On Strong Biclique Covering

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Abstract

A t-strong biclique covering of a graph G is an edge covering $E(G) = \bigcup_{i=1}^{t} E(H_i)$ where each H_i is a set of disjoint biclique; say $H_{i,1}, ..., H_{i,r_i}$ such that the graph G has no edge between $H_{i,k}$ and $H_{i,j}$ for any $1 \leq j < k \leq r_i$.The strong biclique covering index $S(G)$ is the minimum number t for which there exist a t-strong biclique covering of G. In this paper, we study the strong biclique covering index of graphs. The strong biclique covering index of graphs was introduced in [H. Hajiabolhassan, A. Cheraghi, Bounds for Visual Cryptography Scheme, Discrete Applied Mathematics, **158** (2010), 659-665] to study the pixel expansion of visual cryptology. We present a lower bound for the strong biclique covering index of graphs and also we introduce upper bounds for different products of graphs.

Key words: biclique covering; graph products.

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1 Introduction

Throughout the paper the word graph is used for a finite simple graph. A subgraph H of a graph G is said to be induced if for any pair of vertices u and v of H, $\{u, v\}$ is an edge of H if and only if $\{u, v\}$ is an edge of G. Two graphs G and H are called disjoint if they have no vertex in common. An induced matching in a graph is a set of edges such that no two edges in the set are joined by any third edge of the graph. An induced matching is maximum if the number of edges in it is the largest among all possible induced matchings.

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The size of the maximum induced matching is denoted by $m(G)$. A subgraph of G whose edge set is non-empty and forms a complete bipartite graph is called a *biclique* of G. A *biclique cover* $\mathcal{B}$ of G is a collection of bicliques covering $E(G)$ (every edge of G belongs to at least one biclique of the collection). The *biclique covering number* of G, $bc(G)$, is the fewest number of bicliques among all biclique covers of G.

A proper edge-colouring c is called to be strong coloring if no two edges of the same colour lie on a path of length 3. Equivalently, a strong colouring also corresponds to a partition of the edges into induced matchings. The strong chromatic index of G, denoted by $\chi_s(G)$, is the minimum number of colours of any strong colouring of G, for more about strong chromatic index see [5, 7].

A *t-strong biclique covering* of a graph G is an edge covering $E(G) = \bigcup_{i=1}^{t} E(H_i)$ where each H_i is a set of disjoint biclique; say $H_{i,1}, ..., H_{i,r_i}$ such that the graph G has no edge between $H_{i,k}$ and $H_{i,j}$ for any $1 \leq j < k \leq r_i$.The vertex set of each $H_{i,j}$ is to form $(A_{i,j}, B_{i,j})$ that $A_{i,j}$ and $B_{i,j}$ are two sections of biclique. The strong biclique covering index $S(G)$ is the minimum number t for which there exist a t-strong biclique covering of G. Note that strong biclique covering index can be considered as a generalization of biclique covering and strong chromatic index. Moreover, the strong biclique covering number of graphs provides a good upper bound for the pixel expansion of visual cryptography schemes (for more about visual cryptography see [1, 2, 3, 4, 6]).

In this paper, we study the strong biclique covering index of different graph constructions. In the second section, we present a lower bound for the strong biclique covering index of graphs. Next, we derive upper bounds for the strong biclique covering index of graph product in terms of two factors. Finally, we turn our attention to the strong biclique covering index of the Mycielski graph.

The following notations will be used throughout this paper. For a graph G,denote by $\Delta(G)$ and $\chi(G)$ its maximum degree and its chromatic number, respectively. The Cartesian product $G \square H$ of two graphs G and H has vertex set $V(G) \times V(H)$ in which (a, x) is adjacent to (b, y) if and only if either $ab \in E(G)$ and $x = y$ or $xy \in E(H)$ and $a = b$.
The categorical product $G \times H$ has vertex set $V(G) \times V(H)$ in which

(a, x) is adjacent to (b, y) if and only if $ab \in E(G)$ and $xy \in E(H)$. The strong product $G \boxtimes H$ has vertex set $V(G) \times V(H)$ and edge set $E(G \Box H) \bigcup E(G \times H)$. The Cartesian sum $G \otimes H$ has vertex set $V(G) \times V(H)$, in which (a, x) is adjacent to (b, y) if and only if either $ab \in E(G)$ or $xy \in E(H)$.

2 Bounds

In this section, we introduce a lower bound for the strong biclique covering index of a graph based on the size of the maximum induced matching and maximum degree.

Lemma 1. *For every graph G, $S(G) \geq \frac{E(G)}{m(G) \times \Delta(G)^2}$, where $m(G)$ is the size of the maximum induced matching of G.*

Proof. Consider a strong biclique covering of graph G. For instance, asume that $E(G) = \bigcup_{i=1}^{S(G)} E(H_i)$ where each H_i is a set of disjoint biclique; say $H_{i,1}, \ldots, H_{i,r_i}$. It is a simple matter to check that for any $1 \leq i \leq t$, we have $r_i \leq m(G)$. On the other hand, one can see that every biclique has at most $\Delta(G)^2$ edges. Hence,

$$E(G) \leq S(G) \times m(G) \times \Delta(G)^2.$$

Consequently,

$$S(G) \geq \frac{E(G)}{m(G) \times \Delta(G)^2}.$$

■

Now, we introduce some upper bounds for the strong biclique covering index of graph products.

Theorem 1. *Let G and H be two graphs. Then*

$$S(G \Box H) \leq \chi(G)S(H) + \chi(H)S(G).$$

Proof. Let $E(G) = \bigcup_{r=1}^{S(G)} E(T_r)$ (resp. $E(H) = \bigcup_{r=1}^{S(H)} E(L_r)$) be a strong biclique covering of G (resp. H). Moreover, assume that

$f : V(G) \to \{1, 2, \ldots, \chi(G)\}$ (resp. $g : V(H) \to \{1, 2, \ldots, \chi(H)\}$) is a proper coloring of G (resp. H). Also, suppose that $V(G) := \{u_1, \ldots, u_m\}$ and $V(H) := \{v_1, \ldots, v_n\}$. For any $1 \leq i \leq n$ and $1 \leq j \leq m$, set G_i and H_j to be the induced subgraphs on $\{(u_k, v_i) \mid u_k \in V(G)\}$ and $\{(u_j, v_k) \mid v_k \in V(H)\}$, respectively. Note that G_i's and H_j's are isomorphic to G and H, respectively. Hence, assume that $E(G_i) = \bigcup_{r=1}^{S(G)} E(T_r^i)$ (resp. $E(H_i) = \bigcup_{r=1}^{S(H)} E(L_r^i)$) where T_r^i (resp. L_r^i) is isomorphic to T_r (resp. L_r).

For any $1 \leq k \leq S(G)$ and $1 \leq l \leq \chi(H)$, set

$$K_{k,l} := \bigcup_{r \in g^{-1}(l)} T_k^r.$$

Also, for any $1 \leq k \leq S(H)$ and $1 \leq l \leq \chi(G)$, set

$$K'_{k,l} := \bigcup_{r \in f^{-1}(l)} L_k^r.$$

It is easy to see that $(\bigcup_{k,l} K_{k,l}) \bigcup (\bigcup_{k,l} K'_{k,l})$ is a strong biclique covering of $G \square H$ as claimed. ■

Here, we present an upper bound for the strong biclique covering index of categorical product.

Theorem 2. *Let G and H be two graphs. Then*

$$S(G \times H) \leq 2S(G)S(H).$$

Proof. Let $E(G) = \bigcup_{r=1}^{S(G)} E(T_r)$ (resp. $E(H) = \bigcup_{r=1}^{S(H)} E(L_r)$) be a strong biclique covering of G (resp. H). Furthermore, assume that $T_i = \bigcup_{k=1}^{r_i} T_{i,k}$ (resp. $L_j = \bigcup_{l=1}^{r_j} L_{j,l}$) where each $T_{i,k}$ (resp. $L_{j,l}$) is a complete bipartite graph with the vertex set $(A_{i,k}, B_{i,k})$ (resp. $(C_{j,l}, D_{j,l})$). Now, for any $1 \leq i \leq S(G)$ and $1 \leq j \leq S(H)$ define

$$K_{i,j} := \bigcup_k \bigcup_l (A_{i,k} \times C_{j,l}, B_{i,k} \times D_{j,l})$$

and

$$K'_{i,j} := \bigcup_k \bigcup_l (A_{i,k} \times D_{j,l}, B_{i,k} \times C_{j,l}).$$

It is a simple matter to check that $(\bigcup_{i,j} K_{i,j}) \bigcup (\bigcup_{i,j} K_{i,j})$ is a strong biclique covering of $G \times H$ as claimed. ■

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As the edge set of the strong product $G \boxtimes H$ is the union of the edge set of $G \square H$ and of $G \times H$, we have the following corollary.

Corollary 1 *Let G and H be two graphs. Then*

$$S(G \boxtimes H) \leq \chi(G)S(H) + \chi(H)S(G) + 2S(G)S(H).$$

Here, we introduce an upper bound for the strong biclique covering index of Cartesian sum.

Theorem 3. *Let G and H be two graphs. Then*

$$S(G \otimes H) \leq S(G)m(G) + S(H)m(H).$$

Proof. Let $E(G) = \bigcup_{r=1}^{S(G)} E(T_r)$ (resp. $E(H) = \bigcup_{r=1}^{S(H)} E(L_r)$) be a strong biclique covering of G (resp. H). Furthermore, assume that $T_i = \bigcup_{k=1}^{r_i} T_{i,k}$ (resp. $L_j = \bigcup_{l=1}^{r_j} L_{j,l}$) where each $T_{i,k}$ (resp. $L_{j,l}$) is a complete bipartite graph with the vertex set $(A_{i,k}, B_{i,k})$ (resp. $(C_{j,k}, D_{j,k})$). We can construct a strong biclique covering for $G \otimes H$ as follows. Set

$$H_{ij} = (\bigcup_{l,k}(A_{i,j} \times C_{l,k} \bigcup A_{i,j} \times D_{l,k}), \bigcup_{l,k}(B_{i,j} \times C_{l,k} \bigcup B_{i,j} \times D_{l,k}))$$

and

$$H'_{ij} = (\bigcup_{l,k}(A_{l,k} \times C_{i,j} \bigcup B_{l,k} \times C_{i,j}), \bigcup_{l,k}(A_{l,k} \times D_{i,j} \bigcup B_{l,k} \times D_{i,j})).$$

It is easy to see that H_{ij}'s and H'_{ij}'s are biclique. Moreover, one can see that

$$E(G \otimes H) = (\bigcup H_{ij}) \bigcup (\bigcup H'_{ij}).$$

We can consider each H_{ij}'s or H'_{ij}'s as a colour class. On the other hand, every colour class in G (resp.H) has maximum $m(G)$ (resp.$m(H)$) biclique . Then there is at most $m(G)S(G)+m(H)S(H)$ colour class in new covering for $G \otimes H$. ■

The lexicographic product $G[H]$ of graphs G and H is defined as follows. The vertex set of lexicographic product is $V(G[H]) = V(G) \times V(H)$ and $(x_1, y_1)(x_2, y_2) \in E(G[H])$ if either $x_1 = x_2$ and $y_1y_2 \in E(H)$, or $x_1x_2 \in E(G)$. Now,we present an upper bound for the strong biclique covering index of lexicographic product.

Theorem 4. *Let G and H be two graphs, then*

$$S(G[H]) \leq S(G) + S(H)\chi(G).$$

Proof. Let $|V(H)| = n$ and $\overline{K_n}$ be an empty graph with n vertices. In view of definition of lexicographic product, one can consider $G[\overline{K_n}]$ as a spaning subgraph of $G[H]$. Let $E(G) = \bigcup_{r=1}^{S(G)} E(T_r)$ (resp. $E(H) = \bigcup_{r=1}^{S(H)} E(L_r)$) be a strong biclique covering of G (resp. H). Also, assume that $f : V(G) \to \{1, 2, \ldots, \chi(G)\}$ is a proper coloring of G. Define G_i to be the induced subgraph on $f^{-1}(i)$ which is an empty graph. Now, it is readily seen

$$E(G[H]) = (\bigcup_{r=1}^{S(G)} E(T_r[\overline{K_n}])) \bigcup (\bigcup_{i=1}^{\chi(G)} \bigcup_{j=1}^{S(H)} E(G_i[L_j])).$$

One can check the aforementioned covering is a strong biclique covering of $G[H]$. Hence, $S(G[H]) \leq S(G) + S(H)\chi(G)$. ■

For a graph G, let $[V'(G)]$ be a copy of $V(G)$ (i.e. $[V'(G)] = \{v' \ : v \in V(G)\}$ and let z be a new vertex. The Mycielski graph of G, denoted by $M(G)$, has as the vertex set $V(G) \bigcup [V'(G)] \bigcup \{z\}$

, and the edge set $E(G)\bigcup\{xy' \ : xy \in E(G)\}\bigcup\{y'z \ : y' \in V'(G)\}$. In $M(G)$, the new vertex z is called the *root*, and for each $y \in V(G)$, the copy of y, y', is called the twin of y, and vice versa. Here, we present an upper bound for the strong biclique covering index of $M(G)$.

Theorem 5. *For every graph G ,*

$$S(M(G)) \leq 2S(G) + 1.$$

Proof. Assume that $V(G) := \{v_1, \ldots, v_n\}$. Let $E(G) = \bigcup_{r=1}^{S(G)} E(T_r)$ be a strong biclique covering of G. Furthermore, assume that $T_i = \bigcup_{k=1}^{r_i} T_{i,k}$ where each $T_{i,k}$ is a complete bipartite graph with the vertex set $(A_{i,k}, B_{i,k})$.

We construct a new strong biclique covering for $M(G)$ as follows. If we denote the twin vertices of $A_{i,j}$ by $A'_{i,j}$ (resp.$B_{i,j}$ by $B'_{i,j}$) we have;

$$E(M(G)) = (\bigcup_{i=1}^{S(G)} H_i^1)\bigcup(\bigcup_{t=1}^{S(G)} H_t^2)\bigcup K,$$

where $H_i^1 = \bigcup_{j=1}^{r_i} H_{i,j}^1$ such that $H_{i,j}^1$ is a complete bipartite graph with the vertex set $((A_{i,j}\bigcup A'_{i,j}), B_{i,j})$ and $H_t^2 = \bigcup_{l=1}^{r_i} H_{t,l}^2$ where $H_{t,l}^2$ is a complete bipartite graph with the vertex set $(A_{t,l}, (B'_{t,l}\bigcup B_{t,l}))$. Also, K is a star with the vertex set $V(G) := \{z, v'_1, \ldots, v'_n\}$ and the edge set $\{zv'_1, \ldots, zv'_n\}$. ■

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