] *For any vertex x in G, $g(G) \leq g_x(G) + 1$.*

Theorem 1.9. [8] *Let T be a tree with number of pendent vertices k. Then $g_x(T) \geq k - 1$ or k according as x is a pendent or non-pendent vertex.*

Theorem 1.10. [8]

(i) *For the wheel $W_{1,n} = K_1 + C_n (n \geq 4)$, $g_x(W_n) = n$ or $n-3$ according as x is K_1 or x is in C_n.*

(ii) *Let $K_{m,n} (m, n \geq 2)$ be a complete bipartite graph with bipartition (V_1, V_2). Then $g_x(K_{m,n})$ is $m-1$ or $n-1$ according as x is in V_1 or x is in V_2.*

Throughout the following G denotes a connected graph with at least two vertices.

2 The Geo-number of a Graph

Definition 2.1. *Let G be a connected graph. Let x be any vertex of G and S_x be the g_x-set of G. If $S_x \cup \{x\}$ is a g-set of G, then x is called a geo-vertex of G. The set of all geo-vertices of G is called the geo-set of G and the number of geo-vertices of G is called the geo-number of G and it is denoted by $gn(G)$.*

Example 2.2.

(i) *For the graph G given in Figure 2.1, $S_1 = \{v_1, v_4, v_7\}$ and $S_2 = \{v_1, v_5, v_7\}$ are the only two g-sets. The various g_x-sets of the graph G are given in Table 2.1 and $\{v_1, v_5\}$ is the geo-set of G so that $gn(G) = 2$.*

(ii) *For the complete graph* K_p, $g(K_p) = p$ *by Theorem* 1.1 *and so* V *is the* g*-set of* K_p. *For any vertex* x *in* K_p, *the* g_x*-set* S_x *equals* $V - \{x\}$ *and so* $S_x \cup \{x\} = V$. *Thus every vertex of* K_p *is a geo-vertex and hence* $gn(K_p) = p$.

(iii) *For the complete bipartite graph* $K_{m,n}(m, n \geq 5)$ *with partition* (V_1, V_2), $g(K_{m,n}) = 4$ *by Theorem* 1.4. *Let* x *be any vertex of* $K_{m,n}$. *But by Theorem* 1.10(ii), $g_x(K_{m,n})$ *is* $m - 1$ *or* $n - 1$ *according as* x *is in* V_1 *or* x *is in* V_2. *Thus* $S_x \cup \{x\}$ *is not a* g*-set of* G *and hence* $K_{m,n}$ *has no geo-vertex if* $m, n \geq 5$. *Thus* $gn(K_{m,n}) = 0$.

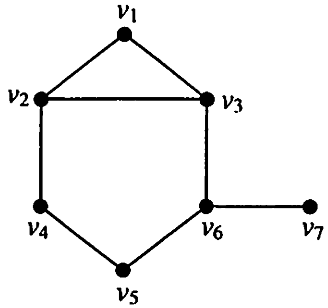

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Figure 2.1

Vertex x	g_x-sets
v_1	$\{v_5, v_7\}$
v_2	$\{v_1, v_5, v_7\}$
v_3	$\{v_1, v_4, v_5, v_7\}$
v_4	$\{v_1, v_3, v_7\}$
v_5	$\{v_1, v_7\}$
v_6	$\{v_1, v_2, v_4, v_7\}$
v_7	$\{v_1, v_2, v_4\}$

Table 2.1

The following theorem is clear from the definition.

Theorem 2.3. *For any connected graph* G, $0 \leq gn(G) \leq p$.

Observation 2.4. *If* G *is a connected graph, then no cut vertex of* G *is a geo-vertex of* G.

The following theorem is an immediate consequence of Observation 2.4.

Theorem 2.5. *If k is the number of cut vertices of a connected graph G, then $gn(G) \leq p - k$.*

Remark 2.6. *The bound in Theorem 2.5 is sharp. For the graph G given in Figure 2.2, $p = 5, k = 1$ and $gn(G) = 4 = p - k$. For the graph G given in Figure 2.1, $p = 7, k = 1$ and $gn(G) = 2 < p - k$ so that the inequality in Theorem 2.5 can be strict.*

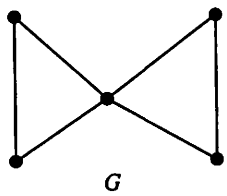

Figure 2.2

The following example gives the geo-numbers of certain special classes of graphs.

Example 2.7.

(i) *If T is a non-trivial tree with number of end vertices k, then $gn(T) = k$.*

(ii) *If G is the complete graph K_p or the cycle C_p, then $gn(G) = p$.*

(iii) *If G is the n-cube $Q_n (n \geq 1)$, then $gn(G) = 2^n$.*

(iv) *If G is the complete bipartite graph $K_{m,n} (m \leq n)$, then*

$$gn(G) = \begin{cases} n & \text{if } 1 = m < n \\ m & \text{if } 1 < m < n \text{ and } m \leq 4 \\ 2m & \text{if } m = n \leq 4 \\ 0 & \text{if } m, n \geq 5. \end{cases}$$

The following observation is an easy consequence of some of the preceeding results.

Observation 2.8. *For a connected graph G,*

(i) *if $g_x(G) = 1$ for some vertex x in G, then $gn(G) \geq 2$.*

(ii) *if $g(G) = 2$, then $gn(G) \geq 2$.*

(iii) *if* $s(G) = 2$, *then* $gn(G) \geq 2$.

(iv) *if every vertex of G is either a cut vertex or a simplicial vertex, then* x *is a geo-vertex of G if and only if* x *is a simplicial vertex of* G.

Since every vertex of a connected block graph is either a cut vertex or a simplicial vertex, the following theorem follows from Observation 2.8(iv).

Theorem 2.9. *If G is a connected block graph with number of simplicial vertices* k, *then* $gn(G) = k$.

Remark 2.10. *In general there is no relation between* $g(G)$ *and* $gn(G)$. *For the graph G given in Figure* 2.1, $g(G) > gn(G)$. *For the cycle* $C_p (p \geq 4)$, $g(C_p) < gn(C_p)$. *For a non-trivial tree T,* $g(T) = gn(T)$.

However, we have the following result.

Theorem 2.11. *If G is a connected graph with unique g-set, then* $gn(G) \leq g(G)$.

Proof. Let S_x be the g_x-set of G and let S be the unique g-set of G. If G is the complete graph K_p, then by Example 2.2(ii) and Theorem 1.1, $gn(G) = g(G) = p$. Now assume that G is not complete. Then by Theorem 1.1, $S \subset V$. For any vertex $x \notin S$, $S_x \cup \{x\} \neq S$ and so x is not a geo-vertex of G. Hence $gn(G) \leq g(G)$. □

Corollary 2.12. *If G is a connected graph with unique g-set, then* $gn(G) \leq g_x(G) + 1$ *for any vertex x in G.*

Proof. This follows from Theorem 1.8 and Theorem 2.11. □

The following theorem gives a realization result for the geo-number of a graph.

Theorem 2.13. *For any integer n such that* $0 \leq n \leq p$ $(p \geq 6)$, *there is a graph G of order p and* $gn(G) = n$.

Proof. Let $n = 0$. For $p = 6$, consider the graph G given in Figure 2.3(i). By Theorem 1.3, $\{x_1, x_3, x_5\}$ is the unique g-set of G and it is easily seen that no vertex of G is a geo-vertex so that $gn(G) = 0$. For $p \geq 7$, let $G = W_{1,p-1}$. Then by Theorem 1.5, $g(G) = \lceil \frac{p-1}{2} \rceil \leq p - 4$. Also, by Theorem 1.10(i), $g_x(G) = p - 1$ or $p - 4$ for any vertex x in G. Hence it follows from the definition of a geo-vertex that no vertex of G is a geo-vertex of G. Thus $gn(G) = 0$.

Let $n = 1$. For $p = 6$, consider the graph G given in Figure 2.3(ii). By Theorem 1.3, $\{x_1, x_3, x_5\}$ is the unique g-set of G and it is easily seen that x_1 is the unique geo-vertex of G so that $gn(G) = 1$. For $p \geq 7$, let

$C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ be a cycle of order 5, $P_2 : x_1, x_2$ be a path of length one and $P_{p-5} : y_1, y_2, \ldots, y_{p-5}$ be a path of length $p-6$. Let G be the graph obtained from C_5, P_2 and P_{p-5} by identifying v_1 in C_5 and x_1 in P_2 and identifying v_2 in C_5 and y_1 in P_{p-5}. The graph G is given in Figure 2.3(iii).

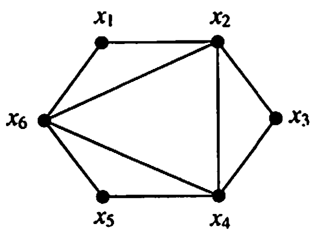

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Figure 2.3(i)

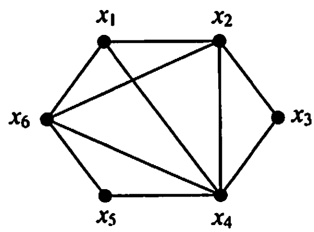

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Figure 2.3(ii)

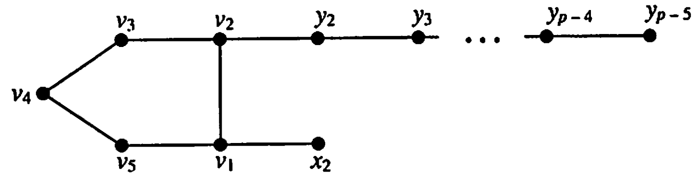

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Figure 2.3(iii)

Let $S = \{x_2, y_{p-5}\}$ be the set of end vertices of G. But $I[S] = S \cup \{v_1, v_2, y_2, y_3, \cdots, y_{p-4}\} \neq V(G)$. This implies that S is not a geodominating set of G. Now by Theorem 1.3, $g(G) > 2$. On the other hand, $I[S \cup \{v_4\}] = V(G)$. Hence $T = S \cup \{v_4\}$ is a geodominating set of G so that $g(G) = 3$. Also it is clear that $I[S \cup \{z\}] \neq V(G)$ for $z \in \{v_3, v_5\}$ and hence T is the unique g-set of G. Thus by Theorem 2.11, $gn(G) \leq 3$. It is easily checked that v_4 is the unique geo-vertex of G and so $gn(G) = 1$.

For $2 \leq n \leq p-1$, the tree T given in Figure 2.3(iv) has $p = k + n$ vertices and it follows from Example 2.7(i) that $gn(T) = n$. For $n = p$, the theorem follows from Example 2.7(ii) by taking $G = K_p$. □

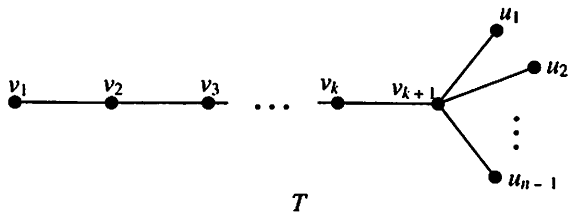


Figure 2.3(iv)

Remark 2.14. *For $2 \leq p \leq 5$, it is straight forward to verify that there is no graph G of order p with $gn(G) = 0$ or $gn(G) = 1$. Thus Theorem 2.13 does not hold for $2 \leq p \leq 5$.*

For every connected graph $G, rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [7] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Ostrand's theorem can be extended so that the geo-number can be prescribed when $a < b \leq 2a$.

Theorem 2.15. *For positive integers r, d and $n \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$ and $gn(G) = n$.*

Proof. If $r = 1$, then $d = 2$. Let $G = K_{1,n}$. Then by Example 2.7(i), $gn(G) = n$. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows:

Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}.

Case 1. Suppose $n = 2$. Let $G = H$. Now $rad\ G = r$, $diam\ G = d$ and G has one end-vertex u_{d-r}. Clearly, $S = \{u_{d-r}, v_{r+1}\}$ is the geodominating set of G so that $g(G) = 2$. Also it is clear that $I[\{u_{d-r}, v_i\}] \neq V(G)$ for $i \neq r+1$ and hence S is the unique g-set of G. Then by Observation 2.8(ii) and Theorem 2.11, $gn(G) = 2$.

Case 2. Suppose $n \geq 3$. Add $n - 2$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H and join each vertex $w_i (1 \leq i \leq n-2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.4.

Now $rad\ G = r$, $diam\ G = d$ and G has $n - 1$ end vertices. Let $S = \{u_{d-r}, w_1, w_2, \ldots, w_{n-2}\}$ be the set of end vertices of G. But $I[S] = S \cup \{u_{d-r-1}\} \neq V(G)$. This implies that S is not a geodominating set of G. Now by Theorem 1.3, $g(G) > n - 1$. On the other hand, $I[S \cup \{v_{r+1}\}] = V(G)$. Hence, $T = S \cup \{v_{r+1}\}$ is a geodominating set of G so that $g(G) = n$. Also it is clear that $I[S \cup \{v_i\}] \neq V(G)$ for $i \neq r+1$ and hence T is the unique g-set of G. Thus by Theorem 2.11, $gn(G) \leq n$. Clearly, $T - \{x\}$ is the g_x-set of G for any vertex x in T and so every vertex of T is a geo-vertex of G. Thus $gn(G) = n$. □

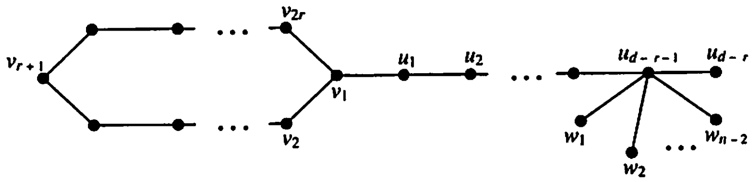


Figure 2.4

The graph G of Figure 2.4 is the smallest graph with the properties described in Theorem 2.15.

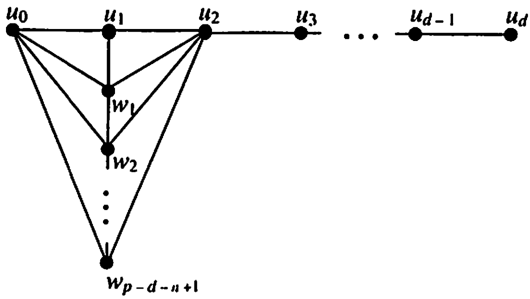
Remark 2.16. *In the case of geo-number $gn(G)$ of G there are graphs for which $gn(G) = p - d + 1$, $gn(G) < p - d + 1$ and $gn(G) > p - d + 1$. For the complete graph $K_p, gn(K_p) = p - d + 1$. For the complete bipartite graph $K_{m,n}(m, n \geq 5)$, $gn(K_{m,n}) < p - d + 1$. For the cycle $C_p(p \geq 4)$, $gn(C_p) > p - d + 1$.*

In the following theorem we construct a graph of prescribed order, diameter and geo-number under suitable conditions.

Theorem 2.17. *If p, d and n are integers such that $3 \leq d \leq p - 1$, $2 \leq n \leq p - 2$ and $p - d - n + 1 \geq 0$, then there exists a graph G of order p, diameter d and $gn(G) = n$.*

Proof. Let $P_{d+1} : u_0, u_1, u_2, \ldots, u_d$ be a path of length d and let $P_{p-d-n+2}$: $w_0, w_1, w_2, \ldots, w_{p-d-n+1}$ be a path of length $p-d-n+1$. The graph H in Figure 2.5(i) is obtained by identifying the vertex w_0 in $P_{p-d-n+2}$ and u_1 in P_{d+1} and joining the vertices $w_1, w_2, \ldots, w_{p-d-n+1}$ to both the vertices u_0 and u_2.

Case 1. Suppose $n = 2$. Let $G = H$. Then G has order p and diameter d. Moreover, the set $S = \{u_0, u_d\}$ is the unique g-set of G and so $g(G) = 2$. Then by Observation 2.8(ii) and Theorem 2.11, $gn(G) = 2$. Thus G has the desired properties.



H

Figure 2.5(i)

Case 2. Suppose $3 \leq n \leq p-2$. Add $n-2$ new vertices $v_1, v_2, \ldots, v_{n-2}$ to H and join these to u_2, there by producing the graph G of Figure 2.5(ii). Then G has order p and diameter d. Let $S = \{u_d, v_1, v_2, \ldots, v_{n-2}\}$ be the set of end vertices of G. But $I[S] = S \cup \{u_2, u_3, \ldots, u_{d-1}\} \neq V(G)$. This implies that S is not a geodominating set of G. Now by Theorem 1.3, $g(G) > n-1$. On the other hand, $I[S \cup \{u_0\}] = V(G)$. Hence $T = S \cup \{u_0\}$ is a geodominating set of G so that $g(G) = n$. Also it is clear that $I[S \cup \{y\}] \neq V(G)$ for $y \in \{u_1, w_1, w_2, \ldots, w_{p-d-n+1}\}$ and hence T is the unique g-set of G. Thus by Theorem 2.11, $gn(G) \leq n$. Clearly, $T - \{x\}$ is the g_x-set of G for any element x in T and so every vertex of T is a geo-vertex of G. Thus $gn(G) = n$. □

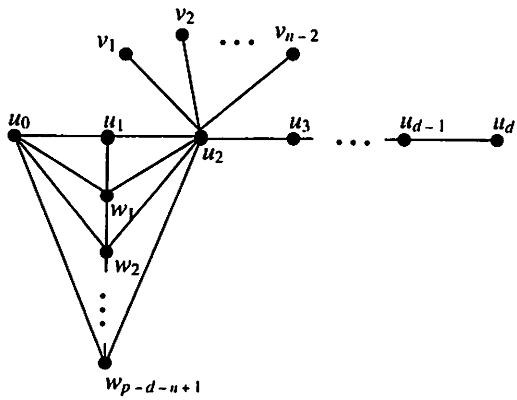


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Figure 2.5(ii)

3 VGR and PVG - Graphs

We now proceed to discuss graphs G for which the x-geodomination number $g_x(G)$ is the same for every vertex x in G and also graphs G for which the g_x-set S_x of G together with x is the same for every vertex x in G.

Definition 3.1. *Let x be any vertex of a connected graph G. If the x-geodomination number $g_x(G)$ is same for every vertex x in G, then G is called a vertex geodomination regular graph or for short VGR-graph.*

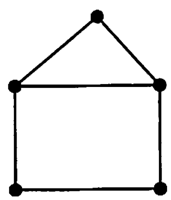
Example 3.2.

(i) *For the complete graph K_p, $g_x(K_p) = p - 1$ for every vertex x in K_p. For the cycle C_p, $g_x(C_p) = 1$ or 2 for every vertex x in C_p according as p is even or odd. Hence the graphs K_p and C_p are VGR-graphs.*

(ii) *Let T be a tree with at least 3 vertices and let k be the number of pendent vertices of T. Then by Theorem 1.9, $g_x(T)$ is $k - 1$ or k according as x is a pendent vertex or not, so that T is not a VGR-graph.*

(iii) *The complete bipartite graph $K_{m,n}$ is a VGR-graph if and only if $m = n$.*

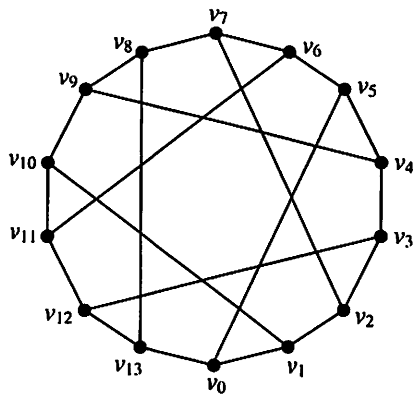
(iv) *For the graph G given in Figure 3.1, $g_x(G) = 2$ for every vertex x in G. Hence G is a VGR-graph.*



G

Figure 3.1

Observation 3.3. *If G is a connected graph such that $gn(G) = p$, then G is a VGR-graph.*



G

Figure 3.2

The converse of Observation 3.3 is false. For the Heawood graph G given in Figure 3.2, $S = \{v_0, v_3, v_7, v_{10}\}$ is a g-set so that $g(G) = 4$. Now $S_{v_0} = \{v_3, v_7, v_9, v_{11}\}$ is the g_{v_0}-*set* of G so that $g_{v_0}(G) = 4$. It is easily checked that $g_x(G) = g_y(G)$ for any two vertices x and y. Thus $g(G) = g_x(G) = 4$ for every vertex x in G so that G is a VGR-graph. Since $g(G) = g_x(G)$ for every vertex x in G, no vertex of G is a geo-vertex and so $gn(G) = 0$.

Definition 3.4. *For any vertex x in a connected graph G, let S_x be the g_x-set of G. If $S_x \cup \{x\}$ is same for every vertex x in G, then G is called a perfect vertex geodomination graph or for short PVG-graph.*

It is clear that every PVG-graph is a VGR-graph. In the following theorem we characterize a PVG-graph.

Theorem 3.5. *A graph G is a PVG-graph if and only if G is complete.*

Proof. Let G be a PVG-graph. Suppose G is not complete. Then there exist at least two vertices x and y in G such that $d(x, y) \geq 2$. Hence there exists a shortest path $x, u_1, u_2, \ldots, u_n, y$ of length at least two joining x and y. Let S_x be the g_x-set of G. By Result 1.6, the internal vertices $u_i (i = 1, 2, \ldots, n)$ of the x-y geodesic do not belong to S_x. Hence $S_x \cup \{x\} \neq S_{u_i} \cup \{u_i\}$ for $i = 1, 2, \ldots, n$ so that G is not a PVG-graph.

Conversely, let G be the complete graph with vertex set $V = \{x_1, x_2, \ldots, x_p\}$. Since every vertex of G is a simplicial vertex, by Theorem 1.7(i), $S_{x_i} = \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p\}$ for $i = 1, 2, \ldots, p$ is the g_{x_i}-set of G. Thus $S_{x_i} \cup \{x_i\} = V$ for $i = 1, 2, \ldots, p$ and so G is a PVG-graph. □

We leave the following problem as an open question.

Problem 3.6.

(i) *Characterize graphs for which every vertex is a geo-vertex.*

(ii) *Characterize graphs for which no vertex is a geo-vertex.*

(iii) *Characterize VGR-graphs.*

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References

[1] F. Buckley, F. Harary, *Distance in Graphs,* Addison-Wesley, Redwood City, CA(1990).

[2] G. Chartrand, F. Harary, H. Swart, and P. Zhang, Geodomination in graphs, *Bull. Inst. Appl.,* **31** (2001), 51 - 59.

[3] G. Chartrand, F. Harary, P. Zhang, On the geodetic number of a graph, *Networks,* **39** (2002), 1 - 6.

[4] G. Chartrand, P. Zhang, The Steiner Number of a graph, *Discrete Math.,* **242**(2002), 41 - 54.

[5] F.Harary, *Graph Theory,* Addison-Wesley, 1969.

[6] F. Harary, E. Loukakis, C. Tsouros, The geodetic number of a graph, *Math. Comput. Modeling,* **17**(11)(1993), 89 - 95.

[7] P. A. Ostrand, Graphs with specified radius and diameter, *Discrete Math.,* 4(1973), 71 - 75.

[8] A. P. Santhakumaran, P. Titus, Vertex Geodomination in Graphs, *Bulletin of Kerala Mathematics Association,* **2**(2) (2005), 45 - 57.

[9] A. P. Santhakumaran, P. Titus, On the Vertex Geodomination Number of a Graph, *Ars Combin.,* (To appear).