An upper bound for the radio number of generalized gear graph *

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Abstract

Let $d(u,v)$ denote the *distance* between two distinct vertices of a connected graph G and $diam(G)$ be the *diameter* of G. A *radio labeling* f of G is an assignment of positive integers to the vertices of G satisfying $d(u,v)+|f(u)-f(v)| \geq diam(G)+1$. The maximum integer in the range of the labeling is its *span*. The *radio number* of G, denoted by $rn(G)$, is the minimum possible span. In [7] M. Farooq et al. found the lower bound for the radio number of generalized gear graph. In this paper we give upper bound for the radio number of generalized gear graph, which coincide with the lower bound found in [7] .

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1 Introduction

A labeling of a graph $G(V,E)$, is a map that carries graph elements to numbers (usually to the positive or non-negative integers). The most common choices of domain are the set of all vertices and edges (*total labelings*), the vertex-set alone (*vertex-labelings*), or the edge-set alone (*edge-labelings*). In this paper we consider a type of vertex labeling known as the *multi-level distance labeling* or *radio labeling* of graphs. Multi-level distance labeling is

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motivated by restrictions inherent in assigning channel frequencies for radio transmitters. We consider simple and connected graphs $G = (V(G), E(G))$. We write $d(u, v)$ for the *distance* between vertices u and v, and use $diam(G)$ to denote the *diameter* of G.

A radio labeling [1] is a one-to-one mapping $f : V(G) \to I\!N$ satisfying the condition,

$$d(u, v) + |f(u) - f(v)| \geq diam(G) + 1$$

for every u, $v \in V(G)$. The span of a labeling f is the maximum integer in the range of f. The *radio number* of G denoted by $rn(G)$ is the lowest span taken over all radio labelings of the graph G.

The *generalized gear graph* $J_{t,n}$ is obtained from a *wheel* graph on $n+1$ vertices by introducing t-vertices between every pair (v_i, v_{i+1}) of adjacent vertices on the n-cycle of wheel. We will denote the central vertex by z. For each $i = 1, 2, \ldots, n$ the tvertices between v_i and v_{i+1} are denoted by, $b_i^j; j = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 2$, and $a_{i+1}^j; j = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 1$, for odd t.

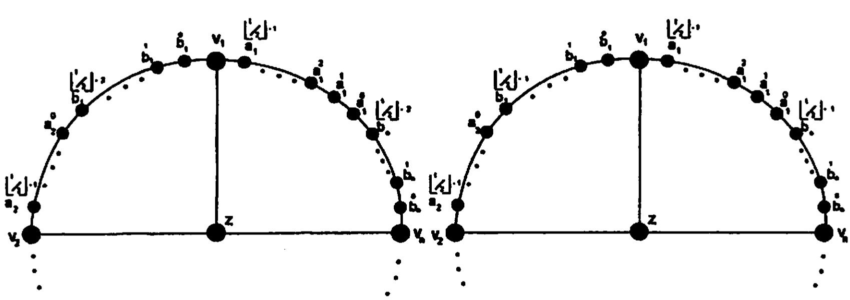
For even t these vertices are denoted by

$b_i^j; j = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 1$, and $a_{i+1}^j; j = 0, 1, \ldots, \lfloor \frac{t}{2} \rfloor - 1$.

The diameter of $J_{t,n}$ for $t \geq 1$, $n > 3$ is given by,

$$diam(J_{t,n}) = \begin{cases} t+2, & \text{t even;} \\ t+3, & \text{t odd.} \end{cases}$$

Some of the contribution of various authors to the area of radio labeling



(a) (b)

Figure 1: $rn(J_{t,n})$ when (a) t is odd and (b) when t is even

of graphs are given below.

Theorem 1.1 [1] *If G is a connected graph of order n and diameter 2, then $n \leq rn(G) \leq 2n - 2$, and for every pair of integers k and n with*

$n \leq k \leq 2n-2$, there exists a connected graph of order n and diameter 2 with $rn(G) = k$.

Theorem 1.2 [2] *The radio number of the complete graph on n vertices is n, i.e. $rn(K_n) = n$.*

Theorem 1.3 [2] $rn(J_{1,n}) = 4n + 2$ *for* $n \geq 4$.

Theorem 1.4 [5] *For odd* $n \geq 3$; $rn(P_n) = \frac{1}{2}(n-1)^2 + 2$;
For even $n \geq 4$, $rn(P_n) = \frac{n^2}{2} - n + 1$.

For a connected graph G of order n, let DG represent the weighted complete graph K_n having $V(K_n) = V(G)$. The length of an edge ij is defined by $l(ij) = d_G(i,j)$, and $hp_{max}(DG)$ denotes the maximum length of a hamiltonian path in DG. Using these notations M. T. Rahim and I. Tomescu found a lower bound for the radio number of a graph G in terms of the length of the maximum hamiltonian path in the corresponding weighted completed graph.

Theorem 1.5 [6] $rn(G) \geq (n-1)(diam(G)+1) - hp_{max}(DG) + 1$.

Theorem 1.6 [7] *For* $t < n-1$, $n \geq 7$, $rn(J_{t,n}) \geq \frac{1}{2}(nt^2 + 4nt + 3n + 4)$.

The most complete survey on the radio labeling of graphs can be found in [3].

2 Upper bound for the radio number of generalized gear graph $J_{t,n}$

2.1 An upper bound for $rn(J_{t,n})$, when $t < (n-1)$

Theorem 2.1 *For* $t < n-1$, $n \geq 7$, $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 3n + 4)$.

Proof: We will define a labeling $f : V(J_{t,n}) \rightarrow \mathbb{N}$ and claim that this labeling is a radio labeling . The span of this labeling will provide an upper bound for the radio number of $J_{t,n}$.
Before defining the labeling we rename the vertices of $J_{t,n}$, by defining a position function $P : V(J_{t,n}) \longrightarrow \{x_0, x_1, \ldots, x_{(t+1)n}\}$. For $n = 2k+1$, P is defined as follows;
$P(z) = x_0$,
$P(a^j_{2i-1}) = x_{i+j(2k+1)}$, for every $i = 1, 2, ..., k+1$, $j = 0, 1, ..., \lceil t/2 \rceil - 1$
$P(a^j_{2i}) = x_{i+j(2k+1)+k+1}$, for every $i = 1, 2, ..., k$, $j = 0, 1, ..., \lceil t/2 \rceil - 1$
$P(b^j_{2i-1}) = x_{i+2k(j+\lceil \frac{t}{2} \rceil)+2j+\lceil \frac{t}{2} \rceil}$, for every $i = 1, 2, ..., k+1$, $j = 0, 1, ..., \lceil t/2 \rceil - 2$
$P(b^j_{2i}) = x_{i+2k(j+\lceil \frac{t}{2} \rceil)+k+2j+1+\lceil \frac{t}{2} \rceil}$, for every $i = 1, 2, ..., k$, $j = 0, 1, ..., \lceil t/2 \rceil - 2$
$P(v_i) = x_{i+tn}$, for $i = 1, 2, ..., n$.

For $n = 2k$, the position function P is defined as,
$P(z) = x_0$,
$P(a^j_{2i-1}) = x_{i+2jk}$, for every $i = 1,2,...,k$, $j = 0,1,...,\lceil t/2\rceil - 1$
$P(a^j_{2i})) = x_{i+k(2j+1)}$, for every $i = 1,2,...,k$, $j = 0,1,...,\lceil t/2\rceil - 1$
$P(b^j_{2i-1}) = x_{i+2k(j+\lceil \frac{t}{2}\rceil)}$, for every $i = 1,2,...,k$, $j = 0,1,...,\lceil t/2\rceil - 1$
$P(b^j_{2i}) = x_{i+2k(j+\lceil \frac{t}{2}\rceil)+k}$, for every $i = 1,2,...,k$, $j = 0,1,...,\lceil t/2\rceil - 1$
$P(v_i) = x_{i+tn}$, for $i = 1,2,...,n$.
Now we define the labeling $f : V(J_{t,n}) \to \mathbb{N}$. We specify the labels $f(x_i)$ so that $i < e$, if and only if
$f(x_i) < f(x_e)$.

$$f(x_i) = \begin{cases} 1, & i=0, \\ A + (2j+1)(i - jn), & i = 1,2,\dots,\lceil t/2\rceil n, \\ B + C + (t-2j)[i - (j + \lceil \frac{t}{2}\rceil)n], & i = \lceil \frac{t}{2}\rceil n + 1,\dots,tn, \\ \frac{n(t^2-1)}{2} + 2 + tn + (t+2)(i - tn), & i = tn+1,\dots,(t+1)n. \end{cases}$$

Where $A = \lceil \frac{t+5}{2}\rceil - 1 + (j-1)[j(t+3) - 1] + jn$,
$B = \{\lceil \frac{t+5}{2}\rceil - 1 + (\lceil \frac{t}{2}\rceil - 2)[\lceil \frac{t}{2}\rceil(t+3) - (t+4)]\}$,
and $C = n(t-1) + j[2t + 3 + n - j(t+3) + 2\lfloor \frac{t}{2}\rfloor(t+2)] + n\lceil \frac{t}{2}\rceil$.

Claim: The labeling f is a valid radio labeling. For this we must show that the radio condition,

$$d(v,u) + |f(v) - f(u)| \geq 1 + diam(J_{t,n}) = t + 4 \quad (1)$$

holds for all pairs of distinct vertices (u,v). For this we consider the following cases.

Case(1). Consider the pairs (z,r), for any vertex $r \neq z$. Consider the pairs (z,r), such that $d(z,r) = 1$, and recall $P(z) = x_0, f(x_0) = 1$, $P(r) = P(v_i) = x_{i+tn}$, $f(x_{i+tn}) = \frac{n(t^2-1)}{2} + 2 + tn + (t+2)i$, where $i = 1,2,...,n$. RHS of condition (1) implies $1 + |\frac{n(t^2-1)}{2} + 2 + tn + (t+2)i - 1| \geq t + 4 + [\frac{n(t^2-1)}{2} + tn] \geq t + 4$. So condition (1) holds for all such pairs. For pairs (z,r) for which $d(z,r) \geq 2$, and $r \in \{a^j_{2i-1}, a^j_{2i}, b^j_{2i-1}, b^j_{2i}\}$, consider the pair (z, a^j_{2i-1}). Now $P(a^j_{2i-1}) = x_{i+2jk}$, $f(x_{i+2jk}) = \lceil \frac{t+5}{2}\rceil - 1 + (j-1)[j(t+3) - 1] + jn + (2j+1)(i + 2jk - jn)$. Hence RHS of condition (1) is $d(v,u) + |f(v) - f(u)| \geq 2 + |\lceil \frac{t+5}{2}\rceil - 1 + (j-1)[j(t+3) - 1] + jn + (2j+1)(i + 2jk - jn) - 1| \geq 2 + \lceil \frac{t+5}{2}\rceil + i + 2t - 2 \geq t + 4$. Condition (1) holds. Analogously it is true for other such pairs.

Case(2). Consider the pairs (a^j_i, a^{j+1}_i), for each $j = 0,1,\dots,\lceil \frac{t}{2}\rceil - 1$, and $i = 1,2,\dots,n$. As $d(a^j_i, a^{j+1}_i) = |j - (j+1)| = 1$, $P(a^j_i) = x_{\frac{i+1}{2}+2jk}$, $P(a^{j+1}_i) = x_{\frac{i+1}{2}+2(j+1)k}$. RHS of condition (1), implies $|j-(j+1)| + |\lceil \frac{t+5}{2}\rceil - 1 + (j-1)[j(t+3)-1] + jn + (2j+1)(\frac{i+1}{2} + j(2k-n)) - [\lceil \frac{t+5}{2}\rceil - 1 + j[(j+1)(t+3)-1] + (j+1)n + (2j+3)(\frac{i+1}{2} + (j+1)(2k-n))]| \geq |1 + 2(t+3)(j+1) + 2k + 1| \geq t+4$. Condition (1) is satisfied . Now for the pairs with $d(a^j_{2i}, a^j_{2i-1}) \geq 4$, $P(a^j_{2i}) = x_{i+k(2j+1)}$, $P(a^j_{2i-1}) = x_{i+2jk}$, and their label difference is $|f(x_{i+k(2j+1)})-f(x_{i+2jk})| \geq t$. Condition (1) implies that $d(a^j_{2i}, a^j_{2i-1})+|f(x_{i+(2j+1)k})-f(x_{i+2jk}| \geq 4+|j-j+1|+t-|j-j+1| = t+4$. So condition (1) holds.

Case(3). Following the same procedure as in Case(2) the condition (1) can be verified for the pairs (b^j_i, b^{j+1}_i), for each $j = 0, 1, \ldots, \lceil \frac{t}{2} \rceil - 1$, and $i = 1, 2, \ldots, n$.

Case(4). For pair, (v_i, v_s), where $i, s = 1, 2, \ldots, n$, and $i \neq s$ the distance is $d(v_i, v_{i+1}) = 2$, $|f(x_{i+tn}) - f(x_{i+1+tn})| \geq t + 2$. RHS of condition (1) implies, $2 + 2 + t = t + 4$, holds. For vertices with $d(v_i, r) \geq 1$, where $i = 1, 2, \ldots, n$, and $r \in \{a^j_{2i-1}, a^j_{2i}, b^j_{2i-1}, b^j_{2i}\}$, $P(v_i) = x_{i+tn}$, $f(x_{i+tn}) = \frac{n(t^2-1)}{2} + 2 + tn + (t + 2)(i - tn)$, and $P(b^j_{2i-1}) = x_{i+2k(j+\lceil \frac{t}{2} \rceil)}$, $f(x_{i+2k(j+\lceil \frac{t}{2} \rceil)}) = \lceil \frac{t+5}{2} \rceil - 1 + (\lceil \frac{t}{2} \rceil - 2)[\lceil \frac{t}{2} \rceil(t + 3) - (t + 4)] + n[t - 1] + j[2t + 3 + 2\lfloor \frac{t}{2} \rfloor(t + 2) - j(t + 3)] + (j + \lceil \frac{t}{2} \rceil)n + (t - 2j)(i + 2k(j + \lceil \frac{t}{2} \rceil) - (j + \lceil \frac{t}{2} \rceil)n)$.

So the radio condition $1 + |\frac{n(t^2-1)}{2} + 2 + tn + (t + 2)(i - tn) - [\lceil \frac{t+5}{2} \rceil - 1 + (\lceil \frac{t}{2} \rceil - 2)[\lceil \frac{t}{2} \rceil(t+3) - (t+4)] + n[t-1] + j[2t+3+2\lfloor \frac{t}{2} \rfloor(t+2) - j(t+3)] + (j + \lceil \frac{t}{2} \rceil)n + (t - 2j)(i + 2k(j + \lceil \frac{t}{2} \rceil) - (j + \lceil \frac{t}{2} \rceil)n)]| \geq 1 + 3 + tn + t + 2 \geq t + 4$, holds. Similarly it is true for all pairs for which $d(v_i, r) \geq 1$.

Case(5). Consider the vertices $r_1, r_2 \in \{a^j_{2i-1}, a^j_{2i}, b^j_{2i-1}, b^j_{2i}\}$, with $d(r_1, r_2) \geq 2$ and suppose $r_1 = a^j_{2i-1}, r_2 = b^j_{2i-1}$.

And their label difference is $|f(x_{i+2jk}) - f((x_{i+2k(j+\lceil \frac{t}{2} \rceil)})| \geq nt$. So the radio condition $d(a^j_{2i-1}, b^j_{2i-1})+|f(x_{i+2jk})-f((x_{i+2k(j+\lceil \frac{t}{2} \rceil)})| \geq 2+nt \geq 4+t$, holds. Analogously it is true for other such pairs.

These above cases establish the claim that f is a radio labeling of $J_{t,n}$.

Hence $rn(J_{t,n}) \leq span(f) = f(x_{(t+1)n}) = \frac{1}{2}(nt^2 + 4nt + 3n + 4)$. ∎

Remark(1): The labeling defined in theorem 2.1 is not a valid radio labeling when $t \geq n - 1$, and $n < 7$. Consider the adjacent pair of vertices (a^0_1, a^1_1). As $P(a^0_1) = x_1, P(a^1_1) = x_{1+2k}$, and $f(x_1) = \lceil \frac{t+5}{2} \rceil + 1, f(x_{1+2k}) = \lceil \frac{t+5}{2} \rceil - 1 + n + 2(1+2k-n)$. RHS of condition (1) implies $1+|\lceil \frac{t+5}{2} \rceil + 1 - (\lceil \frac{t+5}{2} \rceil - 1 + n + 2(1+2k-n))| = 6k+2-2n = n+2$, or $n + 1$. It is easy to see that $n + 2$, or $n + 1 < 4 + t$ for $t \geq n - 1$. Hence the labeling defined in theorem 2.1 is not a valid radio labeling for $J_{t,n}$ when $t \geq n - 1$. For the pair (a^0_1, a^0_n), $d(a^0_1, a^0_n) = t + 1$ and $P(a^0_1) = x_1$, $f(x_1) = \lceil \frac{t+5}{2} \rceil + 1$ for $P(a^0_n) = x_{\frac{n+1}{2}}$, $f(x_{\frac{n+1}{2}}) = \lceil \frac{t+5}{2} \rceil + \frac{n+1}{2}$. Hence $t + 1 + |\lceil \frac{t+5}{2} \rceil + 1 - \lceil \frac{t+5}{2} \rceil 1 \frac{n+1}{2}| = t + \frac{n+1}{2} 4 + t$, for $n < 7$. Hence the the labeling defined in theorem 2.1 is also not a valid radio labeling for $J_{t,n}$ when $n < 7$.

2.2 An upper bound for $J_{t,n}$, when $t \geq (n - 1)$

Theorem 2.2 *For $t \geq n - 1, n \geq 7$, $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 2t + n + 8)$.*

Proof: For this we define $f : V(G) \to \mathbb{N}$ in the following way.

$$f(x_i) = \begin{cases} 1, \text{i=0}, \\ \lceil \frac{t+5}{2} \rceil + i, i = 1, \ldots, n, \ j = 0, \\ A + (2j+3)[i - (j+1)n], i = n+1, \ldots, \lceil \frac{t}{2} \rceil n, \\ B + C + (t - 2j)[i - (j + \lceil \frac{t}{2} \rceil)n], i = \lceil \frac{t}{2} \rceil n + 1, \ldots, tn, \\ D + (t+2)(i - tn), i = tn+1, \ldots, (t+1)n. \end{cases}$$

Where $A = t + 1 - n + \lceil \frac{t+5}{2} \rceil + j^2 + (j+1)[j(t+2) + n]$ $B = \{(n+1)t - 2n + 1 + \lceil \frac{t+5}{2} \rceil + (\lceil \frac{t}{2} \rceil - 2)[(t+3)\lceil \frac{t}{2} \rceil - (t+4)]\}$, for $j = 0, 1, \ldots\ldots \lceil \frac{t}{2} \rceil - 2$, $C = j[2t + 3 - j(t+3) + 2(t+2)\lfloor \frac{t}{2} \rfloor] + (j + \lceil \frac{t}{2} \rceil)n$, and $D = t + 4 + n(t-1) + \frac{n(t^2-1)}{2}$.

Following the same procedure as in theorem (2.1) it can be shown that the above labeling is a valid radio labeling for $J_{t,n}$, for $t \geq n - 1$, and $n \geq 7$. We verify the remark (1) for the pairs (a_1^0, r), where $r \in \{a_i^j, b_i^j, v_i, z\}$, $d(a_1^0, r) \geq 1$, and $P(a_1^0) = x_1$. Consider $r = a_1^j$, where $j = 1, 2, \ldots, \lceil \frac{t}{2} \rceil - 1$, $f(x_1) = \lceil \frac{t+5}{2} \rceil + 1$ and $P(a_1^j) = x_{1+2jk}$ so the label difference $|f(x_1) - f(x_{1+2jk})| \geq t + 3$. Hence the radio condition $d(a_1^0, a_1^j) + |f(x_1) - f(x_{1+2jk})| \geq j + t + 3 \geq 4 + t$ where $j \geq 1$, is satisfied. Similarly it is true for other pairs. Thus for $t \geq n - 1$, and $n \geq 7$, $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 2t + n + 8)$. ∎

2.3 An upper bound for $J_{t,n}$, when $n = 5, 6$.

Theorem 2.3 $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 5n + 2)$. *for* $n = 5, 6$.

Proof: For this we define $f : V(G) \to \mathbb{N}$ in the following way.

$$f(x_i) = \begin{cases} 1, i = 0, \\ \lceil \frac{t+5}{2} \rceil + 1 + 2(i-1), i = 1, \ldots, n, \ j = 0, \\ A + (2j+1)(i - jn), i = n+1, \ldots, (j+1)n, \\ B + C + 3[i - \lceil \frac{t}{2} \rceil n], i = \lceil \frac{t}{2} \rceil n + 1, \ldots, \lceil \frac{t+2}{2} \rceil n, \\ B + D + (2j+3)[i - (j + \lceil \frac{t}{2} \rceil)n], i = \lceil \frac{t+2}{2} \rceil n + 1, \ldots, tn, \\ \frac{n(t^2+1)}{2} + nt + 1 + (t+2)(i - tn), i = tn+1, \ldots, (t+1)n. \end{cases}$$

Where $A = \lceil \frac{t+5}{2} \rceil - 1 + jn + (n-1)[j(j-1) + 1] + (j-1)^2$ $B = \lceil \frac{t+5}{2} \rceil - 1 + (n-1)[(\lceil \frac{t}{2} \rceil - 1)(\lceil \frac{t}{2} \rceil - 2) + 1] + (\lceil \frac{t}{2} \rceil - 2)^2$, and $j = 1, 2, \ldots, \lceil \frac{t}{2} \rceil - 1$, $C = (j + \lceil \frac{t}{2} \rceil)n + (n-1)t + \lfloor \frac{t}{2} \rfloor$, and $D = (j + \lceil \frac{t}{2} \rceil)n + (n-1)t + \lfloor \frac{t}{2} \rfloor + j^2 + 2(n-1)(2j-1)$.

Following the same procedure as in theorem (2.1) it can be shown that the above labeling a is valid radio labeling for $J_{t,n}$, when $n = 5, 6$. We verify the remark (1) for the pairs (a_1^0, r), where $r \in \{a_i^j, b_i^j, v_i, z\}$, $d(a_1^0, r) \geq 1$, and $P(a_1^0) = x_1$. Consider $r = a_1^j$, where $j = 1, 2, \ldots, \lceil \frac{t}{2} \rceil - 1$, $f(x_1) = \lceil \frac{t+5}{2} \rceil + 1$, and $p(a_1^j) = x_{1+2jk}$ so the label difference $|f(x_1) - f(x_{1+2jk})| \geq t + 3$. Hence the radio condition $d(a_1^0, a_1^j) + |f(x_1) - f(x_{1+2jk})| \geq j + t + 3 \geq 4 + t$, where $j \geq 1$, holds. Similarly it is true for other pairs. Thus for $n = 5, 6$, $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 5n + 2)$. ∎

2.4 An upper bound for $J_{t,n}$, when $n = 4$.

Theorem 2.4 *For any t, when $n = 4$, then $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 7n)$.*

Proof: Define $f : V(G) \to \mathbb{N}$ as follows;

$$f(x_i) = \begin{cases} 1, i = 0, \\ \lceil \frac{t+5}{2} \rceil + 1 + 3(i-1), i = 1, \ldots, n,\ j = 0, \\ A + (2j+1)(i - jn), i = n+1, \ldots, (j+1)n, \\ B + C + 3[i - \lceil \frac{t}{2} \rceil n], i = \lceil \frac{t}{2} \rceil n + 1, \ldots, \lceil \frac{t+2}{2} \rceil n, \\ B + D + (2j+3)[i - (j + \lceil \frac{t}{2} \rceil)n], i = \lceil \frac{t+2}{2} \rceil n + 1, \ldots, tn, \\ \frac{n(t^2+1)}{2} + n(t+1) + (t+2)(i - tn), i = (tn+1), \ldots, (t+1)n. \end{cases}$$

Where $A = \lceil \frac{t+5}{2} \rceil + 1 + n(1+j) + (n-1)[j(j-1)] + (j-1)^2$ $B = \lceil \frac{t+5}{2} \rceil + 1 + n + (n-1)[(\lceil \frac{t}{2} \rceil - 1)(\lceil \frac{t}{2} \rceil - 2)] + (\lceil \frac{t}{2} \rceil - 2)^2$.
and $j = 1, 2, \ldots\ldots, \lceil \frac{t}{2} \rceil - 1$, $C = (j + \lceil \frac{t}{2} \rceil)n + (n-1)t + 2 + \lfloor \frac{t}{2} \rfloor$, and $D = (j + \lceil \frac{t}{2} \rceil)n + (n-1)t + 2 + \lfloor \frac{t}{2} \rfloor + j^2 + 2(n-1)(2j-1)$.
Following the same procedure as in theorem (2.1) it can be shown that the above labeling is a valid radio labeling for $J_{t,n}$, when $n = 4$. We verify the remark (1) for the pairs (a_1^0, r), where $r \in \{a_i^j, b_i^j, v_i, z\}$, $d(a_1^0, r) \geq 1$, and $P(a_1^0) = x_1$. Consider $r = a_1^j$, where $j = 1, 2, \ldots, \lceil \frac{t}{2} \rceil - 1$, $f(x_1) = \lceil \frac{t+5}{2} \rceil + 1$ and $P(a_1^j) = x_{1+2jk}$. The label difference $|f(x_1) - f(x_{1+2jk})| \geq t + 3$. Hence the radio condition $d(a_1^0, a_1^j) + |f(x_1) - f(x_{1+2jk})| \geq j + t + 3 \geq 4 + t$, where $j \geq 1$, holds. Similarly it is true for other pairs. Thus for $n = 4$, $rn(J_{t,n}) \leq \frac{1}{2}(nt^2 + 4nt + 7n)$. ∎

Theorem 2.5 *For $t < n-1$, $n \geq 7$, $rn(J_{t,n}) = \frac{1}{2}(nt^2 + 4nt + 3n + 4)$.*

Proof: Combining theorem (1.6) and theorem (2.1) we get the required result. ∎

Open Problem: Calculate the radio number of $J_{t,n}$ when $n \leq 7$ and $t > n - 1$.

2.5 Acknowledgement

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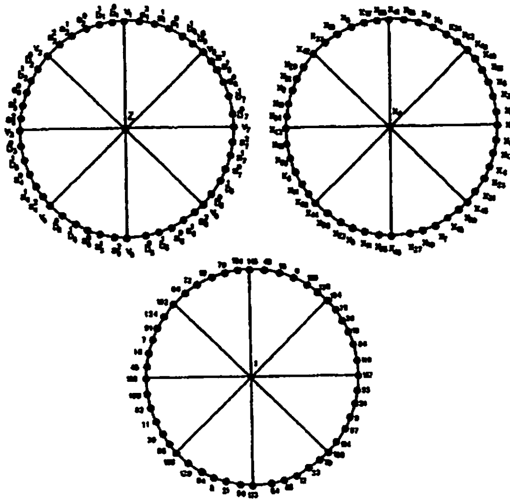

Figure 2: $rn(J_{5,8}) = 194$

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