Lattices generated by orbits of flats under finite affine-singular symplectic groups

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Abstract Let $ASG(2\nu+l,\nu;\mathbb{F}_q)$ be the $(2\nu+l)$-dimensional affine-singular symplectic space over the finite field $\mathbb{F}_q$ and let $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$ be the affine-singular symplectic group of degree $2\nu+l$ over $\mathbb{F}_q$. For any orbit O of flats under $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$, let $\mathcal{L}$ be the set of all flats which are intersections of flats in O such that $O \subseteq \mathcal{L}$ and assume the intersection of the empty set of flats in $ASG(2\nu+l,\nu;\mathbb{F}_q)$ is $\mathbb{F}_q^{(2\nu+l)}$. By ordering $\mathcal{L}$ by ordinary or reverse inclusion, two lattices are obtained. This article discusses the relations between different lattices,classify their geometricity and computes their characteristic polynomial.

Keywords: lattice; affine-singular symplectic groups; orbit; characteristic polynomial

AMS classification: 20G40 51D25

1. Introduction

We first recall some terminologies and definitions about finite posets and lattices[1,2].

Let P be a poset. For $a, b \in P$, we say a covers b, denoted by $b < \cdot\, a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If P has the minimum (resp.maximum) element, then we denote it by 0 (rep.1) and say that P is a finite poset with 0(resp.1). Let P be a finite poset with 0. By a rank function on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b < \cdot\, a$.

A poset P is said to be a lattice if both $a \vee b := sup\{a, b\}$ and $a \wedge b := inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an atom in P, we mean an element in P *covering* 0. We say P is atomic if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a geometric lattice if P admits a rank function r satisfying

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$$r(a \wedge b) + r(a \vee b) \leq r(a) + r(b), \ \forall\ a, b \in P.$$

Let P be a finite poset with 0 and 1. The polynomial

$$\mathcal{X}(P,t) = \sum_{a \in P} \mu(0,a) t^{r(1)-r(a)}$$

is called the characteristic polynomial of P, μ is the *Möbius* function, r is the rank function on P.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two lattices. If there exists a bijection σ from $\mathcal{L}$ to $\mathcal{L}'$ such that

$$\sigma(a \vee b) = \sigma(a) \vee \sigma(b), \quad \sigma(a \wedge b) = \sigma(a) \wedge \sigma(b), \quad \forall\ a, b \in \mathcal{L},$$

then σ is said to be an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$. In this case we call $\mathcal{L}$ is isomorphic to $\mathcal{L}'$, denoted by $\mathcal{L} \simeq \mathcal{L}'$. It is well know that two isomorphic lattices have the same characteristic polynomial.

In the following we introduce the concepts of affine-singular symplectic spaces. Notation and terminology will be adopted from Wan's book[3].

Suppose $\mathbb{F}_q$ is a finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^{(2\nu+l)}$ be the $(2\nu+l)$-dimensional row vector space over $\mathbb{F}_q$ and let

$$K_\nu = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}, \ K_l = \begin{pmatrix} K_\nu & \\ & 0^{(l)} \end{pmatrix}.$$

The singular symplectic group of degree $2\nu + l$ over $\mathbb{F}_q$, denoted by $Sp_{2\nu+l}(\mathbb{F}_q)$, consists of all $(2\nu+l) \times (2\nu+l)$ matrices T over $\mathbb{F}_q$ satisfying $TK_lT^t = K_l$. The vector space $\mathbb{F}_q^{(2\nu+l)}$ together with the right multiplication action of $Sp_{2\nu+l}(\mathbb{F}_q)$ is called the $(2\nu+l)$-dimensional singular symplectic space over $\mathbb{F}_q$. Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu+l)}$, and let E denote the subspace of $\mathbb{F}_q^{(2\nu+l)}$ generated by $e_{2\nu+1}, e_{2\nu+2}, \cdots, e_{2\nu+l}$, where $e_i (2\nu+1 \leq i \leq 2\nu+l)$ is the row vector in $\mathbb{F}_q^{(2\nu+l)}$ whose i-th coordinate is 1 and all other coordinates are 0. An m-dimensional subspace P in the $(2\nu+l)$-dimensional singular symplectic space is said to be of type (m, s, k), if PK_lP^t is of rank $2s$ and $\dim(P \cap E) = k$. It is known that subspaces of type (m, s, k) exist if and only if $0 \leq k \leq l$, $2s \leq m-k \leq \nu+s$. Denote all subspaces of type (m, s, k) by $M(m, s, k; 2\nu+l, \nu)$.

A coset of $\mathbb{F}_q^{(2\nu+l)}$ relative to a subspace P of type (m, s, k) is called a (m, s, k)-flat. The dimension of a flat $U + x$ is defined to be the dimension of the subspace U, denoted by $\dim(U + x)$. In particular, $(0,0,0)$-flats are points, $(1,0,0)$-flats are lines. A flat F_1 is said to be incident with a flat F_2, if F_1 contains or is contained in F_2. The point set $\mathbb{F}_q^{(2\nu+l)}$ with all the flats and the incidence relation among them defined above is said to be the $(2\nu+l)$-dimensional affine-singular symplectic space, which is denoted by $ASG(2\nu+l, \nu; \mathbb{F}_q)$.

Let $\mathcal{L}_n$ denote the set of all flats in affine space $AG(n, \mathbb{F}_q)$ including the empty set. If we partially order $\mathcal{L}_n$ by reverse inclusion, then $\mathcal{L}_n$ is a lattice (see[4]).

Let F_1, F_2 be two flats in $ASG(2\nu+l, \nu; \mathbb{F}_q)$. The set of points belonging to both F_1 and F_2 is called the intersection of F_1 and F_2, which is denoted by $F_1 \cap F_2$. It follows that the intersection of all flats containing two given flats F_1 and F_2 is the minimum flat containing both F_1 and F_2, which is called the join of F_1 and F_2, which is denoted by $F_1 \cup F_2$.

The set of matrices of the form $\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}$, where $T \in Sp_{2\nu+l,\nu}(\mathbb{F}_q)$ and $v \in \mathbb{F}_q^{(2\nu+l)}$, forms a group under matrix multiplication, which is denoted by $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$ and called the affine-singular symplectic group of degree $2\nu + l$ over $\mathbb{F}_q$. Define the action of $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$ on $ASG(2\nu + l, \nu; \mathbb{F}_q)$ as follows:

$$ASG(2\nu + l, \nu; \mathbb{F}_q) \times ASp_{2\nu+l,\nu}(\mathbb{F}_q) \to ASG(2\nu + l, \nu; \mathbb{F}_q)$$

$$\left(x, \begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix}\right) \quad \mapsto xT + v.$$

The above action induces an action on the set of flats in $ASG(2\nu + l, \nu; \mathbb{F}_q)$, i.e., a flat $P + x$ is carried by

$$\begin{pmatrix} T & 0 \\ v & 1 \end{pmatrix} \in ASG(2\nu + l, \nu; \mathbb{F}_q)$$

into the flat $PT + (xT + v)$. It is known that (m, s, k)-flats exist if and only if $0 \leq k \leq l$, $2s \leq m - k \leq \nu + s$, and that the set of flats of the same type form an orbit under $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$. Denote the orbit of (m, s, k)-flats by $O(m, s, k; 2\nu + l, \nu)$.

For any orbit $O(m, s, k; 2\nu+l, \nu)$ of flats under $ASp_{2\nu+l,\nu}(\mathbb{F}_q)$, let $\mathcal{L}(m, s, k; 2\nu+l, \nu)$ be the set of all flats which are intersections of flats in $O(m, s, k; 2\nu + l, \nu)$ and $O(m, s, k; 2\nu + l, \nu) \subseteq \mathcal{L}(m, s, k; 2\nu + l, \nu)$ and assume the intersection of the empty set of flats in $ASG(2\nu + l, \nu; \mathbb{F}_q)$ is $\mathbb{F}_q^{(2\nu+l)}$. By ordering $\mathcal{L}(m, s, k; 2\nu + l, \nu)$, by ordinary or reverse inclusion, two lattices are obtained, denoted by $\mathcal{L}_O(m, s, k; 2\nu+l, \nu)$, $\mathcal{L}_R(m, s, k; 2\nu+l, \nu)$ respectively. In this article, we discusses the inclusion relations between different lattices, classify their geometricity and computes characteristic polynomial of $\mathcal{L}_R(m, s, k; 2\nu + l, \nu)$.

For any two flats $U+x, W+y \in \mathcal{L}_O(m,s,k;2\nu+l,\nu)$,

$$(U+x)\wedge(W+y)=(U+x)\cap(W+y),$$

$$(U+x)\vee(W+y)=\cap\{P+z\in\mathcal{L}_O(m,s,k;2\nu+l,\nu)|P+z\supseteq(U+x)\cup(W+y)\}.$$

Similarly, for any two flats $U+x, W+y \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$,

$$(U+x)\wedge(W+y)=\cap\{P+z\in\mathcal{L}_R(m,s,k;2\nu+l,\nu)|P+z\supseteq(U+x)\cup(W+y)\},$$

$$(U+x)\vee(W+y)=(U+x)\cap(W+y).$$

Therefore, both $\mathcal{L}_O(m,s,k;2\nu+l,\nu)$ and $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ are finite lattices.

The results on the lattices generated by orbits of subspaces under finite classical groups can be found in Wang and Feng[4], Gao and You[5], Huo Liu and Wan[6-8], Huo and Wan[9], Orlik and Solomon[10], Wang and Guo[11],Guo and Nan[12].

2.The inclusion relations between different lattices

Lemma 2.1 Let $2\nu+l>0$, assume that (m,s,k) satisfies $0\le k<l$, $2s\le m-k\le\nu+s$ and $m\ne 2\nu+l$. Then

$$\mathcal{L}_R(m,s,k;2\nu+l,\nu)\supset\mathcal{L}_R(m-1,s,k-1;2\nu+l,\nu).$$

Proof If $l=1$, we know $O(m-1,s,k-1;2\nu+l,\nu)=\emptyset$. then $\mathcal{L}_R(m-1,s,k-1;2\nu+l,\nu)=\{\mathbb{F}_q^{(2\nu+l)}\}\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Let $l\ge 2$, we need only to show that $O(m-1,s,k-1;2\nu+l,\nu)\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$. Let $P+x\in O(m-1,s,k-1;2\nu+l,\nu)$, where $P\in M(m-1,s,k-1;2\nu+l,\nu)$. From the result in [5,Lemma 3.1], we have $P_1,P_2\in M(m,s,k;2\nu+l,\nu)$ such that $P=P_1\cap P_2$, then $P+x=(P_1\cap P_2)+x=(P_1+x)\cap(P_2+x)\in\mathcal{L}_R(m,s,k;2\nu+l,\nu)$. Hence $O(m-1,s,k-1;2\nu+l,\nu)\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Lemma 2.2 Let $2\nu+l>0$, assume that (m,s,k) satisfies $0\le k<l$, $2s\le m-k\le\nu+s$ and $m\ne 2\nu+l$. Then

$$\mathcal{L}_R(m,s,k;2\nu+l,\nu)\supset\mathcal{L}_R(m-1,s,k;2\nu+l,\nu).$$

Proof If $m-k-2s=0$, we have $O(m-1,s,k;2\nu+l,\nu)=\emptyset$. Hence $\mathcal{L}_R(m-1,s,k;2\nu+l,\nu)=\{\mathbb{F}_q^{(2\nu+l)}\}\subset L_R(m,s,k;2\nu+l,\nu)$.

If $m-k-2s>0$, we need only to show that $O(m-1,s,k;2\nu+l,\nu)\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$. Let $P+x\in O(m-1,s,k;2\nu+l,\nu)$, where $P\in M(m-1,s,k;2\nu+l,\nu)$. From the result in [5,Lemma 3.2], there exist two subspaces $P_1,P_2\in M(m,s,k;2\nu+l,\nu)$ such that $P=P_1\cap P_2$. Hence $P+x=(P_1\cap P_2)+x=(P_1+x)\cap(P_2+x)\in\mathcal{L}_R(m,s,k;2\nu+l,\nu)$. We obtain $O(m-1,s,k;2\nu+l,\nu)\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Lemma 2.3 Let $2\nu+l>0$, assume that (m,s,k) satisfies $0\le k<l$, $2s\le m-k\le \nu+s$ and $m\ne 2\nu+l$. Then

$$\mathcal{L}_R(m,s,k;2\nu+l,\nu)\supset \mathcal{L}_R(m-1,s-1,k;2\nu+l,\nu).$$

Proof If $s=0$, we have $O(m-1,s-1,k;2\nu+l,\nu)=\emptyset$. Hence $\mathcal{L}_R(m-1,s-1,k;2\nu+l,\nu)=\{\mathbb{F}_q^{(2\nu+l)}\}\subset L_R(m,s,k;2\nu+l,\nu)$.

If $s>0$, we need only to show that $O(m-1,s-1,k;2\nu+l,\nu)\subset \mathcal{L}_R(m,s,k;2\nu+l,\nu)$. Let $P+x\in O(m-1,s-1,k;2\nu+l,\nu)$, where $P\in M(m-1,s-1,k;2\nu+l,\nu)$. From the result in [5,Lemma 3.3], there exist two subspaces $P_1,P_2\in M(m,s,k;2\nu+l,\nu)$ such that $P=P_1\cap P_2$. Hence $P+x=(P_1\cap P_2)+x=(P_1+x)\cap(P_2+x)\in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$. We obtain $O(m-1,s-1,k;2\nu+l,\nu)\subset \mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Theorem 2.4. Let $2\nu+l>0$, assume that (m,s,k) and (m_1,s_1,k_1) satisfy $0\le k\le l$, $2s\le m-k\le\nu+s$, $m\ne 2\nu+l$ and $0\le k_1\le l$, $2s_1\le m_1-k_1\le \nu+s_1$, $m_1\ne 2\nu+l$. Then $\mathcal{L}_R(m,s,k;2\nu+l,\nu)\supset \mathcal{L}_R(m_1,s_1,k_1;2\nu+l,\nu)$, if and only if

(i) $s=s_1=\nu,\ k_1\le k<l$;

(ii) $s<\nu,\ k_1=k=l,\ m-m_1\ge s-s_1\ge 0$;

(iii) $s<\nu,\ k_1\le k<l,\ (m-k)-(m_1-k_1)\ge s-s_1\ge 0$.

Proof $\Leftarrow$ First, suppose that (m_1,s_1,k_1) satisfies $s=s_1=\nu$, $k_1\le k<l$. We should show that $\mathcal{L}_R(m,\nu,k;2\nu+l,\nu)\supset\mathcal{L}_R(m_1,\nu,k_1;2\nu+l,\nu)$. Let $k-k_1=h$ $(h\ge 0)$. Since $s=s_1=\nu$, $m=2\nu+k$, $m_1=2\nu+k_1$, $m-m_1=k-k_1=h$. By lemma 2.1, we have $\mathcal{L}_R(m,\nu,k;2\nu+l,\nu)\supset \mathcal{L}_R(m-1,\nu,k-1;2\nu+l,\nu)\supset\ldots\supset\mathcal{L}_R(m-h,\nu,k-h;2\nu+l,\nu)=\mathcal{L}_R(m_1,\nu,k_1;2\nu+l,\nu)$.

Next, suppose that (m_1,s_1,k_1) satisfies $s<\nu$, $k_1=k=l$, $m-m_1\ge s-s_1\ge 0$. Let $s-s_1=t$, $m-m_1=t+t'$ $(t,t'\ge 0)$. By lemma 2.3, we have $\mathcal{L}_R(m,s,l;2\nu+l,\nu)\supset\mathcal{L}_R(m-1,s-1,l;2\nu+l,\nu)\supset\ldots\supset \mathcal{L}_R(m-t,s-t,l;2\nu+l,\nu)=\mathcal{L}_R(m_1+t',s_1,l;2\nu+l,\nu)$.

If $t'=0$, then $\mathcal{L}_R(m,s,l;2\nu+l,\nu)\supset\mathcal{L}_R(m_1,s_1,l;2\nu+l,\nu)$.

If $t'>0$, by lemma 2.2, we know $\mathcal{L}_R(m_1+t',s_1,l;2\nu+l,\nu)\supset\mathcal{L}_R(m_1+t'-1,s_1,l;2\nu+l,\nu)\supset\ldots\mathcal{L}_R(m_1,s_1,l;2\nu+l,\nu)$.

Finally, suppose that (m_1,s_1,k_1) satisfies $s<\nu$, $k_1\le k<l$, $(m-k)-(m_1-k_1)\ge s-s_1\ge 0$. Let $t=s-s_1$, $h=k-k_1$ and $m-m_1=t+h+t'(t,t',h\ge 0)$, by lemma 2.1, we have $\mathcal{L}_R(m,s,k;2\nu+l,\nu)\supset\mathcal{L}_R(m-1,s,k-1;2\nu+l,\nu)\supset\ldots\supset\mathcal{L}_R(m-h,s,k-h;2\nu+l,\nu)=\mathcal{L}_R(m_1+t+t',s,k_1;2\nu+l,\nu)$.

By lemma 2.3, we know $\mathcal{L}_R(m_1+t+t',s,k_1;2\nu+l,\nu)\supset\mathcal{L}_R(m_1+t'+t-1,s-1,k_1;2\nu+l,\nu)\supset\ldots\supset\mathcal{L}_R(m_1+t',s-t,k_1;2\nu+l,\nu)=\mathcal{L}_R(m_1+t',s_1,k_1;2\nu+l,\nu)$.

By lemma 2.2, we have $\mathcal{L}_R(m_1+t',s_1,k_1;2\nu+l,\nu)\supset\mathcal{L}_R(m_1+t'-1,s_1,k_1;2\nu+l,\nu)\supset\ldots\supset\mathcal{L}_R(m_1,s_1,k_1;2\nu+l,\nu)$.

$\Rightarrow$ Suppose that $\mathcal{L}_R(m,s,k;2\nu+l,\nu) \supset \mathcal{L}_R(m_1,s_1,k_1;2\nu+l,\nu)$. (m_1,s_1,k_1) satisfies $0 \le k_1 \le l$, $2s_1 \le m_1-k_1 \le \nu+s_1$, $O(m_1,s_1,k_1;2\nu+l,\nu) \ne \emptyset$. As $O(m_1,s_1,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m_1,s_1,k_1;2\nu+l,\nu)$, we have $O(m_1,s_1,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m,s,k;2\nu+l,\nu)$. For any $Q+x \in O(m_1,s_1,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m,s,k;2\nu+l,\nu)$ and $Q+x \ne \mathbb{F}_q^{(2\nu+l)}$.

(1) If $s=\nu$, $k<l$, in this case $Q+x$ is (m_1,ν,k_1)-flat. Let $P+y \in O(m,\nu,k;2\nu+l,\nu)$ such that $Q+x \subset P+y$, $x \in P+y$. There exist $p \in P$, such that $x=p+y$, $y=p-x$, then $P+y=P+x$. Hence $Q \subset P$, $Q \cap E \subset P \cap E$, $k_1 = \dim(Q\cap E) \le \dim(P \cap E) = k$. Therefore, $s=s_1=\nu$, $k_1 \le k < l$.

(2) If $s<\nu$, let $P+y \in O(m,s,k;2\nu+l,\nu)$ such that $Q+x \subset P+y$, $Q \subset P$, then $Q\cap E \subset P \cap E$, $k_1 = \dim(Q \cap E) \le \dim(P\cap E) = k$. If $k=l$, since $Q+x \in \mathcal{L}_R(m,s,l;2\nu+l,\nu)$, we have $Q+x = \cap(P_i+x) = \cap P_i + x, (i \in I, I\ is\ a\ finite\ set)$, hence $Q=\cap P_i (i\in I)$. By [5 Theorem1] proof, we can get the result.

3.Charaterization of flats in $\mathbb{F}_q^{(2\nu+l)}$ contained in $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$

Theorem 3.1. Let $2\nu+l>0$, (m,s,k) satisfies $0 \le k \le l$, $2s \le m-k \le \nu+s$ and $2\nu+l \ne m$, then

(i) $\mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$ consists of $\mathbb{F}_q^{(2\nu+l)}$ and all (m_1,ν,k_1)-flats satisfy $m_1-k_1=2\nu$ and $0 \le k_1 \le k < l$;

(ii) $\mathcal{L}_R(m,s,l;2\nu+l,\nu) (s<\nu)$ consists of $\mathbb{F}_q^{(2\nu+l)}$ and all (m_1,s_1,l)-flats satisfy $m-m_1 \ge s-s_1 \ge 0$ and $2s_1 \le m_1 - l \le \nu+s_1$;

(iii) $\mathcal{L}_R(m,s,k;2\nu+l,\nu)(s<\nu, k<l)$ consists of $\emptyset$, $\mathbb{F}_q^{(2\nu+l)}$ and all (m_1,s_1,k_1)-flats satisfy $\begin{cases} (m-k)-(m_1-k_1) \ge s-s_1 \ge 0, 0 \le k_1 \le k, \\ 0 \le k_1 < l, 2s_1 \le m_1-k_1 \le \nu+s_1. \end{cases}$

Proof (i) By the agreement, $\mathbb{F}_q^{(2\nu+l)} \in \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$. Let $Q+x$ is a (m_1,ν,k_1)-flat with $m_1-k_1=2\nu$ and $0 \le k_1 \le k < l$. By theorem 2.4, we have $\mathcal{L}_R(m_1,\nu,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$. Since $Q+x \in O(m_1,\nu,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$, hence $Q+x \in \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$.

Conversely, if $Q+x \in \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$ and $Q+x \ne \mathbb{F}_q^{(2\nu+l)}$, by the proof of Theorem 2.4, we know $s_1=\nu$ and $0 \le k_1 \le k < l$.

(ii) By the agreement, $\mathbb{F}_q^{(2\nu+l)} \in \mathcal{L}_R(m,s,l;2\nu+l,\nu)$. Let $Q+x$ be a (m_1,s_1,l)-flat with $m-m_1 \ge s-s_1 \ge 0$ and $2s_1 \le m_1-l \le \nu+s_1$. By Theorem 2.4 we have $Q+x \in O(m_1,s_1,l;2\nu+l,\nu) \subset \mathcal{L}_R(m_1,s_1,l;2\nu+l,\nu) \subset \mathcal{L}_R(m,s,l;2\nu+l,\nu)$.

Conversely, if $Q+x \in \mathcal{L}_R(m,s,l;2\nu+l,\nu)$ and $Q+x \ne \mathbb{F}_q^{(2\nu+l)}$, then $Q+x$ is a (m_1,s_1,l)-flat. Since $Q+x$ is the intersection of flats in $O(m,s,l;2\nu+l,\nu)$, there is $P+y \in O(m,s,l;2\nu+l,\nu)$ such that $Q \subset P$. By the proof of the necessity of Theorem 2.4 We know that $k_1 = l$ and (m_1,s_1,l) satisfies

$m-m_1 \geq s-s_1 \geq 0$, $2s_1 \leq m_1 - l \leq \nu + s_1$.

(iii) By the agreement, $\mathbb{F}_q^{(2\nu+l)} \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)(s<\nu, k<l)$. Let $Q+x$ be a (m_1,s_1,k_1)-flat where (m_1,s_1,k_1) satisfies

$$\begin{cases} (m-k)-(m_1-k_1) \geq s-s_1 \geq 0, 0 \leq k_1 \leq k, \\ 0 \leq k_1 < l, 2s_1 \leq m_1 - k_1 \leq \nu + s_1. \end{cases}$$

By Theorem 2.4, we have $Q+x \in O(m_1,s_1,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m_1,s_1,k_1;2\nu+l,\nu) \subset \mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Conversely, if $Q+x \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$ and $Q+x \neq \mathbb{F}_q^{(2\nu+l)}$, then $Q+x$ is a (m_1,s_1,k_1)-flat. Since $Q+x$ is the intersection of flats in $O(m,s,k;2\nu+l,\nu)$, there is $P+y \in O(m,s,k;2\nu+l,\nu)$ such that $Q+x \subset P+y$, $Q \subset P$. By the proof of Theorem 2.4 we know the conclusion.

Corollary 3.2. Let $2\nu+l>0$, assume that (m,s,k) satisfies $0 \leq k < l$ and $2s \leq m-k \leq \nu+s < 2\nu$. Then $\emptyset \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Corollary 3.3. Let $2\nu+l>0$, assume that (m,s,k) satisfies $0 \leq k \leq l$, $2s \leq m-k \leq \nu+s$ and $m \neq 2\nu+l$. If $P+x \in \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$, $P+x \neq \mathbb{F}_q^{(2\nu+l)}$ and $Q+y$ is a (m_1,ν,k_1)-flat contained in $P+x$, then $Q+y \in \mathcal{L}_R(m,\nu,k;2\nu+l,\nu)$.

Corollary 3.4. Let $2\nu+l>0$, assume that (m,s,k) satisfies $0 \leq k \leq l$, $2s \leq m-k \leq \nu+s < 2\nu$ and $m \neq 2\nu+l$. If $P+x \in \mathcal{L}_R(m,s,l;2\nu+l,\nu)$, $P+x \neq \mathbb{F}_q^{(2\nu+l)}$ and $Q+y$ is a (m_1,s_1,l)-flat contained in $P+x$, then $Q+y \in \mathcal{L}_R(m,s,l;2\nu+l,\nu)$.

Corollary 3.5. Let $2\nu+l>0$, assume that (m,s,k) satisfies $0 \leq k < l$, $2s \leq m-k \leq \nu+s < 2\nu$ and $m \neq 2\nu+l$. If $P+x \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$, $P+x \neq \mathbb{F}_q^{(2\nu+l)}$ and $Q+y \subset P+x$, then $Q+y \in \mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

Corollary 3.6. Let $2\nu+l>0$, then $\mathcal{L}_R(2\nu-1+l,\nu,l-1;2\nu+l,\nu) \cup \mathcal{L}_R(2\nu-1+l,\nu-1,l;2\nu+l,\nu) \cup \mathcal{L}_R(2\nu-1+l-1,\nu-1,l-1;2\nu+l,\nu) = \mathcal{L}_R(2\nu+l,\mathbb{F}_q)$, where $\mathcal{L}_R(2\nu+l,\mathbb{F}_q)$ is the lattice consisting of all flats in $\mathbb{F}_q^{(2\nu+l)}$ by the partially ordering of reverse inclusion.

Proof. It is obvious that the left side of equation is contained in $\mathcal{L}_R(2\nu+l,\mathbb{F}_q)$. Now suppose that $P+x \in \mathcal{L}_R(2\nu+l,\mathbb{F}_q)$. If $P+x = \mathbb{F}_q^{(2\nu+l)}$, then $P+x$ is contained in every set on the left of equation. If $P+x \neq \mathbb{F}_q^{(2\nu+l)}$, then $P+x$ is a (m,s,k)-flat. When $k=l$, $s<\nu$. Since (m,s,l) satisfies $l \geq 0$, $2s \leq m-l \leq \nu+s$ and $m \neq 2\nu+l$. Then $(2\nu-1+l-l)-(m-l) = 2\nu-1-(m-l) \geq 2\nu-1-(\nu+s) = (\nu-1)-s \geq 0$, we have $P+x \in \mathcal{L}_R(2\nu-1+l, \nu-1, l; 2\nu+l, \nu)$. When $s=\nu$, $k<l$. Since (m,ν,k) satisfies $2s \le m-k \le \nu+s$ and $m \ne 2\nu+l$, then $m-k=2\nu$. We have $P+x \in \mathcal{L}_R(2\nu+l-1, \nu, l-1; 2\nu+l, \nu)$. When $k<l$ and $s<\nu$, (m,s,k) satisfies $2s \le m-k \le \nu+s$ and $m \ne 2\nu+l$. Then $(2\nu-1+l-1)-(l-1)-(m-k) = 2\nu-1-(m-k) \ge 2\nu-1-(\nu+s) = (\nu-1)-s \ge 0$. So $P+x \in \mathcal{L}_R(2\nu-1+l-1, \nu-1, l-1; 2\nu+l, \nu)$.

Summarizing the above arguments, we obtain the equation.

Corollary 3.7. Let $2\nu \ge 2$. Assume that (m,s,k) satisfies $0 \le k < l$, $2s \le m-k \le \nu+s$ and $m \ne 2\nu+l$. Then $\mathcal{L}_R(m,s,k; 2\nu+l, \nu)$ consists of $\mathbb{F}_q^{(2\nu+l)}$, $\emptyset$ and all (m_1,s_1,k_1)-flats with $0 \le k_1 \le k < l$ and $0 \le s-s_1 \le (m-k)-(m_1-k_1)$.

4. The geometricity of lattices $\mathcal{L}_R(m,s,k; 2\nu+l, \nu)$ and $\mathcal{L}_O(m,s,k; 2\nu+l, \nu)$

By Corollary 3.3, 3.4 we easily obtain the following two lemmas.

Lemma 4.1 If $0<k<l$, then

(i) $\mathcal{L}_R(2\nu+k, \nu, k; 2\nu+l, \nu) \simeq \mathcal{L}_R(k,l)$, $\mathcal{L}_O(2\nu+k, \nu, k; 2\nu+l, \nu) \simeq \mathcal{L}_O(k,l)$.

(ii) $\mathcal{L}_R(k,0,k; 2\nu+l, \nu) \simeq \mathcal{L}_R(k,l)$, $\mathcal{L}_O(k,0,k; 2\nu+l, \nu) \simeq \mathcal{L}_O(k,l)$.

Lemma 4.2 If $0 \le s < \nu$ and $2s \le m-l \le \nu+s$, then

(i) $\mathcal{L}_R(m,s,l; 2\nu+l, \nu) \simeq \mathcal{L}_R(m-l, s; 2\nu)$, $\mathcal{L}_O(m,s,l; 2\nu+l, \nu) \simeq \mathcal{L}_O(m-l, s; 2\nu)$.

(ii) $\mathcal{L}_R(m,s,0; 2\nu+l, \nu) \simeq \mathcal{L}_R(m,s; 2\nu)$, $\mathcal{L}_O(m,s,0; 2\nu+l, \nu) \simeq \mathcal{L}_O(m,s; 2\nu)$.

Theorem 4.3 Let $2\nu+l>0$, assume that (m,s,k) satisfies $0 \le k \le l$, $2s \le m-k \le \nu+s$ and $m \ne 2\nu+l$. Then

(i) For $k=0$, $\mathcal{L}_O(m,s,k; 2\nu+l, \nu)$ is a finite geometric lattice when $m=0,1,2\nu-1$.

(ii) For $k=l$, $\mathcal{L}_O(m,s,k; 2\nu+l, \nu)$ is a finite geometric lattice when $m-l=0,1,2\nu-1$.

(iii) $\mathcal{L}_O(k,0,k; 2\nu+l, \nu)$ is a geometric lattice when $0<k<l$.

(iv) $\mathcal{L}_O(2\nu+k, \nu, k; 2\nu+l, \nu)$ is a finite geometric lattice when $0<k<l$.

(v) For $0<s<\nu$, $\mathcal{L}_O(m,s,k; 2\nu+l, \nu)$ is not a finite geometric lattice when $0<k<l$.

Proof For any flat $U+x \in \mathcal{L}_O(m,s,k; 2\nu+l, \nu)$,define

$$r_O(U+x)=\begin{cases} 0, & \text{if } U+x=\emptyset,\\ m+2, & \text{if } U+x=\mathbb{F}_q^{(2\nu+l)},\\ \dim(U+x)+1, & \text{otherwise.}\end{cases}$$

(i) If $k=0$, $\mathcal{L}_O(m,s,0;2\nu+l,\nu)\simeq\mathcal{L}_O(m,s;2\nu)$ [12,Theorem 3.2].

(ii) If $k=l$, $\mathcal{L}_O(m,s,l;2\nu+l,\nu)\simeq\mathcal{L}_O(m-l,s;2\nu)$ [12,Theorem 3.2].

(iii),(iv),(v) If $0<k<l$. By Theorem 3.1 all $(0,0,0)$-flats contained in $\mathcal{L}_O(m,s,k;2\nu+l,\nu)$. For any $F\in\mathcal{L}_O(m,s,k;2\nu+l,\nu)\setminus\emptyset$, F is union of some $(0,0,0)$-flats. It is obvious that $\mathcal{L}_O(m,s,k;2\nu+l,\nu)$ is a finite atomic lattice.

We distinguish the following two cases.

1. $m-k=0$, then $\mathcal{L}_O(m,s,k;2\nu+l,\nu)=\mathcal{L}_O(k,0,k;2\nu+l,\nu)\simeq\mathcal{L}_O(k,l)$ [4,Theorem 1.1].

2. $0<2s\le m-k\le\nu+s$.

(1) when $s=\nu$, $\mathcal{L}_O(2\nu+k,\nu,k;2\nu+l,\nu)\simeq\mathcal{L}_O(k,l)$ [4,Theorem 1.1].

(2) when $0<s<\nu$, let $F_1=V_1+x_1$ and $F_2=V_2+x_2$ be $(m-1,s-1,k)$-flat and $(k+1,0,k)$-flat. When $F_1\vee F_2=\mathbb{F}_q^{(2\nu+l)}$, let

$$V_1=\left(\begin{array}{ccccccccc} I^{(s-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I^{(s)} & 0 & 0 & 0 & 0\\ 0 & 0 & I^{(m-k-2s)} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0\end{array}\right),$$
$$\qquad\quad s-1 \quad 1 \quad m-k-2s \quad \nu+s-m+k \quad s \quad m-k-2s \quad \nu+s-m+k \quad k \quad l-k$$

$$V_2=\left(\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k)} & 0\end{array}\right).$$
$$\qquad\quad s-1 \quad 1 \quad m-k-2s \quad 1 \quad \nu+s-m+k \quad s \quad m-k-2s \quad \nu+s-m+k \quad k \quad l-k$$

Then $r_O(F_1\vee F_2)+r_O(F_1\wedge F_2)=m+3+\dim(V_1\cap V_2)>r_O(F_1)+r_O(F_2)=m+k+2$.

Hence $\mathcal{L}_O(m,s,k;2\nu+l,\nu)$ is not a geometric lattice in this condition.

Theorem 4.4 Let $2\nu+l>0$. Assume that (m,s,k) satisfies $0\le k\le l$, $2s\le m-k\le\nu+s$ and $m\ne 2\nu+l$. Then

(i) For $k=0$, $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ is a finite geometric lattice when $m=0$.

(ii) For $k=l$, $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ is a finite geometric lattice when $m-l=0$.

(iii) $\mathcal{L}_R(k,0,k;2\nu+l,\nu)$ is a finite geometric lattice when $k=0$.

(iv) $\mathcal{L}_R(2\nu+k,\nu,k;2\nu+l,\nu)$ is a finite geometric lattice when $k=0$.

(v) For $0<s<\nu$, $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ is not a finite geometric lattice when $0<k<l$.

Proof For any flat $U+x\in\mathcal{L}_R(m,s,k;2\nu+l,\nu)$, define

$$r_R(U+x)=\begin{cases} m+2, & \text{if } U+x=\emptyset,\\ 0, & \text{if } U+x=\mathbb{F}_q^{(2\nu+l)},\\ m+1-\dim F, & \text{otherwise.}\end{cases}$$

(i) If $k=0$, $\mathcal{L}_R(m,s,0;2\nu+l,\nu)\simeq\mathcal{L}_R(m,s;2\nu)$ [12,Theorem 3.3].

(ii) If $k=l$, $\mathcal{L}_R(m,s,l;2\nu+l,\nu)\simeq\mathcal{L}_R(m-l,s;2\nu)$ [12,Theorem 3.3].

(iii),(iv),(v) If $0<k<l$. By Theorem 3.1 all (m,s,k)-flats contained in $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$. For any $F\in\mathcal{L}_R(m,s,k;2\nu+l,\nu)\setminus\emptyset$, F is union of some (m,s,k)-flats. It is obvious that $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ is a finite atomic lattice.

We distinguish the following two cases.

1. $m-k=0$, then $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$=$\mathcal{L}_R(k,0,k;2\nu+l,\nu)\simeq\mathcal{L}_R(k,l)$ [4,Theorem 1.2].

2. $m-k>0$, $0<2s\le m-k\le\nu+s$.

(1) when $s=\nu$, $\mathcal{L}_R(2\nu+k,\nu,k;2\nu+l,\nu)\simeq\mathcal{L}_R(k,l)$ [4,Theorem 1.2].

(2) when $0<s<\nu$,

let $F_1=V_1+x_1=<e_1,\dots,e_s,e_{s+1},\dots,e_{s+m-k-2s},e_{\nu+1},\dots,e_{\nu+s},e_{2\nu+1},\dots,e_{2\nu+k}>+x_1$, and $F_2=V_2+x_2=<e_{\nu-s+1},\dots,e_{\nu-s+s},e_{\nu+(\nu-m+k+s)+1},\dots,e_{\nu+\nu-s},e_{\nu+\nu-s+1},\dots,e_{2\nu},e_{2\nu+1},\dots,e_{2\nu+k}>+x_2$. $F_1,F_2\in O(m,s,k;2\nu+l,\nu)\subset\mathcal{L}_R(m,s,k;2\nu+l,\nu)$.

If $m-k=2s$ and $s\le\nu-1$. When $m-k>2s$, $s\le\nu-2$ and $F_1\cap F_2\ne\emptyset$. Since $F_1\cup F_2=(V_1+V_2+<x_2-x_1>)+x_1$, $x_2-x_1\in V_1+V_2$, so $\dim(F_1\cup F_2)\ge m+2$, $\dim(F_1\cap F_2)$ =$\dim(V_1\cap V_2)\le m-2$. Clearly $F_1\wedge F_2=\mathbb{F}_q^{(2\nu+l)}$.

Then $r_R(F_1\vee F_2)+r_R(F_1\wedge F_2)\ge 3> r_R(F_1)+r_R(F_2)$= 2.

Hence $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$ is not a geometric lattice in this condition.

5. Characteristic polynomial of lattice $\mathcal{L}_R(m,s,k;2\nu+l,\nu)$

Theorem 5.1. Let $2s\le m-k\le\nu+s$, $0\le k\le l$. Then

$\mathcal{X}(\mathcal{L}_R(m,s,k;2\nu+l,\nu),t)=\sum\limits_{s_1=0}^{s}\sum\limits_{k_1=k+1}^{l}\sum\limits_{m_1=2s_1+k_1}^{\nu+s_1+k_1}q^{2\nu+l-m_1}N(m_1,s_1,k_1;2\nu+l,\nu)\mathcal{X}(\mathcal{L}_n,t)+$
$\sum\limits_{s_1=s+1}^{\nu}\sum\limits_{k_1=0}^{k}\sum\limits_{m_1=2s_1+k_1}^{\nu+s_1+k_1}q^{2\nu+l-m_1}N(m_1,s_1,k_1;2\nu+l,\nu)\mathcal{X}(\mathcal{L}_n,t)+\sum\limits_{s_1=0}^{s}\sum\limits_{k_1=0}^{k}$
$\sum\limits_{m_1=m-k-s+s_1+k_1+1}^{\nu+s_1+k_1}q^{2\nu+l-m_1}N(m_1,s_1,k_1;2\nu+l,\nu)\mathcal{X}(\mathcal{L}_n,t)$, where $N(m_1,s_1,k_1;2\nu+l,\nu)$ is the number of type (m_1,s_1,k_1) subspaces in $\mathbb{F}_q^{(2\nu+l)}$.

Proof For convenience, we write $V=\mathbb{F}_q^{(2\nu+l)}$, $\mathcal{L}=\mathcal{L}_R(m,s,k;2\nu+l,\nu)$, $\mathcal{L}_0=\mathcal{L}_R(2\nu+l,V)$ where $\mathcal{L}_R(2\nu+l,V)$ is all flats in $ASG(2\nu+l,\nu;\mathbb{F}_q)$.

For m-dimensional flat $U+x \in \mathcal{L}$, define

$$\mathcal{L}^{U+x} = \{W+y \in \mathcal{L} | W+y \geq U+x\}$$

$$\mathcal{L}_0^{U+x} = \{W+y \in \mathcal{L}_0 | W+y \geq U+x\}.$$

Clearly, $\mathcal{L}^V = \mathcal{L}$. For $U+x \in \mathcal{L} \setminus V$, by Corollary 3.5 we get $\mathcal{L}^{U+x} = \mathcal{L}_0^{U+x}$. Therefore, the characteristic polynomial of $\mathcal{L}$ is

$$\mathcal{X}(\mathcal{L}^V, t) = \mathcal{X}(\mathcal{L}, t) = \sum_{U+x \in \mathcal{L}} \mu(V, U+x) t^{r_R(\emptyset) - r_R(U+x)}.$$

$\mathcal{L}_0^V = \mathcal{L}_0$,

$$\mathcal{X}(\mathcal{L}_0^V, t) = \mathcal{X}(\mathcal{L}_0, t) = \sum_{U+x \in \mathcal{L}_0} \mu(V, U+x) t^{r_R(\emptyset) - r_R(U+x)}.$$

From *Möbius* inversion formula

$$t^{m+2} = \sum_{U+x \in \mathcal{L}^V} \mathcal{X}(\mathcal{L}^{U+x}, t) = \sum_{U+x \in \mathcal{L}} \mathcal{X}(\mathcal{L}^{U+x}, t),$$

$$t^{m+2} = \sum_{U+x \in \mathcal{L}_0^V} \mathcal{X}(\mathcal{L}_0^{U+x}, t) = \sum_{U+x \in \mathcal{L}_0} \mathcal{X}(\mathcal{L}_0^{U+x}, t).$$

Thus,

$$\begin{aligned}\mathcal{X}(\mathcal{L}, t) = \mathcal{X}(\mathcal{L}^V, t) &= t^{m+2} - \sum_{U+x \in \mathcal{L} \setminus V} \mathcal{X}(\mathcal{L}^{U+x}, t) \\ &= \sum_{U+x \in \mathcal{L}_0} \mathcal{X}(\mathcal{L}_0^{U+x}, t) - \sum_{U+x \in \mathcal{L} \setminus V} \mathcal{X}(\mathcal{L}^{U+x}, t) \\ &= \sum_{U+x \in (\mathcal{L}_0 \setminus \mathcal{L} \cup V)} \mathcal{X}(\mathcal{L}_0^{U+x}, t).\end{aligned}$$

By Corollary 3.5, $U+x \in (\mathcal{L}_0 \setminus \mathcal{L} \cup V)$ if and only if $\{U+x \in \mathcal{L}_0 | U+x \text{ is } (m_1, s_1, k_1)\text{-flat}, l \geq k_1 > k\} \cup \{U+x \in \mathcal{L}_0 | U+x \text{ is } (m_1, s_1, k_1)\text{-flat}, s < s_1\} \cup \{U+x \in \mathcal{L}_0 | U+x \text{ is } (m_1, s_1, k_1)\text{-flat}, (m-k) - (m_1 - k_1) < s - s_1\}$.
Thus,

$$\begin{aligned}\mathcal{X}(\mathcal{L}_R(m, s, k; 2\nu+l, \nu), t) =& \sum_{s_1=0}^{s} \sum_{k_1=k+1}^{l} \sum_{m_1=2s_1+k_1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu+l, \nu) \mathcal{X}(\mathcal{L}_0^{U+x}, t) \\ &+ \sum_{s_1=s+1}^{\nu} \sum_{k_1=0}^{k} \sum_{m_1=2s_1+k_1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu+l, \nu) \mathcal{X}(\mathcal{L}_0^{U+x}, t) \\ &+ \sum_{s_1=0}^{s} \sum_{k_1=0}^{k} \sum_{m_1=m-k-s+s_1+k_1+1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu+l, \nu) \mathcal{X}(\mathcal{L}_0^{U+x}, t),\end{aligned}$$

where $N(m_1, s_1, k_1; 2\nu + l, \nu)$ is the number of type (m_1, s_1, k_1) subspaces in $\mathbb{F}_q^{(2\nu+l)}$.

It is a routine to show that $\mathcal{L}_0^{U+x} \simeq \mathcal{L}_n$ where n=dim$(U + x) - k$. Hence both the lattices $\mathcal{L}_0^{U+x}$ and $\mathcal{L}_n$ have the same characteristic polynomial.

Hence
$\mathcal{X}(\mathcal{L}_R(m, s, k; 2\nu+l, \nu), t) = \sum_{s_1=0}^{s} \sum_{k_1=k+1}^{l} \sum_{m_1=2s_1+k_1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu+l, \nu)\mathcal{X}(\mathcal{L}_n, t) +$
$\sum_{s_1=s+1}^{\nu} \sum_{k_1=0}^{k} \sum_{m_1=2s_1+k_1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu + l, \nu)\mathcal{X}(\mathcal{L}_n, t) +$
$\sum_{s_1=0}^{s} \sum_{k_1=0}^{k} \sum_{m_1=m-k-s+s_1+k_1+1}^{\nu+s_1+k_1} q^{2\nu+l-m_1} N(m_1, s_1, k_1; 2\nu+l, \nu)\mathcal{X}(\mathcal{L}_n, t)$, where $N(m_1, s_1, k_1; 2\nu + l, \nu)$ is the number of type (m_1, s_1, k_1) subspaces in $\mathbb{F}_q^{(2\nu+l)}$.$\square$

Acknowledgements This work is supported by the National Natural Science Foundation of China under Grant No.61179026 and the Fundamental Research Funds for the Central Universities under Grant No.ZXH2012K003.

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