On super edge magic deficiency of kite graphs

Ali Ahmad[1], M.K. Siddiqui[2], M.F. Nadeem[2], M. Imran[3]

[1] *College of computer and information system, Jazan University, Jazan, KSA.*
[2] *Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore, Pakistan.*
[3] *Center for Advanced Mathematics and Physics (CAMP), National University of Sciences and Technology (NUST), H-12 Sector, Islamabad, Pakistan.*
E-mail: ahmadsms@gmail.com, kamransiddiqui75@gmail.com, mfaisalnadeem@ymail.com, imrandhab@gmail.com

Abstract

Motivated by Kotzig and Rosas concept of edge magic deficiency, Figueroa-Centeno, Ichishima and Muntaner-Batle defined a similar concept for super edge magic total labelings. The *super edge magic deficiency* of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$, has a super edge magic total labeling or it is equal to $+\infty$ if there exists no such n. In this paper, we study the super edge magic deficiency of kite graphs.

Keywords : edge magic labeling, super edge magic labeling, super edge magic deficiency, path, cycle, kite graphs .
MR (2000) Subject Classification : 05C78.

1 Introduction and Definitions

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. An *edge magic labeling* of a graph G is a bijection $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that $f(x) + f(xy) + f(y)$ constant, for every edge $xy \in E(G)$. A graph with an edge magic labeling is called *edge magic graph.* An edge magic labeling ϕ is called *super edge magic* if $\phi(V(G)) = \{1, 2, \ldots, p\}$. A graph with super edge magic labeling is called a *super edge magic graph.*

In [14], Kotzig and Rosa proved that for any graph G there exists an edge magic graph H such that $H \cong G \cup nK_1$ for some nonnegative integer n. This fact leads to the concept of edge magic deficiency of a graph G, which is the minimum nonnegative integer n such that $G \cup nK_1$ is edge magic and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = min\{n \geq 0 : G \cup nK_1 \text{ is edge magic}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge magic deficiency of a graph G with n vertices, $\mu(G) \leq F_{n+2} - 2 - n - \frac{1}{2}n(n-1)$, where F_n is the nth Fibonacci number. Motivated by Kotzig and Rosa's concept of edge magic deficiency, Figueroa-Centeno *et al* [8] defined a similar concept for super edge magic labelings. The super edge magic deficiency of a graph G, which is denoted by $\mu_s(G)$, is the minimum nonnegative integer n such that $G \cup nK_1$ has a super edge magic labeling or $+\infty$ if there exists no such n, formally defined as:

$$Let\ M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge magic graph}\}, then$$

$$\mu_s(G) = \begin{cases} min\ M(G), & \text{if } M(G) \neq \phi; \\ +\infty, & \text{if } M(G) = \phi. \end{cases}$$

As a consequence of the above two definitions, we note that for every graph G, $\mu(G) \leq \mu_s(G)$.

In [8, 9], Figueroa-Centeno *et al* provided the exact values for the super edge magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs, and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. They proved that

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ 1, & \text{if } n \equiv 0 \pmod 4 \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

For more detail, the results on edge magic and super edge magic labeling of some graphs can be seen in [3, 4, 5, 6, 8, 11, 12, 15] and a complete survey [10].

In this paper, we discuss the super edge magic deficiency μ_s of kite graphs. In particular, we show that (n,t)−kite has (a) super edge magic deficiency one for all odd $n \geq 5$ and even $t \geq 4$; (b) super edge magic deficiency less than or equals to one for all odd $n \geq 5$ and $t \equiv 3, 7 \pmod 8$; super edge magic deficiency less than or equals to one for all $n \geq 10$, $n \equiv 2 \pmod 4$ and $t = 4$; super edge magic deficiency one for all $n \geq 10$, $n \equiv 2 \pmod 4$ and $t = 5$.

In [17] Wallis posed the problem of investigating the edge magic properties of C_n with the path of length t attached to one vertex. Kim and Park [13] call such a graph an (n,t)−kite. The following proposition, proved by Ahmad and Muntaner-Batle [2], show that for an (n,t)−kite to be super edge-magic, n and t must have same parity.

Proposition 1. *[2] Let $G = (n,t)-$kite. If G is super edge-magic then n and t have the same parity.*

In proving our results, we frequently use the following lemma:

Lemma 1. *[7] A graph G with p vertices and q edges is super edge magic total if and only if there exists a bijective function $\phi : V(G) \to \{1, 2, \cdots, p\}$ such that the set $S = \{\phi(x) + \phi(y) : xy \in E(G)\}$ consists of q consecutive integers. In such a case, ϕ extends to a super edge magic total labeling of G.*

Kim and Park [13] proved that an $(n,1)-$kite is super edge-magic if and only if n is odd and an $(n,3)-$kite is super edge magic if and only if n is odd and at least 5. Also, Park, Choi and Bae [16] proved that an $(n,2)-$kite is super edge magic if and only if n is even. From the Proposition 1, $(n,t)-$kite is not super edge magic if n is odd and t is even. In the next theorem, we give the exact value of super edge magic deficiency of $(n,t)-$kite graph, for all odd $n \geq 5$ odd and even $t \geq 4$.

Theorem 1. *For all odd $n \geq 5$ and even $t \geq 4$, let G be an $(n,t)-$kite graph. Then $\mu_s(G) = 1$.*

Proof. Let $x_1, x_2, \ldots, x_n, x_1$ be a vertex sequence of C_n and let $y_1, y_2, \ldots, y_t$ be the vertices of the path (the tail). Let $G^* = G \cup K_1$, the vertex set and edge set of G^* are defined as:

$$V(G^*) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq t\} \cup \{z\}$$

$$E(G^*) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_j y_{j+1} : 1 \leq j \leq t-1\} \cup \{x_n x_1, y_1 x_n\}$$

By Proposition 1, $G = (n,t)$-kite graph is not super edge-magic for n odd and t even. Therefore

$$\mu_s(G) \geq 1.$$

To prove $\mu_s(G) \leq 1$, we define the labeling $\phi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 1\}$ of the graph G^* in the following two cases:

Case 1 When $t \equiv 2 \pmod 4$,

$$\phi(x_i) = \begin{cases} \frac{n+i+t}{2}, & \text{for odd } i;\ 1 \leq i \leq n, \\ \frac{i}{2}, & \text{for even } i;\ 1 \leq i \leq n, \end{cases}$$

$$\phi(y_j) = \begin{cases} \frac{n+j}{2}, & \text{for odd } j;\ 1 \leq j \leq t, \\ \frac{2n+t+j}{2}, & \text{for even } j;\ 2 \leq j \leq \frac{t}{2}, \\ \frac{2n+t+2+j}{2}, & \text{for even } j;\ \frac{t}{2}+1 \leq j \leq t, \end{cases}$$

The isolated vertex z, is labeled as $\phi(z) = \frac{4n+3t+2}{4}$.
Case 2 When $t \equiv 0 \pmod 4$, we define the labeling for $t = 4$ and $t \geq 6$.
For $t = 4$,

$$\phi(x_i) = \begin{cases} \frac{n-1}{2}, & \text{for } i = 1, \\ n+2, & \text{for } i = 2, \\ \frac{n+2+i}{2}, & \text{for odd } i;\ 3 \leq i \leq n, \\ \frac{i-2}{2}, & \text{for even } i;\ 3 \leq i \leq n, \end{cases}$$

$$\phi(y_j) = \begin{cases} \frac{n-j+4}{2}, & \text{for } j = 1,3, \\ n+5, & \text{for } j = 2, \\ n+3, & \text{for } j = 4, \end{cases}$$

The isolated vertex z, is labeled as $\phi(z) = n + 4$.
For all even $t \geq 6$,

$$\phi(x_i) = \begin{cases} \frac{2n+t}{2}, & \text{for } i = 1, \\ \frac{i-1}{2}, & \text{for odd } i;\ 2 \leq i \leq n, \\ \frac{n+t+i-1}{2}, & \text{for even } i;\ 2 \leq i \leq n, \end{cases}$$

$$\phi(y_j) = \begin{cases} \frac{n+j-1}{2}, & \text{for even } j;\ 1 \leq j \leq t, \\ \frac{t+2n+j+1}{2}, & \text{for odd } j;\ 1 \leq j \leq \frac{t}{2}, \\ \frac{t+2n+j+3}{2}, & \text{for odd } j;\ \frac{t}{2}+1 \leq i \leq t, \end{cases}$$

The isolated vertex z, is labeled as $\phi(z) = \frac{4n+3t+4}{4}$.
One can see that all edge sums generated by the above formula form the following set of q consecutive integers: $\{\frac{n+t+3}{2}, \frac{n+t+3}{2}+1, \ldots, \frac{3n+3t+1}{2}\}$. Therefore by using Lemma 1, ϕ can be extended to a super edge magic total labeling. Hence, the graph G^* admits a super edge magic total labeling. □

In [2], Ahmad *et al.* determined the exact value of super edge magic deficiency of (n,t)-kite graph for all odd n; $t \equiv 0, 1 \pmod 4$, and also showed the upper bound for all odd n, $t \equiv 2, 3 \pmod 4$. In the next theorem, we improve the upper bound for all odd n and $t \equiv 3, 7 \pmod 8), t \neq 11$.

Theorem 2. *For all odd $n \geq 5$ and $t \geq 5, t \neq 11, t \equiv 3, 7 \pmod 8$, let G be a kite graph. Then $\mu_s(G) \leq 1$.*

Proof. Let $x_1, x_2, \ldots, x_n, x_1$ be a vertex sequence of C_n and let $y_1, y_2, \ldots, y_t$ be the vertices of the path (the tail). Let $G^* = G \cup K_1$, the vertex set and edge set of G^* are defined as:

$$V(G^*) = \{x_i : 1 \le i \le n\} \cup \{y_j : 1 \le j \le t\} \cup \{z\}$$

$$E(G^*) = \{x_ix_{i+1} : 1 \le i \le n-1\} \cup \{y_jy_{j+1} : 1 \le j \le t-1\} \cup \{x_nx_1, y_1x_n\}$$

To prove $\mu_s(G) \le 1$, we define the labeling $\phi : V(G^*) \to \{1, 2, \ldots, |V(G)| + 1\}$ of the graph G^* in the following two cases:

Case 1 When $t = 8s + 7, s \ge 0$

$$\phi(x_i) = \begin{cases} \frac{i}{2}, & \text{for even } i;\ 1 \le i \le n, \\ \frac{n+i+t+1}{2}, & \text{for odd } i;\ 1 \le i \le n, \end{cases}$$

$$\phi(y_j) = \begin{cases} \frac{n+j}{2}, & \text{for odd } j;\ 1 \le j \le \frac{t-1}{2}, \\ \frac{n+t-j+5+8s}{2}, & \text{for odd } j;\ \frac{t+3+8s}{2} \le j \le \frac{t+7+8s}{2}, \\ & \text{where } 0 \le s \le \frac{t-7}{8} \\ \frac{2n+t+j+1}{2}, & \text{for even } j;\ 1 \le j \le \frac{t-3}{2}, \\ \frac{4n+t+4j+1}{4}, & \text{for even } j;\ \frac{t+1}{2} \le j \le \frac{t+5}{2} \\ \frac{2n+2t-j+14+8s}{2}, & \text{for even } j;\ \frac{t+9+8s}{2} \le j \le \frac{t+13+8s}{2} \\ & \text{where } 0 \le s < \frac{t-7}{8} \end{cases}$$

The isolated vertex z is labeled as $\phi(z) = \frac{4n+3t+7}{4}$.

Case 2 When $t = 8s + 3, s \ge 2$,

$$\phi(x_i) = \begin{cases} \frac{i}{2}, & \text{for even } i;\ 1 \le i \le n, \\ \frac{n+t+1+i}{2}, & \text{for odd } i;\ 1 \le i \le n, \end{cases}$$

$$\phi(y_j) = \begin{cases} \frac{n+j}{2}, & \text{for odd } j;\ 1 \le j \le \frac{t-1}{2}, \\ \frac{n+t-j+5}{2}, & \text{for odd } j;\ \frac{t+3}{2} \le j \le \frac{t+7}{2}, \\ \frac{n+t-j+11}{2}, & \text{for odd } j;\ j = \frac{t+11}{2}, \\ \frac{n+t-j+17+8s}{2}, & \text{for odd } j;\ \frac{t+15+8s}{2} \le j \le \frac{t+19+8s}{2}, \\ & \text{where } \ 0 \le s \le \frac{t-19}{8} \\ \frac{2n+t+j+1}{2}, & \text{for even } j;\ 1 \le j \le \frac{t-3}{2} \\ \frac{4n+t+4j+1}{4}, & \text{for even } j;\ \frac{t+1}{2} \le j \le \frac{t+5}{2}, \\ \frac{2n+2t-j+14+8s}{2}, & \text{for even } j;\ \frac{t+9+8s}{2} \le j \le \frac{t+13+8s}{2}, \\ & \text{where } \ 0 \le s \le \frac{t-19}{8} \\ n+t+1, & \text{for } \ j = t-1 \end{cases}$$

The isolated vertex z under the labeling ϕ is labeled as $\phi(z) = \frac{4n+3t+7}{4}$.

One can see that the set all edge sums of both cases generated by the above formula forms a set q consecutive integers: $\{\frac{n+t+4}{2}, \frac{n+t+6}{2}, \frac{n+t+8}{2}, \dots, \frac{6n+5t+8s+11}{4}\}$, $s = \frac{t-7}{8}$; $\{\frac{n+t+4}{2}, \frac{n+t+6}{2}, \frac{n+t+8}{2}, \dots, \frac{6n+5t+8s+23}{4}\}$, $s = \frac{t-19}{8}$, respectively. Therefore by Lemma 1, ϕ can be extended to a super edge magic total labeling. This shows that $\mu_s(G) \leq 1$, which completes the proof. □

Ahmad *et al.* [1] found the exact value of super edge magic deficiency of $(n,t)-$kite graph for n even and $t = 1, 3$. In the following theorem, we found the upper bound and exact value of super edge magic deficiency of $(n,t)-$kite graph for $n \equiv 2 \pmod 4$, $t = 4$ and $t = 5$, respectively.

Theorem 3. *For $n \geq 10$ and $n \equiv 2 \pmod 4$, the super edge magic deficiency of $G = (n,t)$-kite graph is*

$$\mu_s(G) \begin{cases} \leq 1, & for \;\; t = 4 \\ = 1, & for \;\; t = 5 \end{cases}$$

Proof. Let $n \equiv 2 \pmod 4$ be a nonnegative integer. Let $G = (n,t)$-kite graph. Recall the vertex set and edge set of G $\{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq t\}$, and $\{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_j y_{j+1} : 1 \leq j \leq t-1\} \cup \{x_n x_1, x_n y_1\}$, respectively. To prove $\mu_s(G) \leq 1$ for $t = 4, 5$, according to Lemma 1 it is sufficient to prove that there exists a vertex labeling with the property that the edge-sums under this labeling are consecutive q integers. It is easy to see that the following labeling $\phi : V(G \cup K_1) \to \{1, 2, \dots, |V(G)| + 1\}$ has the desired property.
The labeling of $G = (6,4)$-kite graph is shown in Figure 1.

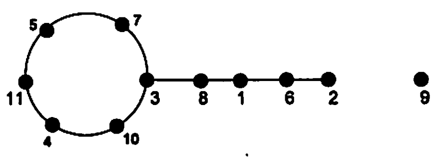

Figure 1: An illustration for the labeling of $G = (6,4)$-kite graph

Here, we label $G \cup K_1$ where $V(K_1) = \{z\}$ as follows:

$$\phi(x_i) = \begin{cases} \frac{i+5}{2}, & \text{for odd } i;\ 1 \le i \le n-1, \\ \frac{n+8}{2}, & \text{for } \ i = n \text{ and } t = 4, \\ \frac{n+10}{2}, & \text{for } \ i = n \text{ and } t = 5, \\ \frac{n+i+10}{2}, & \text{for even } i;\ 1 \le i \le \frac{n-4}{2} \text{ and } t = 4, \\ \frac{n+i+12}{2}, & \text{for even } i;\ 1 \le i \le \frac{n-4}{2} \text{ and } t = 5, \\ \frac{n+i+12}{2}, & \text{for even } i;\ \frac{n-4}{2}+1 \le i \le n-1 \text{ and } , t = 4 \\ \frac{n+i+14}{2}, & \text{for even } i;\ \frac{n-4}{2}+1 \le i \le n-1 \text{ and } , t = 5 \end{cases}$$

For $t = 4$

$$\phi(y_j) = \begin{cases} \frac{n+10}{2}, & \text{for } j = 1 \\ \frac{j}{2}, & \text{for } j = 2, 4 \\ \frac{n+6}{2}, & \text{for } j = 3 \end{cases}$$

For $t = 5$

$$\phi(y_j) = \begin{cases} \frac{n+12}{2}, & \text{for } j = 1 \\ \frac{j}{2}, & \text{for } j = 2, 4, \\ \frac{n+j+3}{2}, & \text{for } j = 3, 5, \end{cases}$$

The isolated vertex z, is labeled as

$$\phi(z) = \begin{cases} \frac{3n+18}{4}, & \text{for } t = 4, \\ \frac{3n+22}{4}, & \text{for } t = 5, \end{cases}$$

One can see that the set all edge sums generated by the above formula forms a set q consecutive integers: $\{\frac{n+8}{2}, \frac{n+10}{2}, \ldots, \frac{3n+14}{2}\}$; $\{\frac{n+8}{2}, \frac{n+10}{2}, \ldots, \frac{3n+16}{2}\}$ for $t = 4; 5$, respectively. Therefore by Lemma 1, ϕ can be extended to a super edge magic total labeling. This shows that $\mu_s(G) \le 1$.

By Proposition 1, $G = (n, 5)$-kite graph is not super edge magic for n even. Therefore $\mu_s(G = (n, 5) - \text{kite}) \ge 1$. Which completes the proof.

□

References

[1] A. Ahmad, I. Javaid and M. F. Nadeem, Further results on super edge magic deficiency of unicyclic graphs, *Ars Combin.*, **99**(2011), 129-138.

[2] A. Ahmad and F. A. Muntaner-Batle, On Super Edge-Magic Deficiency of Unicyclic Graphs, Preprint.

[3] A. Ahmad, A. Q. Baig and M. Imran, On super edge-magicness of graphs, *Utilitas Math.*, to appear.

[4] A. Q. Baig, Super edge-magic deficiency of forests, PhD Thesis (2011).

[5] A. Q. Baig, E. T. Baskoro and A. Semaničová–Feňovčíková, On the super edgemagic deficiency of a star forest, *Ars Combin.*, to appear.

[6] A. Q. Baig, A. Ahmad, E. T. Baskoro and R. Simanjuntak, On the super edgemagic deficiency of forests, *Utilitas Math.*, to appear.

[7] R. M. Figueroa, R. Ichishima and F. A. Muntaner-Batle, The place of super edge-magic labeling among other classes of labeling, *Discrete Math.* **231**(2001), 153-168.

[8] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, On the super edge magic deficiency of graphs, *Electron. Notes Discrete Math.*, 11, 2002.

[9] R. M. Figueroa-Centeno, R. Ichishima, F. A. Muntaner-Batle, On the super Edge-Magic Deficiency of Graphs. *Ars Combin.*, **78**(2006), 33-46.

[10] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* 18 #DS6, http://www.combinatorics.org/surveys/ds6.pdf (2010).

[11] M. Hussain, E. T. Baskoro, K. Ali, On super antimagic total labeling of Harary graph, *Ars combin.*, to appear.

[12] M. Hussain, E. T. Baskoro, Slamin, On super edge-magic total labeling of ba- nana trees, *Utilitas Math.*, **79**(2009), 243-251.

[13] S.-R. Kim and J. Y. Park, On super edge-magic graphs, *Ars Combin.*, **81**(2006), 113-127.

[14] A. Kotzig and A. Rosa, Magic valuaton of finite graphs, *Canad. Math. Bull.* 13(4)(1970), 451-461.

[15] A.A.G. Ngurah, R. Simanjuntak, and E.T. Baskoro, On the super edge-magic deficiencies of graphs, *Australas. J. Combin.*, **40**(2008), 3-14.

[16] J. Y. Park, J. H. Choi and J-H. Bae, On super edge-magic labeling of some graphs, *Bull. Korean Math. Soc.*, **45**(2008), No.1, 11-21.

[17] W. D. Wallis, Magic Graphs, Birkhäuser, Boston, 2001.

[18] W. D. Wallis, E. T. Baskoro, M. Miller, and Slamin, Edge-magic total labelings, *Australas. J. Combin.*, **22**(2000), 177-190.