# The Wiener-type indices of the corona of two graphs*

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**Abstract.** The corona of two graphs $G$ and $H$, written as $G \odot H$, is the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and then joining the $ith$ vertex of $G$ to every vertex in the $ith$ copy of $H$. In this paper, we present the explicit formulae for the Wiener, hyper-Wiener and reverse-Wiener indices of the corona of two graphs.

MSC: 05C99

Key words: corona graph; Wiener index; hyper-Wiener index; reverse- Wiener index

# 1 Introduction

We consider finite undirected connected graphs without loops or multiple edges. Corona graphs were introduced by Frucht and Harary in 1970 [8]. The corona of two graphs $G$ and $H$ (where $G$ has $n$ vertices ), written as $G \odot H$, is defined as the graph obtained by taking one copy of $G$ and $n$ copies of $H$, and then joining by an edge the $i$th vertex of $G$ to every vertex in the $i$th copy of $H$. In 2002, Barrientos [2] first studied the graceful labelings of the corona graphs; soon after, some results on the corona graphs are obtained in succession. Lai and Chang [17] gave the exact values of the profiles of coronas $G \odot H$; Kwong and Lee [14] investigated the integer-magic spectra of the coronas of some specific graphs including paths, cycles, complete graphs and stars; the basis number of the corona of graphs is determined by Shakhatreh and Al-Rhayyel [20]; Barik et al. [1] and Kojima [13] investigated the spectrum and the bandwidth of the corona of two graphs respectively; Rodríguez-Velázquez et al. [19] studied

*Supported by NSFC (No. 11061034, 11061035) and XJEDU2010I01, the Natural Science Foundation of Xinjiang University (XY110102) and Research Fund for the Doctoral Program of Xinjiang University (BS110103).

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the relationship between the partition dimension of $G \odot H$ and several parameters of the graphs $G \odot H$, $G$ and $H$, including the metric dimension of $G \odot H$, the partition dimension of $G$ and the partition dimension of $H$. In this work we consider some Wiener-type indices of the corona graphs.

In the rest of this section, we present some basic concepts and notations. Let $G = (V, E)$ be a connected graph. For vertices $x, y$ of $G$, we denote by $deg_G(x)$ and $d_G(x, y)$ the degree of $x$ and the distance between vertices $x$ and $y$ of $G$, respectively. Recall that the distance $d_G(x, y)$ is the length of the shortest path joining $x$ and $y$ in $G$, while the diameter of G is defined as $D(G) = max\{d_G(x, y)|x, y \in V(G)\}$. The distance of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the sum of distances between $v$ and all other vertices of $G$.

The Wiener index of a graph G is defined by:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v).$$

First studied by Wiener in 1947 for acyclic molecular graphs G [21], the Wiener index is one of the most popular topological indices in combinatorial chemistry.

The hyper-Wiener index of acyclic graphs was introduced by Randić in 1993. It is one of the recently conceived distance-based graph invariants, used as a structure-descriptor for predicting physicochemical properties of organic compounds (often significant for pharmacology, agriculture and environmental protection). Klein et al.[15] generalized this extension to cyclic structures as

$$WW(G) = \frac{1}{2}W(G) + \frac{1}{2} \sum_{\{u,v\}\subseteq V(G)} d_G^2(u, v),$$

where $d_G^2(u, v) = d_G(u, v)^2$. We encourage the reader to consult [4, 5, 9, 11, 12, 16, 23] for the mathematical properties of hyper-Wiener index and its applications in chemistry.

The reverse-Wiener index was proposed by Balaban et al. in 2000 [3], it turns out that this index is important for a reverse problem and also found applications in modeling of structure-property relations [3, 10]. The reverse-Wiener index is defined as follows:

$$\Lambda(G) = \frac{1}{2}n(n-1)D(G) - W(G)$$

where $n$ is the number of vertices and $D(G)$ is the diameter of $G$. Some mathematical properties of the reverse-Wiener index may be found in [6, 18, 22].

In this paper, we present the explicit formulae for the aforementioned Wiener-type indices of the corona graphs.

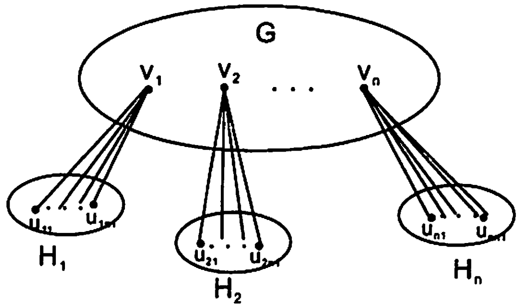


Fig. 1: Corona of two graphs $G$ and $H$, where $H_i$ is the copy of $H$, for $i \in \{1, 2, \cdots, n\}$.

# 2 Main results

## 2.1 Wiener index

In this subsection, we give an exact formula for the Wiener index of the corona of two graphs.

**Theorem 2.1** *Let $G$ and $H$ be two graphs of orders $n$ and $n_1$, respectively. Then $W(G \odot H) = (n_1+1)^2 W(G) + n^2(n_1^2+n_1) - n(n_1+m_1)$, where $m_1$ is the number of edges of $H$.*

**Proof.** Set $V(G)=\{v_1, v_2, \cdots, v_n\}$ and $V(H_i) = \{u_{i1}, u_{i2}, \cdots, u_{in_1}\}$, $i \in \{1, 2, \cdots, n\}$(see Figure1). We partition the set of unordered pairs of vertices of $G \odot H$ into four subsets: $V_1$, $V_2$, $V_3$, and $V_4$, where $V_1=\{\{u,v\} \subseteq V(G \odot H) |\ u, v \in V(H_i),\ i \in \{1, 2, \cdots, n\}\}$, $V_2=\{\{u,v\} \subseteq V(G \odot H) |\ v \in V(G), u \in V(H_i), i \in \{1, 2, \cdots, n\}\}$, $V_3=\{\{u,v\} \subseteq V(G \odot H) |\ u, v \in V(G)\}$, and $V_4=\{\{u,v\} \subseteq V(G \odot H) |\ u \in V(H_i), v \in V(H_j),\ i, j \in \{1, 2, \cdots, n\}$ and $i \neq j\}$. Then $W(G \odot H) = \sum_{i=1}^{4} \sum_{\{u,v\} \in V_i} d_G(u, v)$, and we distinguish the following cases:

Case 1. $\{u, v\} \in V_1$. Suppose that $u_{ik}, u_{il} \in V(H_i)$, for $i \in \{1, 2, \cdots, n\}$, and $k, l \in \{1, 2, \cdots, n_1\}$, then the summation of distances between $u_{ik}$ and $u_{il}$ for $\{u_{ik}, u_{il}\} \in V_1$ is

$$\begin{aligned} A &:= n \sum_{\{u_{ik}, u_{il}\} \subseteq V(H_i)} d_{G \odot H}(u_{ik}, u_{il}) \\ &= n(m_1 + 2(\binom{n_1}{2}) - m_1) = nn_1^2 - n(n_1 + m_1). \end{aligned}$$

Case 2. $\{u, v\} \in V_2$. Suppose that $v_i \in V(G), u_{jk} \in V(H_j)$, for $i, j \in \{1, 2, \cdots n\}, k \in \{1, 2, \cdots n_1\}$, then the summation of distances between $v_i$

and $u_{jk}$ for $\{v_i, u_{jk}\} \in V_2$ is

$$B := \sum_{v_i \in V(G), u_{jk} \in V(H_j)} d_{G \odot H}(v_i, u_{jk}) = \sum_{v_i \in V(G), u_{jk} \in V(H_j)} (d_G(v_i, v_j) + 1)$$
$$= nn_1 + (2W(G) + n(n-1))n_1 = 2n_1(W(G) + n^2 n_1.$$

Case 3. $\{u, v\} \in V_3$. Suppose that $v_i, v_j \in V(G)$, for $i, j \in \{1, 2, \cdots, n\}$, then the summation of distances between $v_i$ and $v_j$ for $\{v_i, v_j\} \in V_3$ is

$$C := \sum_{\{v_i, v_j\} \subseteq V(G)} d_{G \odot H}(v_i, v_j) = W(G).$$

Case 4. $\{u, v\} \in V_4$. Suppose that $u_{ik} \in V(H_i), u_{jl} \in V(H_j)$, for $i, j \in \{1, 2, \cdots n\} (i \neq j), k, l \in \{1, 2, \cdots n_1\}$, then the summation of distances between $u_{ik}$ and $u_{jl}$ for $\{u_{ik}, u_{jl}\} \in V_4$ is

$$D := \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j), i \neq j}} d_{G \odot H}(u_{ik}, u_{jl}) = \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j) i \neq j}} (d_G(v_i, v_j) + 2)$$
$$= n_1^2 (W(G) + 2\binom{n}{2}) = n_1^2 W(G) + n^2 n_1^2 - nn_1^2.$$

Hence,

$$W(G \odot H) = A + B + C + D$$
$$= (n_1 + 1)^2 W(G) + n^2(n_1^2 + n_1) - n(n_1 + m_1).$$

## 2.2 Hyper-Wiener index

Next we will give an explicit formula for the hyper-Wiener index of $G \odot H$ in terms of the (hyper-)Wiener, the order and the size of the factor graphs.

**Theorem 2.2** *Let $G$ and $H$ be two graphs with $|V(G)| = n, |V(H)| = n_1$ and $|E(H)| = m_1$. Then $WW(G \odot H) = (n_1^2 + 2n_1)W(G) + (n_1 + 1)^2 WW(G) + n^2(\frac{3}{2}n_1^2 + n_1) - n(\frac{3}{2}n_1 + 2m_1)$.*

**Proof.** Set $V(G)=\{v_1, v_2, \cdots, v_n\}$ and $V(H_i)= \{u_{i1}, u_{i2}, \cdots, u_{in_1}\}$, $i \in \{1, 2, \cdots, n\}$ (see Figure 1). We partition the set of unordered pair of vertices of $V(G \odot H)$ into four subsets: $V_1$, $V_2$, $V_3$, and $V_4$, where $V_1=\{\{u, v\} \subseteq V(G \odot H)|\ u, v \in V(H_i),\ i \in \{1, 2, \cdots, n\}\}$, $V_2=\{\{u, v\} \subseteq V(G \odot H)|\ v \in V(G), u \in V(H_i), i \in \{1, 2, \cdots, n\}\}$, $V_3=\{\{u, v\} \subseteq V(G \odot H)|\ u, v \in V(G)\}$, and $V_4=\{\{u, v\} \subseteq V(G \odot H)|\ u \in V(H_i), v \in V(H_j),\ i, j \in \{1, 2, \cdots, n\}$ and $i \neq j\}$. Then by definition, we have

$$WW(G \odot H) = \frac{1}{2} W(G \odot H) + \frac{1}{2} \sum_{i=1}^{4} \sum_{\{u,v\} \in V_i} d^2_{G \odot H}(u, v).$$

Case 1. $\{u, v\} \in V_1$. Suppose that $u_{ik}, u_{il} \in V(H_i)$, for $i \in \{1, 2, \cdots, n\}$, and $k, l \in \{1, 2, \cdots, n_1\}$, then the summation of distances between $u_{ik}$ and $u_{il}$ for $\{u_{ik}, u_{il}\} \in V_1$ is

$$\begin{aligned} A &:= \frac{n}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{il} \in V(H_i)}} d_{G \odot H}(u_{ik}, u_{il}) + \frac{n}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{il} \in V(H_i)}} d^2_{G \odot H}(u_{ik}, u_{il}) \\ &= \frac{n}{2}[m_1 + 2(\frac{1}{2} n_1(n_1 - 1) - m_1)] + \frac{n}{2}[m_1 + 4(\frac{1}{2} n_1(n_1 - 1) - m_1)] \\ &= \frac{3n}{2} n_1(n_1 - 1) - 2nm_1. \end{aligned}$$

Case 2. $\{u, v\} \in V_2$. Suppose that $v_i \in V(G), u_{jk} \in V(H_j)$, for $i, j \in \{1, 2, \cdots n\}, k \in \{1, 2, \cdots n_1\}$, then the summation of distances between $v_i$ and $u_{jk}$ for $\{v_i, u_{jk}\} \in V_2$ is

$$\begin{aligned} B &:= \frac{1}{2} \sum_{\substack{v_i \in V(G) \\ u_{jk} \in V(H_j)}} d_{G \odot H}(v_i, u_{jk}) + \frac{1}{2} \sum_{\substack{v_i \in V(G) \\ u_{jk} \in V(H_j)}} d^2_{G \odot H}(v_i, u_{jk}) \\ &= \frac{1}{2} \sum_{\substack{v_i \in V(G) \\ u_{jk} \in V(H_j)}} (d_G(v_i, v_j) + 1) + \frac{1}{2} \sum_{\substack{v_i \in V(G) \\ u_{jk} \in V(H_j)}} (d_G(v_i, v_j) + 1)^2 \\ &= 2n_1(WW(G) + W(G)) + n^2 n_1. \end{aligned}$$

Case 3. $\{u, v\} \in V_3$. Suppose that $v_i, v_j \in V(G)$, for $i, j \in \{1, 2, \cdots, n\}$, then the summation of distances between $v_i$ and $v_j$ for $\{v_i, v_j\} \in V_3$ is

$$C := \frac{1}{2} \sum_{\{v_i, v_j\} \subseteq V(G)} d_{G \odot H}(v_i, v_j) + \frac{1}{2} \sum_{\{v_i, v_j\} \subseteq V(G)} d^2_{G \odot H}(v_i, v_j)$$

$$= \frac{1}{2} \sum_{\{v_i, v_j\} \subseteq V(G)} d_G(v_i, v_j) + \frac{1}{2} \sum_{\{v_i, v_j\} \subseteq V(G)} d^2_G(v_i, v_j) = WW(G).$$

Case 4. $\{u, v\} \in V_4$. Suppose that $u_{ik} \in V(H_i), u_{jl} \in V(H_j)$, for $i, j \in \{1, 2, \cdots n\}(i \neq j), k, l \in \{1, 2, \cdots n_1\}$, then the summation of distances between $u_{ik}$ and $u_{jl}$ for $\{u_{ik}, u_{jl}\} \in V_4$ is

$$\begin{aligned} D &:= \frac{1}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j), i \neq j}} d_{G \odot H}(u_{ik}, u_{jl}) + \frac{1}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j), i \neq j}} d^2_{G \odot H}(u_{ik}, u_{jl}) \\ &= \frac{1}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j), i \neq j}} (d_G(v_i, v_j) + 2) + \frac{1}{2} \sum_{\substack{u_{ik} \in V(H_i) \\ u_{jl} \in V(H_j), i \neq j}} (d_G(v_i, v_j) + 2)^2 \\ &= n_1^2(WW(G) + W(G)) + \frac{3}{2} n_1^2 n(n-1). \end{aligned}$$

Now we can compute the hyper-Wiener index of $G \odot H$:

$$\begin{aligned} WW(G \odot H) &= A + B + C + D \\ &= WW(G)(n_1 + 1)^2 + W(G)(n_1^2 + 2n_1) \\ &\quad + n^2(\frac{3}{2} n_1^2 + n_1) - n(\frac{3}{2} n_1 + 2m_1). \end{aligned}$$

## 2.3 Reverse-Wiener index

By definition it is easy to see that $D(G \odot H) = D(G) + 2$, then by Theorem 2.1 and by a simple calculation, we have the following result on the reverse-Wiener index of the corona of two graphs.

**Theorem 2.3** *Let $G$ and $H$ be two graphs with $|V(G)| = n, |V(H)| = n_1$ and $|E(H)| = m_1$. Then* $\Lambda(G \odot H) = \frac{1}{2} D(G)[n^2(n_1 + 1)^2 - n(n_1 + 1)] - (n_1 + 1)^2 W(G) + n^2(n_1 + 1) + n(m_1 - 1)$.

**Acknowledgments**

The authors are thankful to the anonymous referee for the helpful remarks.

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