Hamiltonian Properties of the 3-(γ,2)-Critical Graphs *

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Abstract

Ewa Wojcicka (Journal of Graph Theory, 14(1990), 205-215) showed that every connected, 3-γ-critical graph on more than 6 vertices has a Hamiltonian path. Henning et al. (Discrete Mathematics, 161(1996), 175-184) defined a graph G to be k-(γ,d)-critical graph if $\gamma(G) = k$ and $\gamma(G+uv) = k-1$ for each pair u, v of nonadjacent vertices of G that are at distance at most d apart. They asked if a 3-(γ,2)-critical graph must contain a dominating path. In this paper, we show that every connected, 3-$(\gamma, 2)$-critical graph must contain a dominating path. Further we show that every connected, 3-$(\gamma, 2)$-critical graph on more than 6 vertices has a Hamiltonian path.

Keywords: *edge-critical, dominating path, Hamiltonian path*

1 Introduction

We consider only finite connected undirected graphs without loops or multiple edges. Terminology not presented here can be found in [1].

The neighborhood and the closed neighborhood of a vertex $x \in V$ are respectively $N(x) = \{y \in V(G) : xy \in E(G)\}$ and $N[x] = N(x) \cup \{x\}$.

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Let deg(x) = $|N(x)|$ and $\delta(G) = min\{deg(x) : x \in V(G)\}$. For a vertex set $S \subseteq V(G)$, $N[S] = \cup_{x \in S} N[x]$. Let $G[S]$ denote a subgraph induced by S in G. We write $u \perp v$ if u is adjacent to v, and $u \pm v$ otherwise.

A path P in G is a Hamilton path if P contains every vertex of G. Some classes of graphs which contain Hamilton path were researched on [10, 14].

A set $S \subseteq V(G)$ is a dominating set if for each $v \in V(G)$ either $v \in S$ or v is adjacent to some $w \in S$. That is, S is a dominating set if and only if $N[S] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinalities of dominating sets.

A graph G is k-edge-domination-critical (or just k-γ-critical) if $\gamma(G) = k$, and for every nonadjacent pair of vertices v and u, $\gamma(G + vu) = k - 1$. Hamiltonian properties of k-γ-critical graphs were researched in [2-8, 13, 15, 16].

Wojcicka[15] showed following two theorems:

Theorem 1.1. Every connected 3-γ-critical graph on more than 6 vertices has a Hamiltonian path. ■

Theorem 1.2. Every connected 3-γ-critical graph has a dominating cycle. ■

Henning[9] et al. introduced the concept of the k-(γ,d)-critical graphs : Let $d_G(u, v)$ be the distance of two vertices u and v of G. When no confusion is possible, we may write $d(u, v)$ instead of $d_G(u, v)$. A graph G is k-(γ,d)-critical if $\gamma(G) = k$, and for every nonadjacent pair of vertices v and u with $d(u, v) \leq d$, $\gamma(G + vu) = k - 1$.

They showed following results:

Theorem 1.3. The diameter of a connected 3-(γ,2)-critical graph is at most 4. ■

Observation 1.4. A graph is a 3-(γ,2)-critical graph of diameter 2 if and only if it is a 3-γ-critical of diameter 2. ■

They exhibited a class $\mathcal{H}$ of 3-(γ,2)-critical graphs with diameter 4. Let $H_1 \cong K_r (r \geq 2), H_2 \cong K_s (s \geq 1)$ and let H_3 be obtained from a complete graph $K_{2m} (m \geq 2)$ by removing the edges of a 1-factor. Let $u \in V(H_1)$ and let $v \in V(H_3)$. Let G be obtained from the disjoint union of H_1, H_2 and H_3 by joining every vertex of H_2 to every vertex of $H_1 \cup H_3$ distinct from u and v.

Theorem 1.5. G is a 3-(γ,2)-critical graph of diameter 4 if and only if

$G \in \mathcal{H}$. ■

Observation 1.6. Every 3-(γ,2)-critical graph of diameter 4 has a Hamiltonian path. ■

Let the double star $S(m,n)$ be the graph obtained from the disjoint union of stars $K_{1,m}$ and $K_{1,n}(m,n \geq 1)$ by joining the two central vertices. Let $\overline{G}$ is the complement of G.

Theorem 1.7. A connected graph G is a 2-(γ,2)-critical if and only if either $\overline{G} \cong \bigcup_{i=1}^{m} K_{1,n_i}(m \geq 1, n_i \geq 1)$ or $\overline{G} \cong S(m,n)(m,n \geq 1)$. ■

Theorem 1.8. A graph G is a 2-γ-critical if and only if $\overline{G} \cong \bigcup_{i=1}^{m} K_{1,n_i}(m \geq 1, n_i \geq 1)$. ■

They posed some open problems:

1. Characterization of 3-(γ,2)-critical graphs of diameter 3.
2. Is it true that a 3-(γ,2)-critical graph always possesses a dominating path?

In this paper, we study connected 3-(γ,2)-critical graphs of diameter 3. In section 2, we characterize 3-(γ,2)-critical graphs of diameter 3 and $\delta = 1$. In section 3, we prove that every connected, 3-(γ, 2)-critical graph of diameter 3 must contain a dominating path. In section 4, we prove that every connected, 3-(γ, 2)-critical graph of diameter 3 on more than 6 vertices has a Hamiltonian path.

2 3-(γ,2)-Critical Graphs with $\delta = 1$

In this section, we study 3-(γ,2)-critical graphs with $\delta = 1$.

Sumner and Blitch studied some properties of 3-γ-critical graphs with $\delta = 1$ in [11, 12]. We prove that 3-(γ,2)-critical graphs have similar properties.

Lemma 2.1. Let G be a 3-(γ,2)-critical graph, thus every vertex $v \in V(G)$ is adjacent to at most one endpoint of G.
Proof. If v is adjacent to at least two endpoints of G, say $x, y \in V(G)$, then the distance of x and y is 2 and $\gamma(G + xy) = 3$, a contradiction. So v is adjacent to at most one endpoint of G. ■

By Lemma 2.1, we have

Lemma 2.2. A connected 3-(γ,2)-critical graph has at most three endpoints. ■

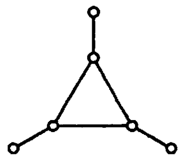

Figure 2.1. The $3-(\gamma,2)$-critical graph with three endpoints

There is only one 3-(γ,2)-critical graph with three endpoints, shown in Figure 1, since for $|V(G)| \geq 7$, we have

Lemma 2.3. A 3-(γ,2)-critical graph G with at least seven vertices has at most two endpoints.

Proof. Suppose G has three endpoints, say $v_i, i = 1,2,3$. Let $N(v_i) = \{u_i\}$ $(i = 1,2,3)$. Let S be an arbitrary dominating set of G with $|S| = 3$, then $S = \{w_i : w_i \in \{u_i, v_i\}, i = 1,2,3\}$. Since G has at least seven vertices, there exists a vertex $x \in V(G) - \{u_i, v_i : \ i = 1,2,3\}$. Then there exists a $w_i = u_i$, such that $u_i \perp x$. Since $d(v_i, x) = 2$, we have $\gamma(G+v_i x) = 3$, a contradiction. ■

Observation 2.4. The diameter of a connected 3-(γ,2)-critical graph with $\delta = 1$ is 3 or 4. ■

Hence Henning[9] et al. characterized 3-(γ,2)-critical graph of diameter 4, we only discuss 3-(γ,2)-critical graph of diameter 3 and $\delta = 1$.

Let $A = \{u \in V(G) : \ \deg(u) = 1\}$.

Theorem 2.5. Let G be a graph with $|A| = 1$, u is the endpoint of G, $N(u) = \{v\}$, $V_1 = \{x \ : \ x \in V(G) \setminus u, d(x,v) = 1\}$ and $V_2 = \{y \ : \ y \in V(G), d(y,v) = 2\}$, then G is a connected 3-(γ,2)-critical if and only if

(1) $G[V_1]$ is a complete graph,

(2) Every vertex $x \in V_1$ is adjacent to $|V_2| - 1$ vertices of V_2,

(3) For every $x \in V_2$, there exists a vertex $y \in V_1 \cup V_2$ such that y dominates $V_2 \setminus x$,

(4) $G[V_2]$ is 2-γ-critical.

Proof. *Necessity.*

(1) For any $x, y \in V_1$, if $x \pm y$, then $d(x,y) = 2$. Since G is 3-(γ,2)-critical, $\gamma(G+xy) = 2$. Let S be a dominating set of $G+xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{u,v\}, w_2 \in \{x,y\}\}$. Hence w_2 dominates V_2, $\{v, w_2\}$ dominates G, a contradiction.

(2) Let x be an arbitrary vertex of V_1. If x is adjacent to all vertices

of V_2, then $\{v, x\}$ is a dominating set of G, a contradiction. If x is not adjacent to at least two vertices of V_2, say y, z, then since $y \in V_2$, there is a vertex $w \in V_1$ such that $y \perp w$. By (1), $G[V_1]$ is a complete graph, $d(x,y) = 2$, $\gamma(G + xy) = 2$. Let S be a dominating set of $G + xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{u, v\}, w_2 \in \{x, y\}\}$. Since z can be dominated by neither w_1 nor x, we have $w_2 = y$. Hence y dominates V_2, $\{v, y\}$ dominates G, a contradiction.

(3) For every $x \in V_2$, $d(x, v) = 2$. Hence $\gamma(G + xv) = 2$. Let S be a dominating set of $G + xv$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{u, v\}, w_2 \in V_1 \cup V_2\}$. Since w_1 can not dominate any vertex of $V_2 \setminus x$, the vertex $y = w_2 \in V_1 \cup V_2$ dominates $V_2 \setminus x$.

(4) If there exists a vertex $x \in V_2$ such that x dominates $G[V_2]$, then x, v dominate G and $\gamma(G) = 2$, a contradiction. Hence $|N[x]| < |V_2|$ for every vertex $x \in V_2$, $|V_2| \geq 2$ and $\gamma(G[V_2]) \geq 2$.

If $|V_2| = 2$, then since $|N[x]| < |V_2|$ for every vertex $x \in V_2$, $G[V_2]$ contains two independent vertices. By Theorem 1.8, $G[V_2]$ is 2-γ-critical. Hence we need only consider the case of $|V_2| \geq 3$.

For any pair nonadjacent vertices x, y of V_2, by (3), we have $d_G(x, y) = 2$. Hence $\gamma(G + xy) = 2$. Let S be a dominating set of $G + xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{u, v\}, w_2 \in \{x, y\}\}$. Hence x dominates $V_2 \setminus y$ or y dominates $V_2 \setminus x$, i.e. $\gamma(G[V_2] + xy) = 1$. Hence $G[V_2]$ is 2-γ-critical.

Sufficiency.
Let x be a vertex of V_1. By (1), x dominates V_1. By (2), there exists a vertex $y \in V_2$ such that $x \pm y$ and x dominates $V_2 \backslash y$. Hence $\{v, x, y\}$ dominates G and $\gamma(G) \leq 3$. Let S be an arbitrary dominating set of G, then S contains at least one vertex of $\{u, v\}$ and at least one vertex of $V_1 \cup V_2$. By (2) and (4), any single vertex of $V_1 \cup V_2$ can not dominate V_2, hence $|S| \geq 3$, i.e. $\gamma(G) = 3$.

Let $E_2 = \{xy : \ d(x, y) = 2\}$. For any $xy \in E_2$, there are three cases:

Case 1. Suppose $x = u$ and $y \in V_1$. By (1), y dominates V_1. By (2), there is a vertex of V_2, say z, such that $z \pm y$. Then $\{y, z\}$ dominates $G + xy$.

Case 2. Suppose $x = v$ and $y \in V_2$. By (3), there is a vertex $z \in V_1 \cup V_2$, which dominates $V_2 \setminus y$. Then $\{v, z\}$ dominates $G + xy$.

Case 3. Suppose $x, y \in V_2$, then by (4), $\{v, w : w \in \{x, y\}\}$ dominates $G + xy$.

So, G be 3-(γ,2)-critical. ■

Theorem 2.6. Let G be a graph with $|A| = 2$, for $i = 1$ or 2, u_i is an endpoint of G, $N(u_i) = v_i$, $V_1^i = \{x : x \in V(G) - \{u_1, u_2, v_1, v_2\}, d(x, v_i) = 1\}$ and $V_2^i = \{y : y \in V(G) - V_1^1 - V_1^2 - \{u_1, u_2\}, d(y, v_i) = 2\}$. Then G is 3-(γ,2)-critical if and only if

(1) $G[V_1^1] = G[V_1^2] = G[V_1]$ is a complete graph,

(2) $G[V_2^1] = G[V_2^2] = G[V_2]$ is a vertex which is adjacent to all vertices of V_1.

Proof. *Necessity.*

(1) Since the diameter of G is 3, $v_1 \perp v_2$. Assume that there exists a vertex $x \in V_1^1$ and $x \notin V_1^2$, then $d(x, v_2) = 2$ and $\gamma(G + xv_2) = 2$. Let S be a dominating set of $G + xv_2$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x, v_2\}\}, w_2 \in \{u_1, v_1\}\}$. If $w_1 = x$, then w_2 has to dominate both u_1 and u_2, a contradiction to Lemma 2.1, hence $w_1 = v_2$. Since w_2 has to dominate $V(G) - V_2^1 - \{u_2, v_1, v_2, x\}$, we have $w_2 = v_1$. Thus v_1, v_2 would be a dominating set of G, a contradiction. Hence $G[V_1^1] \subseteq G[V_1^2]$. By a similar argument, we have $G[V_1^2] \subseteq G[V_1^1]$. Hence $G[V_1^1] = G[V_1^2] = G[V_1]$.

For any $x, y \in V_1$, if $x \pm y$, then $d(x, y) = 2$. Since G is 3-(γ,2)-critical, $\gamma(G + xy) = 2$. Let S be a dominating set of $G + xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x, y\}\}$. Since $w_1 \pm u_1$ and $w_1 \pm u_2$, w_2 has to dominate both u_1 and u_2, a contradiction to Lemma 2.1.

(2) By (1), $G[V_1^1] = G[V_1^2]$, we have $G[V_2^1] = G[V_2^2] = G[V_2]$. If $V_2 = \emptyset$, then $\{v_1, v_2\}$ would be a dominating set of G, a contradiction. Hence $|V_2| \geq 1$. If $|V_2| \geq 2$, let $x \in V_2$, then $V_2 \backslash x \neq \emptyset$. Since $d(x, v_1) = 2$, $\gamma(G + xv_1) = 2$. Let S be a dominating set of $G + xv_1$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x, v_1\}, w_2 \in \{u_2, v_2\}\}$. If $w_1 = x$, then w_2 has to dominate both u_1 and u_2, a contradiction to Lemma 2.1, hence $w_1 = v_1$. Thus, any vertex of $V_2 \backslash x$ would not be dominated by $S = \{v_1, w_2\}$, a contradiction. Hence $G[V_2]$ contains exact one vertex.

Let $V_2 = \{x\}$. If there is a vertex $y \in V_1$ such that $y \pm x$, then $\gamma(G + xy) = 2$. Let S be a dominating set of $G + xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x, y\}\}$. Since $w_1 \pm u_1$ and $w_1 \pm u_2$, w_2 has to dominate both u_1 and u_2, a contradiction to Lemma 2.1.

Sufficiency.

Since the dominating set of G must contain $w_1 \in \{v_1, u_1\}$, $w_2 \in \{v_2, u_2\}$ and $w_3 \in V_1 \cup V_2$, $\gamma(G) \geq 3$. Let $V_2 = \{z\}$, from (1) and (2), $\{z, v_1, v_2\}$ dominates G. So $\gamma(G) = 3$.

Let $E_2 = \{xy : d(x, y) = 2\}$, $w \in V_1$. For any $xy \in E_2$, there are six cases:

Case 1. $x = u_1$ and $y = v_2$, then $\{v_2, z\}$ dominates $G + xy$.

Case 2. $x = u_2$ and $y = v_1$, then $\{v_1, z\}$ dominates $G + xy$.

Case 3. $x = u_1$ and $y = w$, then $\{w, v_2\}$ dominates $G + xy$.

Case 4. $x = u_2$ and $y = w$, then $\{w, v_1\}$ dominates $G + xy$.

Case 5. $x = v_1$ and $y = z$, then $\{v_1, v_2\}$ dominates $G + xy$.

Case 6. $x = v_2$ and $y = z$, then $\{v_1, v_2\}$ dominates $G + xy$.

So, G is 3-(γ,2)-critical. ■

Theorem 2.7. Let G be a 3-(γ,2)-critical graph of diameter 3 with $\delta \geq 2$, then G is 2-connected.
Proof. Assume that G has a cut-point u, then $d(u, v) \leq 2$ for any $v \in V(G)$. Let $V_i = \{x : x \in V(G), d(x, u) = i\}, (i = 1, 2)$.

Since the diameter of G is 3, there exists a connected component of $G \backslash u$, say C_1, with $C_1 \subseteq V_1$, and there exists a connected component of $G \backslash u$, say C_2, with $C_2 \cap V_2 \neq \emptyset$. For any $x \in C_1$ and $y \in C_2 \cap V_1$, $\gamma(G + xy) = 2$.

Let S be a dominating set of $G + xy$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x, y\}\}$. If $w_1 = x$, then w_1 can not dominate any vertices of V_2, w_2 dominates V_2, $\{u, w_2\}$ dominates G, a contradiction. If $w_1 = y$, then w_2 dominates $C_1 \backslash x$. Hence $w_2 \in u \cup C_1$, and w_2 can not dominate any vertices of V_2, y dominates V_2, $\{u, y\}$ dominates G, a contradiction. ■

Lemma 2.8. Let G be a 3-(γ,2)-critical graph of diameter 3 and let $A = \{x \in V(G) : deg(x) = 1\}$. If $|A| = 2$, then G has a Hamiltonian path.
Proof. For $i = 1$ or 2, let u_i be the endpoint of G, $N(u_i) = v_i$, $V_1^i = \{x : x \in V(G) - \{u_1, u_2, v_1, v_2\}, d(x, v_i) = 1\}$ and $V_2^i = \{y : y \in V(G), d(y, v_i) = 2\}$, then by Theorem 2.6, $G[V_1^1] = G[V_1^2] = G[V_1]$ is a complete graph and $G[V_2^1] = G[V_2^2] = G[V_2]$ is a vertex which is adjacent to all vertices of V_1. Let $V(G[V_1]) = \{v_{1,1}, v_{1,2}, \cdots, v_{1,|V(G[V_1])|}\}$, $V(G[V_2]) = \{v_{2,1}\}$, then $P = u_1 v_1 v_{1,1} v_{2,1} v_{1,2} \cdots v_{1,|V(G[V_1])|} v_2 u_2$ be a Hamiltonian path of G. ■

Lemma 2.9. Let G be a 3-(γ,2)-critical graph of diameter 3 and let $A = \{x \in V(G) : deg(x) = 1\}$. If $|A| = 1$, then G has a Hamiltonian path.
Proof. Let u be the endpoint of G, $N(u) = \{v\}$, $V_1 = \{x : x \in V(G) \setminus u, d(x, v) = 1\}$ and $V_2 = \{y : y \in V(G), d(y, v) = 2\}$, then by Theorem 2.5,

(1) $G[V_1]$ is a complete graph,

(2) Every vertex $x \in V_1$ is adjacent to $|V_2| - 1$ vertices of V_2,

(3) For every $x \in V_2$, there exists a vertex $y \in V_1 \cup V_2$ such that y dominates $V_2 \setminus x$,

(4) $G[V_2]$ is 2-γ-critical.
By Theorem 1.8, we have $G[V_2]$ is $\overline{G[V_2]} \cong \bigcup_{i=1}^{m} K_{1,n_i}(m \geq 1, n_i \geq 1)$. Let r_i be the root of K_{1,n_i} and $l_{i,j}$ be a leaf of K_{1,n_i}. Let $V(G[V_1]) = \{v_{1,1}, v_{1,2}, \cdots, v_{1,|V(G[V_1])|}\}$.

Case 1. $m = 1$. Since $r_1 \in V_2$, r_1 is adjacent to a vertex of V_1, say $v_{1,1}$. From (2), $v_{1,1}$ is adjacent to $|V_2| - 1$ vertices of V_2. Without loss of generality, we may assume that $v_{1,1} \pm l_{1,1}$. Since $l_{1,1} \in V_2$, $l_{1,1}$ is adjacent to a vertex of V_1, say $v_{1,|V(G[V_1])|}$, then $P = uvv_{1,1}r_1v_{1,2} \cdots v_{1,|V(G[V_1])|}l_{1,1} \cdots l_{1,n_1}$ be a Hamiltonian path of G.

Case 2. $m > 1$. Since $r_1 \in V_2$, r_1 is adjacent to a vertex of V_1, say $v_{1,|V(G[V_1])|}$. Then $P = uvv_{1,1}v_{1,2} \cdots v_{1,|V(G[V_1])|}r_1 \cdots r_m l_{1,1} \cdots l_{1,n_1} \cdots l_{m,1} \cdots l_{m,n_m}$ is a Hamiltonian path of G. ■

By Lemmas 2.3,2.8 and 2.9, we have

Theorem 2.10. Let G be a 3-(γ,2)-critical graph of diameter 3 with $\delta = 1$ and $n \geq 7$, then G has a Hamiltonian path. ■

3 Dominating Path in 3-(γ,2)-Critical Graphs

Henning[9] et al. observed that a 3-(γ,2)-critical graph does not necessarily have a dominating cycle. They asked if a connected 3-(γ,2)-critical graph has always a dominating path. In this section, we prove that every longest path in a connected 3-(γ,2)-critical graph is a dominating path.

Let $P = x_1x_2 \cdots x_n$ be a path in G. We will write $P^{\rightarrow}$ if the order of vertices in P is to be considered from x_1 to x_n, or $P^{\leftarrow}$ if the order is in the opposite direction. For x_i, $x_j \in P$, $i < j$, we write $x_iP^{\rightarrow}x_j$ to indicate the segment on P originating at x_i and terminating at x_j, and we write $x_iP^{\leftarrow}x_j$ to denote the same segment in the opposite direction. For x belonging to P, we denote by x^+ the vertex on $P^{\rightarrow}$ that immediately follows x, and denote by x^- the vertex that precedes x on $P^{\rightarrow}$.

Observation 3.1. Let $P = x_1x_2 \cdots x_n$ be any longest path in a connected 3-(γ,2)-critical graph G of diameter 3, where $x_1 = a$ and $x_n = b$. If there is a vertex $x \in V(G) - V(P)$, then $x \pm a, x \pm b$ and $a \pm b$. ■

Lemma 3.2. There exists a vertex $y \in V(G) - V(P)$ such that $|Y| = |N(y) \cap V(P)| \geq 2$.
Proof. By contrary, suppose that for every $y^t \in V(G) - V(P)$, $Y^t = N(y^t) \cap V(P) = \{y_1^t\}$. By Theorem 2.7, G is 2-connected, hence every

component of $G[V(G) - V(P)]$ contains at least two vertices, and there is at least one pair of vertices, say $y^i, y^j \in V(G) - V(P)$ such that $y^i \perp y^j$ and $y_1^i \neq y_1^j$. By Observation 3.1, we have $y^i \pm a$, $y^i \pm b$, $y^j \pm a$ and $y^j \pm b$.

If $y_1^{i+} \perp a$, then $bP^{\leftarrow} y_1^{i+} aP^{\rightarrow} y_1^i y^i y^j$ would be a longer dominating path(see Figure 3.1.(1)), hence $y_1^{i+} \pm a$. If $y_1^{i+} \perp b$, then $aP^{\rightarrow} y_1^i y^i y^j y_1^j P^{\rightarrow} b y_1^{i+} P^{\rightarrow} y_1^{j-}$ would be a longer dominating path(see Figure 3.1.(2)), hence $y_1^{i+} \pm b$.

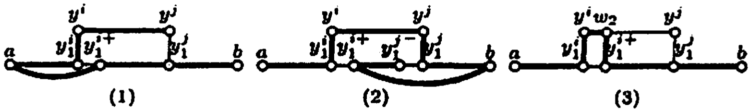

Figure 3.1.

Since $d(y^i, y_1^{i+}) = 2$, $\gamma(G + y^i y_1^{i+}) = 2$. Let S be a dominating set of $G + y^i y_1^{i+}$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{y^i, y_1^{i+}\}\}$.

Since $y^i \pm a$, $y^i \pm b$, $y_1^{i+} \pm a$, $y_1^{i+} \pm b$, it follows that w_2 has to dominate $\{a, b\}$. By Observation 3.1, $a \pm b$, hence $d(a, b) = 2$, $\gamma(G + ab) = 2$. Let S be a dominating set of $G + ab$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{a, b\}\}$. Without loss of generality, we may assume $w_1 = a$. Since $a \pm y^i$, $a \pm y^j$, $a \pm y_1^{i+}$, w_2 has to dominate $\{y^i, y^j, y_1^{i+}\}$. If $w_2 \in V(G) - V(P)$, then $aP^{\rightarrow} y_1^i y^i w_2 y_1^{i+} P^{\rightarrow} b$ would be a longer dominating path(see Figure 3.1.(3)). Hence $w_2 \in V(P)$. Thus, $w_2 = y_1^i$ and $w_2 = y_1^j$, a contradiction to $y_1^i \neq y_1^j$. Hence there is at least one vertex of $\{y^i, y^j\}$ satisfies the conclusion of the Lemma. ■

Since P is a longest path of G, we have

Observation 3.3.

(1) For all $i \in \{1, \cdots k\}$, $a \pm y_i^+$, $b \pm y_i^-$, $y \pm y_i^+$ and $y \pm y_i^-$.

(2) For $i \neq j$, $i, j \in \{1, \cdots k\}$, $y_i^+ \pm y_j^+$ and $y_i^- \pm y_j^-$.

(3) For all $i \in \{2, \cdots k\}$, $a \pm y_i^-$. For all $i \in \{1, \cdots, k-1\}$, $b \pm y_i^+$.

(4) If y_i dominates one vertex of $\{a, b\}$, then $y_i^+ \pm y_i^-$. ■

Lemma 3.4. Every longest path in a connected 3-(γ,2)-critical graph G of diameter 3 with $\delta \geq 2$ is a dominating path.

Proof. Let $P = x_1 x_2 \cdots x_n$ be any longest path in a 3-(γ,2)-critical graph of diameter 3, where $x_1 = a$ and $x_n = b$. If P is not a dominating path, then there exists a vertex u that can not be dominated by P. By Lemma 3.2 and Observation 3.3, there exist $x \in V(G) - V(P)$ and $x_i \in P$ such that $u \perp x$, $x \perp x_i$ and $x_i \perp a$. Similarly, there exist $y \in V(G) - V(P)$ and $x_j \in P$ such that $u \perp y$, $y \perp x_j$ and $x_j \perp b$.

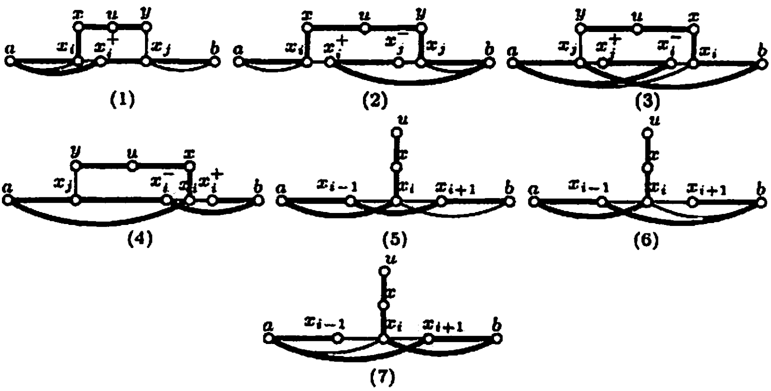

Figure 3.2.

If $i \neq j$, then $|i-j| \geq 4$(otherwise $aP^{\rightarrow}x_i xuyx_j P^{\rightarrow}b$ would be a longer path), $i \neq 1, 2$(otherwise $yuxx_i P^{\rightarrow}b$ would be a longer path) and $j \neq n-1, n$(otherwise $aP^{\rightarrow}x_j yux$ would be a longer path).

Case 1. $i < j$.

If $a \perp x_i^+$, then $yuxx_i P^{\leftarrow}ax_i^+ P^{\rightarrow}b$ would be a longer path(see Figure 3.2.(1)). So, $a \pm x_i^+$. Hence $d(a, x_i^+) = 2$, $\gamma(G + ax_i^+) = 2$. Let S be a dominating set of $G + ax_i^+$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{a, x_i^+\}, w_2 \in V(G) - V(P)\}$. By Observation 3.1, $b \pm w_2$ and $b \pm a$, b has to be dominated by $w_1 = x_i^+$, thus, $x_j^- P^{\leftarrow} x_i^+ bP^{\leftarrow} x_j yuxx_i P^{\leftarrow} a$ would be a longer path(see Figure 3.2.(2)).

Case 2. $i > j$.

If $a \perp x_i^-$, then $yuxx_i P^{\rightarrow} bx_j P^{\leftarrow} ax_i^- P^{\leftarrow} x_j^+$ would be a longer path(see Figure 3.2.(3)). So, $a \pm x_i^-$. Hence $d(a, x_i^-) = 2$, $\gamma(G + ax_i^-) = 2$. Let S be a dominating set of $G + ax_i^-$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{a, x_i^-\}, w_2 \in V(G) - V(P)\}$. By Observation 3.1, $b \pm w_2$ and $b \pm a$, b has to be dominated by $w_1 = x_i^-$, thus, $yuxx_i aP^{\rightarrow} x_i^- bP^{\leftarrow} x_i^+$ would be a longer path(see Figure 3.2.(4)).

Case 3. $i = j$.

If $x_{i-1} \perp x_{i+1}$, then $uxx_i aP^{\rightarrow} x_{i-1} x_{i+1} P^{\rightarrow} b$ would be a longer path(see Figure 3.2.(5)). So, $x_{i-1} \pm x_{i+1}$. $d(x_{i+1}, x_{i-1}) = 2$, $\gamma(G + x_{i+1}x_{i-1}) = 2$. Let S be a dominating set of $G + x_{i+1}x_{i-1}$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{x_{i+1}, x_{i-1}\}, w_2 \in V(G) - V(P)\}$. Since $w_2 \in V(G) - V(P)\}$, by Observation 3.1, $w_2 \pm a$ and $w_2 \pm b$, it follows that w_1 has to dominate

$\{a,b\}$. If $w_1 = x_{i-1}$, then $x_{i-1} \perp b$, $uxx_iaP^{\rightarrow}x_{i-1}bP^{\leftarrow}x_{i+1}$ would be a longer path(see Figure 3.2.(6)), a contradiction. If $w_1 = x_{i+1}$, then $a \perp x_{i+1}$, then $uxx_ibP^{\leftarrow}x_{i+1}aP^{\rightarrow}x_{i-1}$ would be a longer path(see Figure 3.2.(7)), a contradiction. ■

From Theorem 1.3, Observation 1.4, Theorem 2.10, Lemma 3.4 and Observation 1.6, we have

Theorem 3.5. Every longest path in a connected 3-(γ,2)-critical graph is a dominating path. ■

4 Hamiltonian Paths

Throughout this section G will be assumed to be a 3-(γ,2)-critical graph of diameter 3 with order more than 6, and $P^{\rightarrow} = x_1x_2\cdots x_t$ will be a longest dominating path in G by Theorem 3.5, where $x_1 = a$ and $x_t = b$. If $V(G) = V(P)$, then there is nothing to prove and P is a Hamiltonian path of G. So, we will also assume that $V(P)$ is properly contained in $V(G)$ and we will let y denote a vertex of G that is not on P. We will set $Y = N(y) \cap V(P) = \{y_1, \cdots, y_k\}$, ordered so that if $i < j$ then y_i precedes y_j on $P^{\rightarrow}$.

For $\delta = 1$, by Theorem 2.10, G has a Hamiltonian path. In the following, we consider the case for $\delta \geq 2$.

Let $B = \{v \in Y : v^+ \pm v^-\}$.

Lemma 4.1. $B \neq \emptyset$.
Proof. By Observation 3.3(1) $a \pm y_1^+$, hence $d(a, y_1^+) = 2$ or 3.

Case 1. $d(a, y_1^+) = 2$, then $\gamma(G + ay_1^+) = 2$. Let S be a dominating set of $G + ay_1^+$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{a, y_1^+\}\}$. By Observation 3.1 and 3.3, $a \pm b$, $a \pm y$ and $y_1^+ \pm b$, $y_1^+ \pm y$, w_2 has to dominate b and y. By Observation 3.1, $w_2 \notin V(G) - V(P)$, hence $w_2 \in Y$. By Observation 3.3(4), $w_2^- \pm w_2^+$, we have $w_2 \in B$.

Case 2. $d(a, y_1^+) = 3$. Since $d(y, y_2^-) = 2$, $\gamma(G + yy_2^-) = 2$. Let S be a dominating set of $G + yy_2^-$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{y, y_2^-\}\}$. By Observation 3.1 and 4.2, $y \pm a$, $y \pm y_1^-$ and $y_2^- \pm a$, $y_2^- \pm y_1^-$, hence w_2 has to dominate $\{a, y_1^-\}$, $d(a, y_1^-) \leq 2$. If $y_1^- \pm y_1^+$, then $y_1 \in B$, and the lemma is proved. If $y_1^- \perp y_1^+$, then since $d(a, y_1^+) = 3$, we have $a \pm y_1^-$. Hence $d(a, y_1^-) = 2$, $\gamma(G + ay_1^-) = 2$. Let S be a dominating set of $G + ay_1^-$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{a, y_1^-\}\}$. By Observation 3.1 and

3.3, $a \pm b$, $a \pm y$ and $y_1^- \pm b$, $y_1^- \pm y$, it forces that w_2 dominates b and y. By Observation 3.1, $w_2 \notin V(G) - V(P)$, hence $w_2 \in Y$. By Observation 3.3(4), $w_2^- \pm w_2^+$, we have $w_2 \in B$. ■

Let G^* be a directed graph with $V(G^*) = B$ and vw is an arc in G^* if and only if there exists a domination set $S = \{w_1, w : w_1 \in \{v^-, v^+\}\}$ of $G + v^-v^+$.

Lemma 4.2. If $r \in B$, then there exists $w \in B - \{r\}$ such that rw is an arc in G^*, and w dominates at least one of the end points of P.
Proof. Since $r \in B$, we have $d(r^-, r^+) = 2$ and $\gamma(G + r^-r^+) = 2$. Let S be a dominating set of $G + r^-r^+$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{r^-, r^+\}\}$. If $w_2 \perp r^-$ and $w_2 \perp r^+$, then $\{w_1, w_2\}$ would be a dominating set of G, a contradiction. Hence w_2 is not adjacent to at least one of r^- and r^+, so $w_2 \neq r$. By Observation 3.3 neither of r^- nor r^+ can dominate both of the endpoints of P and $r^- \pm y$, $r^+ \pm y$, it forces that w_2 dominates both y and at least one of the endpoints of P. By Observation 3.1, $w_2 \notin V(G) - V(P)$, hence $w_2 \in Y$. By Observation 3.3(4), $w_2^- \pm w_2^+$, we have $w_2 \in B - \{r\}$ and w_2 dominates at least one of the endpoints of P. ■

Lemma 4.3. If $r \in B$ dominates one of the endpoints of P, and both sw and rw are arcs in G^*, then $r = s$.
Proof. By contrary, suppose that there exists $r, s, w \in B$, such that both rw and sw are arcs in G^*, and $r \neq s$. Then there exists a dominating set $S_1 = \{w_{1,1}, w : w_{1,1} \in \{r^-, r^+\}\}$ of $G + r^-r^+$ and a dominating set $S_2 = \{w_{2,1}, w : w_{2,1} \in \{s^-, s^+\}\}$ of $G + s^-s^+$. Without loss of generality, assume that r follows s on P. There are four cases:

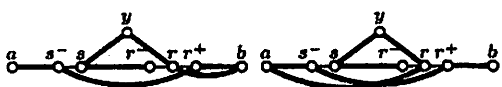

Figure 4.1.

Case 1. $w_{1,1} = r^+$ and $w_{2,1} = s^+$. If $w \perp s^-$, then since $\{s^+, w\}$ dominates $G + s^-s^+$, $\{s^+, w\}$ would be a dominating set of G, a contradiction. Hence $w \pm s^-$. Since $\{r^+, w\}$ dominates $G + r^-r^+$, $r^+ \perp s^-$. If $r \perp b$, then $r^-P^{\leftarrow}syrbP^{\leftarrow}r^+s^-P^{\leftarrow}a$ would be a longer dominating path(see Figure 4.1.(1)). If $r \perp a$, then $r^-P^{\leftarrow}syraP^{\rightarrow}s^-r^+p^{\rightarrow}b$ would be a longer dominating path(see Figure 4.1.(2)).

Case 2. $w_{1,1} = r^-$ and $w_{2,1} = s^-$. Case 2 is symmetric to Case 1.

Case 3. $w_{1,1} = r^-$ and $w_{2,1} = s^+$. If $w \perp r^+$, then since $\{r^-, w\}$ dominates

$G+r^-r^+$, $\{r^-,w\}$ would be a dominating set of G, a contradiction. Hence $w \pm r^+$. Since $\{s^+,w\}$ dominates $G+s^-s^+$, $s^+ \perp r^+$, which contradicts the conclusion of Observation 3.3(2).

Case 4. $w_{1,1}=r^+$ and $w_{2,1}=s^-$. If $w \perp r^-$, then since $\{r^+,w\}$ dominates $G+r^-r^+$, $\{r^+,w\}$ would be a dominating set of G, a contradiction. Hence $w \pm r^-$. Since $\{s^-,w\}$ dominates $G+s^-s^+$, $s^- \perp r^-$, which contradicts the conclusion of Observation 3.3(2). ■

Lemma 4.4. For each $w \in B$, there exists $v \in B-\{w\}$ such that vw is an arc in G^*.

Proof. By contrary, suppose that there is a vertex $v_0 \in B$ such that vv_0 is not an arc in G^* for every $v \in B-\{v_0\}$.

By Lemma 4.2 there exists $v_1 \in B-\{v_0\}$ such that v_0v_1 is an arc in G^*, and v_1 dominates one of the endpoints of P.

By Lemma 4.2 there exists $v_2 \in B-\{v_1\}$ such that v_1v_2 is an arc in G^*, and v_2 dominates one of the endpoints of P. If $v_2=v_0$, then v_1v_0 would be an arc in G^*, a contradiction. Hence, $v_2 \neq v_0$.

By Lemma 4.2 there exists $v_3 \in B-\{v_2\}$ such that v_2v_3 is an arc in G^*, and v_3 dominates one of the endpoints of P. If $v_3=v_0$, then v_2v_0 would be an arc in G^*, a contradiction. Hence, $v_3 \neq v_0$.

So, we can get $v_1,v_2,v_3,\ldots,v_t$ such that $v_i \neq v_0 (t \geq 3$ and $1 \leq i \leq t)$. Then, let $Q=v_0v_1\cdots v_t$ be a longest directed path in G^* such that for all $i \in \{1,\cdots,t\}$, the v_i is distinct and each of them dominates one of the endpoints of P.

Since $v_t \in B$, by Lemma 4.2 there exists $v_{t+1} \in B-\{v_t\}$ such that v_tv_{t+1} is an arc in G^*, and v_{t+1} dominates one of the endpoints of P. If $v_{t+1}=v_0$, then v_tv_0 would be an arc in G^*, a contradiction. Hence, $v_{t+1}=v_i$ for some $i \in \{1,\cdots,t-1\}$. Then v_tv_{t+1} and $v_{i-1}v_{t+1}$ would be distinct arcs in G^*, a contradiction to Lemma 4.3. ■

By Lemma 4.4 and Observation 3.3, we have

Corollary 4.5. For each $w \in B$, let y_iw be an arc in G^*, then
(1) If $\{y_i^+,w\}$ dominates $G+y_i^-y_i^+$ and $i<k$, then w dominates a and b.
(2) If $\{y_i^-,w\}$ dominates $G+y_i^-y_i^+$ and $i>1$, then w dominates a and b.
(3) If $i=1$ and $\{y_1^-,w\}$ dominates $G+y_1^-y_1^+$, then w dominates b.
(4) If $i=k$ and $\{y_k^+,w\}$ dominates $G+y_k^-y_k^+$, then w dominates a. ■

By Corollary 4.5, we have

Corollary 4.6.
(1) If $w \in B$, then w dominates one of the endpoints of P.
(2) If $w \in B$ and $w \pm b$, then $\{y_k^+, w\}$ dominates $G + y_k^- y_k^+$.
(3) If $w \in B$ and $w \pm a$, then $\{y_1^-, w\}$ dominates $G + y_1^- y_1^+$. ∎

Now we prove the main result.

Lemma 4.7. If G is a 3-(γ,2)-critical graph of diameter 3 with $\delta \geq 2$, then G has a Hamiltonian path.
Proof. Let P be a longest dominating path in G. If $V(P) \neq V(G)$, then by Lemma 3.2, there exists $y \in V(G) - V(P)$ such that $|Y| = |N(y) \cap V(P)| \geq 2$. By Observation 3.3, $y \pm y_k^-$. Hence $d(y, y_k^-) = 2$, $\gamma(G + yy_k^-) = 2$. Let S be a dominating set of $G + yy_k^-$ with $|S| = 2$, then $S = \{w_1, w_2 : w_1 \in \{y, y_k^-\}\}$. By Observation 3.1 and 3.3, $y \pm a$, $y \pm b$ and $y_k^- \pm a$, $y_k^- \pm b$, it forces that w_2 dominates $\{a, b\}$. Thus $d(a,b) \leq 2$. By Observation 3.1, $a \pm b$, we have $d(a,b) = 2$, $\gamma(G + ab) = 2$.

Let S_1 be a dominating set of $G + ab$ with $|S_1| = 2$, then $S_1 = \{w_{1,1}, w_{1,2} : w_{1,1} \in \{a, b\}\}$. By Observation 3.1 and 3.3, $a \pm y$, $a \pm y_1^+$, $a \pm y_k^-$ and $b \pm y$, $b \pm y_1^+$, $b \pm y_k^-$, it forces that $w_{1,2}$ dominates $\{y, y_1^+, y_k^-\}$. By Observation 3.3, $y \pm y_1^+$, $w_{1,2}$ is distinct from y and y_1^+. If $w_{1,2} \in V(G) - V(P)$, then $aP^{\rightarrow}y_1 y w_{1,2} y_1^+ P^{\rightarrow} b$ would be a longer dominating path(see Figure 4.2.(1)). Hence $w_{1,2} \in Y$.

If $w_{1,2} \in B$ and $w_{1,2} \pm b$, then by Corollary 4.6(2), we have $\{y_k^+, w_{1,2}\}$ dominates $G + y_k^- y_k^+$. Since $w_{1,2}$ dominates y_k^-, $\{y_k^+, w_{1,2}\}$ would be a dominating set of G, a contradiction. If $w_{1,2} \in B$ and $w_{1,2} \pm a$, then by Corollary 4.6(3), we have $\{y_1^-, w_{1,2}\}$ dominates $G + y_1^- y_1^+$. Since $w_{1,2}$ dominates y_1^+, $\{y_1^-, w_{1,2}\}$ would be a dominating set of G, a contradiction. Hence, if $w_{1,2} \in B$, then $w_{1,2}$ dominates $\{a, b\}$, $\{w_{1,1}, w_{1,2}\}$ would be a dominating set of G, a contradiction. So, $w_{1,2} \in Y - B$, $w_{1,2}^+ \perp w_{1,2}^-$.

Case 1. $w_{1,1} = a$. By Observation 3.1, $a \pm b$, hence $w_{1,2} \perp b$. If $w_{1,2} = y_1$, then $aP^{\rightarrow}y_1^- y_1^+ P^{\rightarrow} y_k y y_1 b P^{\leftarrow} y_k^+$ would be a longer dominating path(see Figure 4.2.(2)). If $w_{1,2} = y_k$, then $aP^{\rightarrow}y_k^- y_k^+ P^{\rightarrow} b y_k y$ would be a longer dominating path(see Figure 4.2.(3)). If $w_{1,2} = y_i$ and $1 < i < k$, then $aP^{\rightarrow}y_i^- y_i^+ P^{\rightarrow} b y_i y$ would be a longer dominating path (see Figure 4.2.(4)).

Case 2. $w_{1,1} = b$. By an argument similar to the proof of Case 1, we can get a longer dominating path (see Figure 4.2.(7)).

By Cases 1-2, $w_{1,2} \notin Y - B$, a contradiction. Hence, $V(P) = V(G)$, i.e. P is a Hamiltonian path. ∎

From Theorem 1.3, Observation 1.4, Theorem 2.10, Lemma 4.7 and Observation 1.6, we have

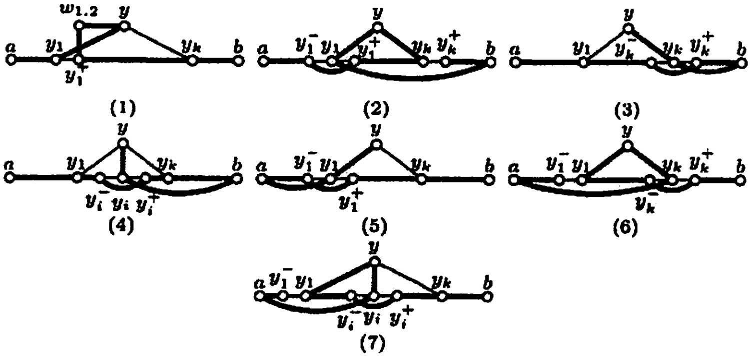

Figure 4.2.

Theorem 4.8. Every connected 3-(γ,2)-critical graph on more than 6 vertices has a Hamiltonian path. ■

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