L-Presheaves and Their Stalks

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Abstract. The main aim of this paper is to present the idea of L-presheaves on a topological space X. Categorical properties of L-presheaves are studied. The nature of L-presheaves locally in the neighbourhood of some point is summarized. This aim required constructing the notions of category of L-sets, L-direct systems and their L-limits and L-functors with their L-natural transformations. We prove that the "L-stalk" is an L-functor from the category of L-presheaves to the category of L-sets.

1. Introduction.

The rise and development of new fields in this century, for example, general systems theory, language theory, robotics and artificial intelligence, force Mathematicians to be engaged also in the problem of specification of non-precise notions. In 2001-2005, Radwan, Haussein and Hashem published their papers [12], [13] which opened the development of the modified sheaf theory called "fuzzy sheaf theory" in which [0,1] was the measuring grade.

It is known that sheaf theory is a broad generalization of a part of algebraic topology e.g. singular homology theory, see ref.[14]. Sheaves play a fundamental role in the study of cohomology theory; [14], [16] and of commutative and noncommutative algebraic geometry; [3], [4] and [17]. Even in case of graded and filtered levels; [18], [9], [10] and [11]. Sheaf theory depends in its construction and its applications on the theory of categories, functors and ....; [16], [4], [17], [19]. Hence, sheaf theory provides a language for the discussion of geometric objects of many diferent kinds and has main applications in topology and (more especially) in modern algebraic geometry. It has been used in the solution of several long-standing problems.

Theory of lattices is not only fundamental tools in the general theory of rings; [15], but it represents the scale of membership grades in fuzzy set theory and its applications, fuzzy commutative algebra and fuzzy topology theory; [5], [6], [1], [7], [2] and [8]. In the present paper, the infinitely distributive lattice ,L, is the main type of structure with which we shall be considering and using instead of [0,1].

The results here in this paper strengthen [12], [13]. We deal with the concept of what we call category of L-sets , L_S, which may be viewed as a category of solutions for L-presheaves. Of course, the notions of L-direct limit for L-direct system of L-sets is applied to construct L-stalks for L-presheaf which summarize the nature of L-presheaf in the neighbourhoods of the points of the space X. Also we deal with L-presheaf morphism between the L-presheaves.

2. Generalities.

To reach our aim in this paper we need some preparations about a category of L-sets, category of L-direct systems and L-direct limits of L-direct systems which will be used later in the framework of L-presheaves.

2.1 The category of L-sets "L_S".

It can be constructed as follows:

s_1. The objects are (L-sets) of the form (A, μ_A) where A is a set and $\mu_A : A \longrightarrow L$ is a map from A into the lattice L. The object (A, μ_A) is denoted by μ_A.

s_2. For each two objects μ_A, μ_B a set $Hom(\mu_A, \mu_B)$ of morphisms (maps) from μ_A to μ_B. A morphism from μ_A to μ_B is a map $f \equiv \mu_{A\rightarrow B} : A \times B \longrightarrow L$

s_3. For each μ_A, μ_B, μ_C a function (composition) from $Hom(\mu_B, \mu_C) \times Hom(\mu_A, \mu_B)$ to $Hom(\mu_A, \mu_C)$ which is a map $\mu_{A\rightarrow C} : A \times C \longrightarrow L$ such that

$$\mu_{A\rightarrow C}(a,c) = \bigvee_{b\in B} [\mu_{A\rightarrow B}(a,b) \wedge \mu_{B\rightarrow C}(b,c)]; (a,c) \in A \times C.$$

This composition $\mu_{A\rightarrow C} = \mu_{B\rightarrow C} \circ \mu_{A\rightarrow B}$ is called the joint-meet composition or the lattice composition for which we have

s_4. Associative law is satisfied for the L-morphisms, i.e., for any $\mu_{A\rightarrow B} \in Hom(\mu_A, \mu_B), \mu_{B\rightarrow C} \in Hom(\mu_B, \mu_C), \mu_{C\rightarrow D} \in Hom(\mu_C, \mu_D)$ it follows from the distributive condition on L that $\mu_{C\rightarrow D} \circ (\mu_{B\rightarrow C} \circ \mu_{A\rightarrow B}) = (\mu_{C\rightarrow D} \circ \mu_{B\rightarrow C}) \circ \mu_{A\rightarrow B}$.

s_5. For every object $\mu_A \in L_S$ there exists a morphism $I_{\mu_A} \in Hom(\mu_A, \mu_A)$ called the identity of μ_A so that for all $\mu_{A\rightarrow B} \in Hom(\mu_A, \mu_B), \mu_{A\rightarrow B} \circ I_{\mu_A} = I_{\mu_B} \circ \mu_{A\rightarrow B} = \mu_{A\rightarrow B}$. It is noted that I_{μ_A} may be defined in terms of μ_A.

2.1.1 Remarks and examples.

i. See examples in [12], [13] in case of $L = [0,1]$, $\vee = max$, $\wedge = min$.

ii. Let $\alpha \in L$. Define $\mu_A^\alpha : A \to L$ by $\mu_A^\alpha(a) = \alpha, \forall a \in A$. Then (A, μ_A^α) is called the constant L-set.
iii. Recall from [7], let (A, μ_A) be L-set, $\alpha \in L$. Define $(A, \mu_A)^\alpha = \{a \in A : \mu_A(a) \geq \alpha\} \subseteq A$. Then, for $\alpha \leq \beta$ in $L, (A, \mu_A)^\beta \subseteq (A, \mu_A)^\alpha$, and $(A, \mu_A) = (B, \mu_B) \iff A = B, \mu_A = \mu_B$. If $(A, \mu_A) = (B, \mu_B)$, then $(A, \mu_A)^\alpha = (A, \mu_B)^\alpha, \forall \alpha \in L$.
iv. L can be embeded into a geometrical factor lattice $e(L)$ up to an equivlence relation. The later lattice is applied well in algebraic geometry, e.g. constructing the L-germs of sheaves over L. We donot go into this here.
v. Let $\{(A_i, \mu_{A_i}); i \in I \neq \phi \text{ indexed set }\}$ be a family of L-sets, $A = \prod_{i \in I} A_i$ the cartesian product of its members. Then A can be mad into an L-set by:

$$\mu_A : A \longrightarrow L;\ \mu_A((a_i)) = \bigwedge_{i \in I} \mu_{A_i}(a_i),\ \ foreach\,(a_i) \in A.$$

2.2 L-direct systems and their limits.

In this section we construct the second preparation for our aim, that is the category of L-direct systems over L_S (hence forth denoted by $Di(L_S)$). It can be defined in the following manner:

d_1. A direct system of L-sets $\{D, \pi\}$ indexed by a directed set I is a function which assigns to each $i \in I$ an L-set $(D, \mu_D)_i = (D_i, \mu_{D_i})$, and to each pair $i \leq j$ in I an L-morphism $\pi_{ij} = \mu_{D_i \to D_j} : D_i \times D_j \longrightarrow L$ such that, for each $i \in I\ \pi_{ii} : D_i \times D_i \longrightarrow L$ is the identity and for $i \leq j \leq k$ in $I\ \pi_{ik} = \pi_{jk} \circ \pi_{ij}$.

d_2. For each two L-direct systems on $L_S\ \{D, \pi\}, \{D', \pi'\}$ indexed by I, I' respectively a map Φ from $\{D, \pi\}$ to $\{D', \pi'\}$ which consists of an ordered preserving map $\varphi : I \longrightarrow I'$ and for each $i \in I$ a map (L-morphism) $\varphi_i : D_i \times D'_{\varphi(i)} \longrightarrow L$ such that, if $i \leq j$ in I, then we have

$$\varphi_j \circ \pi_{ij}(: D_i \times D'_{\varphi(j)} \longrightarrow L) = \pi_{\varphi(i)\varphi(j)} \circ \varphi_i(: D_i \times D'_{\varphi(j)} \longrightarrow L)$$

d_3. If $\{D, \pi\}$ is L-direct system indexed by I, then the identity $I_{\{D,\pi\}}$ is defined to consist of the identity map on I and the L-identity map

$$I_{\mu_{D_i}} = I_{D_i \to D_i} : D_i \times D_i \longrightarrow L \text{ in } L_S.$$

d_4. If $\Phi : \{D, \pi\} \longrightarrow \{D', \pi'\}, \Phi' : \{D', \pi'\} \longrightarrow \{D'', \pi''\}$ are two L-direct system morphisms on L_S, then their composition from $\{D, \pi\}$ to $\{D'', \pi''\}$ can be defined to consist of the compositions $\varphi \circ \varphi' : I \longrightarrow I''$ and $\varphi'_{\varphi(i)} \circ \varphi_i : D_i \times D_{\varphi' \circ \varphi(i)} \longrightarrow L, i \in I$.

Again we come back our preparation, given an L-direct system $\{D, \pi\}$ in $D_i(L_S)$ over L_S as above, an L-target (denoted by $t(D)$) for $\{D, \pi\}$ is a pair $(E, \{\sigma_i : D_i \times E \longrightarrow L;\ i \in I\})$ consisting of an L-set (E, μ_E) and a collection of L-morphisms $\{\sigma_i : D_i \times E \longrightarrow L, i \in I\}$ such that, for each $i \leq j$ in I $\sigma_i = \sigma_j \circ \pi_{ij}$. An L-direct limit (denoted by $\ell(D)$) for $\{D, \pi\}$ is an L-target satisfying that if $(E', \{\sigma'_i : D_i \times E' \longrightarrow L,\ i \in I\})$ is another L-target for $\{D, \pi\}$, then there is unique L-morphism $\varphi : E' \times E \longrightarrow L$ such that $\sigma_i = \sigma'_i \circ \varphi$.

2.2.1 Proposition.

Constructions and notions are as above:

i. Every L-direct system of L-sets has an L-direct limit.

ii. Any two L-direct limits for an L-direct system are naturally isomorphic.

Proof.

i. Let $\{D, \pi\}$ be a member in $Di(L_S)$ indexed by I and $F = \coprod_{i \in I} D_i$ the L-set of disjoint union of the L-sets D_i's; [7]. On F we define the relation " $\sim_L$ " by: $d_i \sim_L d_j \Longleftrightarrow \exists k \geq i, j$ in I such that $\pi_{ik}(d_i, d_k) = \pi(d_j, d_k)$. We have $i \geq i, i$ in I such that $\pi_{ii}(d_i, d_i) = \pi_{ii}(d_i, d_i)$. Therefore, " $\sim_L$ " is a reflexivity relation. It is clearly that " $\sim_L$ " is symmetric. Finally, if $d_i \sim_L d_j, d_j \sim_L d_k$, then we have $n \geq m, \rho$ where $m \geq i, j$ and $\rho \geq j, k$ such that $\pi_{in}(d_i, d_n) = \pi_{kn}(d_k, d_n)$. So " $\sim_L$ " is an equivalence relation on F. Denote by E the resultant factor L-set $F/\sim_L$ and $\sigma_i : D_j \times E \longrightarrow L$ the composite of the inclusion $D_i \times F \longrightarrow L$ and the natural map $F \times E \longrightarrow L$. Therefore $(E, \{\sigma_i : D_i \times E \longrightarrow L, i \in I\})$ is L-direct limit for $\{D, \pi\}$.

ii. Let $(E, \{\sigma_i : D_i \times E \longrightarrow L; i \in I\}), (E', \sigma'_i : D_i \times E' \longrightarrow L; i \in I\})$ be two L-limits for $\{D, \pi\}$, over L_S, indexed by I. The universality of both targets implies the existence of unique L-morphisms $\varphi : E \times E' \longrightarrow L$ and $\varphi' : E' \times E \longrightarrow L$ such that $\varphi \circ \sigma_i = \sigma'_i$ and $\varphi' \circ \sigma'_i = \sigma_i$. The uniqueness of $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ implies $\varphi \circ \varphi' = I_{E'}, \varphi' \circ \varphi = I_E$, which proves the assertion in ii. □

2.2.2. Remark.

As in [7] it is clear that we can define and construct the same concepts relative to any structure, e.g., of L-rings, L-groups, L-ringed spaces, L-schemes, L-modules, etc. This will give us the ability to construct a new L-commutative (noncommutative) algebraic geometry.

3. L-presheaves and their stalks.

In this section, we give definitions of L-presheaves of L-sets, and of L-morphisms between them. We apply the notion in 2.2 to construct the L-stalks of an L-presheaf. This construction allows us to study some properties of L-presheaf and of its stalks.

We introduce, at first, the notion of L-functor with solutions in the category L_S which may be viewed as a key of the concept of L-presheaf over a topological space X in case that *open*(X) will considered as a category.

An L-functor F, from an arbitrary category $\mathcal{A}$ to L_S, is given by:

t_1. For each an object $A \in \mathcal{A}$ an L-set $F(A)$ is L_S.

t_2. For each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ an L-morphism (a single-valued map) $F(f) : F(A) \times F(B) \longrightarrow L$ which is a morphism from $F(A)$ to $F(A)$, object to the following conditions:

t_3. For each $A \in \mathcal{A}$ $F(I_A) = I_{F(A)} : F(A) \times F(A) \longrightarrow L$.

t_4. For each two morphisms $f : A \longrightarrow B, g : B \longrightarrow C$ in $\mathcal{A}$ $F(g \circ f) = F(g) \circ F(f) : F(A) \times F(C) \longrightarrow L$ whenever $g \circ f$ is defined in $\mathcal{A}$.

3.1 Example.

Let $X = [0,1] \subset \mathbb{R}$ with the usual induced topology, open $(X) = \mathcal{A}$. We can define a $[0,1]$-functor F from $\mathcal{A}$ to $[0,1]_S$ by putting:

$$F(X) = (\mathbb{Z}^+, \mu_{\mathbb{Z}^+} : \mathbb{Z}^+ \longrightarrow [0,1]; \mu_{\mathbb{Z}^+}(n) = \frac{1}{n}, n \in \mathbb{Z}^+),$$
$$F(U) = (\{1\}, \mu_{\{1\}} : \{1\} \longrightarrow [0,1]; \mu_{\{1\}}(1) = 1) \forall U \in \text{open}(X)$$

and ρ_{UV} except ρ_{XU} being identities $\forall V \subseteq U$ in open (X).

If F, G are two L-functors from an arbitrary category $\mathcal{A}$ to L_S, an L-natural transformation T from F to G (i.e. $T : F \times G \longrightarrow L$) is specified by giving for each $A \in \mathcal{A}$ an L-morphism $T_A : F(A) \times G(A) \longrightarrow L$, in such a way that whenever

$$f : A \longrightarrow B \text{ in } \mathcal{A}, T_B \circ F(f) = G(f) \circ T_A.$$

An L-natural transformations can be composed $(T_1 \circ T_2)_A = T_{1A} \circ T_{2A}$, and the L-functors F, G are called L-naturally isomorphic iff $\exists\ T_1 : F \times G \longrightarrow L$ and $T_2 : G \times F \longrightarrow L$ such that $T_1 \circ T_2 = I_G$ and $T_2 \circ T_1 = I_F$. Note that L in the above definition may be viewed as a constant functor.

Now, let X be an ordinary topological space determined by its collection *open*(X) of open subsets. The family *open*(X) is partially ordered by inclusion, hence may be regarded as a category. An L-presheaf of L-sets on X is an L-functor $\underline{F}_X$ from open $(X)^{op}$ to L_S, then for each $U \in$ open(X), we are given an L-set $\underline{F}_X(U) \equiv (\underline{F}_X(U), \mu_{\underline{F}_X(U)})$, and for any smaller $V \subseteq U$ in open(X) an L-morphism $\rho_{UU} = \mu_{\underline{F}_X(U) \rightarrow \underline{F}_X(U)} : \underline{F}_X(U) \times \underline{F}_X(V) \longrightarrow L$ in L_S such that :

b_1. For all $U \in$ open(X) $\rho_{UU} = \mu_{\underline{F}_X(U) \rightarrow \underline{F}_X(U)} = I_{\underline{F}_X(U)}$ in L_S.

b_2. Whenever $W \subseteq V \subseteq U$ in open(X) $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ as the joint-meet composition in L_S.

The elements in $\underline{F}_X(U)$ are called L-sections and the elements in $\underline{F}_X(X)$ are called L-globle sections.

3.2 Example and remark.

i. Let (A, μ_A) be an L-set in L_S. Then the constant L-presheaf $\underline{F_A}_X$ on a topological space X is given by $\underline{F_A}_X(U) = (A, \mu_A) \forall U \in$ open(X) and $\rho_{U_1 U_2} = I_{A \to A} : A \times A \longrightarrow L \forall U_2 \subseteq U_1$ in *open*(X).

ii. The L-presheaf F can be defined over an arbitrary lattice. Hence a generalized sheaf theory again will come.

It is not difficult, by using the notions in Section 1.2, to prove the following result:-

3.3 Lemma.

Let X be a topological space, *open*$(X)_x$ the directed set, by inclusion, of all open sets containing x. Then, for an L-presheaf $\underline{F}_X$ on $X, \{\underline{F}_X(U), \rho_{UV}; V \subseteq U$ in *open*$(X)_x\}$ forms an L-direct system of L-sets.

The L-stalk $\underline{F}_{X,x}$ of $\underline{F}_X$ at $x \in X$ is the L-direct limit for the L-direct system in the above lemma. The elements in $\underline{F}_{X,x}$ are called L-germs. Any element is denoted z_x for some L-section $z \in \underline{F}_X(U), x \in U \in$ *open*$(X)_x$ with $\mu_{\underline{F}_X(U)}(z) \in L$. Two members z_x, z'_x in $\underline{F}_{X,x}$, for some $z \in \underline{F}_X(U), z' \in \underline{F}_X(V)$, are equal $\Longleftrightarrow \exists W \subseteq U \cap V$ such that $\rho_{UW}(z, z'') = \rho_{VW}(z', z'')$; $z'' \in \underline{F}_X(W)$. In example 3.2 above $\underline{F}_{X,x} = (A, \mu_A)$.

An L-presheaf morphism f from $\underline{F}_X$ to $\underline{G}_X$ is given by L-morphisms $f(U) : \underline{F}_X(U) \times \underline{G}_X(U) \longrightarrow L$ for each $U \in$ *open*(X), such that whenever $V \subseteq U$ in *open*(X), $\rho'_{UV} \circ f(U) = f(V) \circ \rho_{UV}$. Composition of such L-presheaf morphisms is defined as: if $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ and $g : \underline{G}_X \times \underline{H}_X \longrightarrow \underline{L}_X$, then

$$g \circ f : \underline{F}_X \times \underline{H}_X \longrightarrow \underline{L}_X \,;\; g \circ f(U) = g(U) \circ f(U)$$

for each $U \in$ *open*(X). From Section 2.1 and the local property we can deduce that " $\circ$ " is an associative. Here, $\underline{L}_X$ can be regarded as constant presheaf on X. The identity $I_{\underline{F}_X} : \underline{F}_X \times \underline{F}_X \longrightarrow \underline{L}_X$ is given by $I_{\underline{F}_X}(U) = I_{\underline{F}_X(U)}$ for each $U \in$open(X). Again, by Section 2.1 and the local property we can verify the properties of $I_{\underline{F}_X}$. An L-presheaf morphism $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ is L-presheaf isomorphism $\Longleftrightarrow \exists g : \underline{G}_X \times \underline{F}_X \longrightarrow \underline{L}_X$ such that $g \circ f = I_{\underline{F}_X}$ and $f \circ g = I_{\underline{G}_X}$. This statement is local, hence $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ is an L-presheaf isomorphism $\Longleftrightarrow f(U), \forall U \in$open$(X)$, is L-isomorphism in L_S.

3.4 Proposition.

Notions and notations are as above:

i. Let X be a topological space, $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ L-presheaf morphism. Then f induces, for each $x \in X$, an L-morphism of stalks $f_x : \underline{F}_{X,x} \times \underline{G}_{X,x} \longrightarrow L$ in L_S.

ii. If $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X, g : \underline{G}_X \times \underline{H}_X \longrightarrow \underline{L}_X$ are two L-presheaf morphisms, then for each $x \in X$, we have

$$(g \circ f)_x = g_x \circ f_x$$

iii. If $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X, g : \underline{G}_X \times \underline{H}_X \longrightarrow \underline{L}_X, h : \underline{H}_X \times \underline{R}_X \longrightarrow \underline{L}_X$ are L-presheaf morphisms, then for each $x \in X$, we have

$$h_x \circ (g_x \circ f_x) = (h_x \circ g_x) \circ f_x$$

iv. If $I_{\underline{F}_X} : \underline{F}_X \times \underline{F}_X \longrightarrow \underline{L}_X$ is the L-presheaf identity, then for each $x \in X$, we have

$$\left(I_{\underline{F}_X}\right)_x = I_{\underline{F}_{X,x}}$$

Therefore, the L-stalk is an L-functor from the category of L-presheaves on X to the category of L-sets L_S.

Proof.

The proof we present here is some what general and in more detials:
(i) Let $x \in X$, one may define f_x as: for $(z_x, z'_x) \in \underline{F}_{X,x} \times \underline{G}_{X,x}; z \in \underline{F}_X(U), z' \in \underline{G}_X(U), x \in U \in open(X)$, put $f_x(z_x, z'_x) = (f(U)(z, z'))_x$. It is not difficult to show that f_x is well defined. For, let $(z_x, z'_x) = (h_x, h'_x)$, where $h \in \underline{F}_X(V), h' \in \underline{G}_X(V), x \in V \in \text{open}(X)$, then $\exists W \subseteq V \cap U$ containing x such that $\rho_{UW}(z, z') = \rho_{VW}(h, h')$, then $f(W)\rho_{UW}(z, z') = f(W)\rho_{VW}(h, h')$. Since f is L-presheaf morphism, then $(f(U)(z, z'))_x = (f(V)(h, h'))_x$ which proves that f_x is well defined.
(ii) If $(z, z'') \in \underline{F}_X(U) \times \underline{H}_X(U), U \in open(X)$, then we do obtain

$$(g \circ f)(U)(z, z'') = (g(U) \circ f(U)(z, z'') = \bigvee_{z' \in \underline{G}_X(U)} [f(U)(z, z') \wedge g(U)(z', z'')].$$

Therefore,

$$\begin{aligned}
(g \circ f)_x(z_x, z''_x) &= ((g \circ f)(U)(z, z''))_x \\
&= \left(\bigvee_{z' \in \underline{G}_X(U)} [f(U)(z, z') \wedge g(U)(z', z'')] \right)_x \\
&= \bigvee_{z'_x \in \underline{G}_{X,x}} [(f(U)(z, z'))_x \wedge (g(U)(z', z''))_x] \\
&= \bigvee_{z'_x \in \underline{G}_{X,x}} [f_x(z_x, z'_x) \wedge g_x(z'_x, z''_x)] \\
&= g_x \circ f_x(z_x, z''_x)
\end{aligned}$$

which proves that, for each $(z_x, z''_x) \in \underline{F}_{X,x} \times \underline{H}_{X,x}, (g \circ f)_x = g_x \circ f_x$
(iii) From above, since $(h \circ (g \circ f))_x = ((h \circ g) \circ f)_x$, we obtain

$$h_x \circ (g_x \circ f_x) = (h_x \circ g_x) \circ f_x.$$

(iv) The statement is local, for, let $f : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ be an L-presheaf morphism. Then for each $U \in open(X)$, we have

$$(I_{\underline{F}_X} \circ f)(U) = I_{\underline{F}_X}(U) \circ f(U) = I_{\underline{F}_X(U)} \circ f(U) = f(U)$$

in L_S. In similar, if $f : \underline{G}_X \times \underline{F}_X \longrightarrow \underline{L}_X$ is an L-morphism of presheaves $\underline{G}_X, \underline{F}_X$ over X, then for every $U \in openX$, we have

$$(f \circ I_{\underline{F}_X})(U) = f(U) \circ I_{\underline{F}_X}(U) = f(U) \circ I_{\underline{F}_X(U)} = f(U)$$

in L_S. Now, if $(z_x, z'_x) \in \underline{F}_{X,x} \times \underline{F}_{X,x}$ where

$$(z, z') \in \underline{F}_X(U) \times \underline{F}_X(U), x \in U \in \; open(X),$$

then we have

$$\left(I_{\underline{F}_X}\right)_x (z_x, z'_x) = (I_{\underline{F}_X}(U)(z, z'))_x = (I_{\underline{F}_X(U)}(z, z'))_x = I_{\underline{F}_{X,x}}(z_x, z'_x)$$

which proves that, for each $x \in X, (I_{\underline{F}_X})_x = I_{\underline{F}_{X,x}}$.

Now, we study L-presheaves which satisfy additional property concerning their L-section. The constant $L = [0,1]$-presheaf mentioned above satisfies this property.

Let X be a topological space and $\underline{F}_X$ an L-presheaf over X. $\underline{F}_X$ is said to satisfy the property "Mos" iff it satisfies the following condition: For every $U \in open(X)$, every open covering $\{U_\alpha\}$ of U (i.e. each $U_\alpha \in open(X)$) and every $(z_1, z_{\alpha_1}), (z_2, z_{\alpha_2}) \in \underline{F}_X(U) \times \underline{F}_X(U_\alpha)$, it follows from $\rho_{UU_\alpha}(z_1, z_{\alpha_1}) = \rho_{UU_\alpha}(z_2, z_{\alpha_2})$ that $(z_1, z_{\alpha_1}) = (z_2, z_{\alpha_2})$ i.e. $z_1 = z_2$ and $z_{\alpha_1} = z_{\alpha_2}$.

3.5 Proposition.

Notions and notations are as above:

i. If $\underline{F}_X, \underline{G}_X$ are L-presheaves over $X, \underline{G}_X$ satisfies the "Mos" property and $f, g : \underline{F}_X \times \underline{G}_X \longrightarrow \underline{L}_X$ are two L-presheaf morphisms such that for each $x \in X$, $f_x = g_x$, then $f = g$.

ii. Let $\underline{F}_X$ be an L-presheaf over X satisfying the property "Mos". Then for any $U \in$ open$(X), z, z' \in \underline{F}_X(U)$ and $z_x = z'_x \forall x \in U$, we have $z = z'$.

Proof.

(i) Let $U \in \text{open}(X)$ and $(z, z') \in \underline{F}_X(U) \times \underline{G}_X(U)$. We have to prove that $f(U)(z, z') = g(U)(z, z')$. Now, $\forall x \in U$, $f_x(z_x, z'_x) = g_x(z_x, z'_x)$ that is $(f(U)(z, z'))_x = (g(U)(z, z'))_x$ and $\exists W \subseteq U \cap U$ containing x such that $\rho_{UW}(f(U)(z, z')) = \rho_{UW}((g(U)(z, z'))$. Applying the property "Mos" for $\underline{G}_X$ to W's of U we see that $f(U)(z, z') = g(U)(z, z')$ which proves that $f = g$.

(ii) If $z_x = z'_x; x \in U \in \text{open}(X)$, then $\exists W \subseteq U \cap U = U$ containing x such that $\rho_{UW}(z, z'') = \rho_{UW}(z', z''); z'' \in \underline{F}_X(W)$. Applying the property "Mos" for $\underline{F}_X$ to W's of U we see that $(z, z'') = (z', z'')$ so that $z = z'$. □

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