Decomposition of Complete Graphs Into Paths of Length Three and Triangles

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Abstract. Let P_{k+1} denote a path of length k and let C_k denote a cycle of length k. A triangle is a cycle of length three. As usual K_n denotes the complete graph on n vertices. It is shown that for all nonnegative integers p and q and for all positive integers n, K_n can be decomposed into p copies of P_4 and q copies of C_3 if and only if $3(p+q) = e(K_n)$, $p \neq 1$ if n is odd, and $p \geq \frac{n}{2}$ if n is even.

1 Introduction

All graphs considered here are finite and undirected, unless otherwise noted. For the standard graph-theoretic terminology the reader is referred to [2].

As usual K_n denotes the complete graph on n vertices. A complete m-partite graph with m-partition $(V_1, V_2, \ldots, V_m)$ and $|V_i| = n_i$ is denoted by $K_{n_1,n_2,\ldots,n_m}$. In the case $m = 2$ we speak about complete bipartite graphs; if $m = 3$ we get, by analogy, complete tripartite graphs. Let P_{k+1} denote a path of length k and let C_k denote a cycle of length k. A triangle is a cycle of length three. Let $L = \{H_1, H_2, \ldots, H_r\}$ be a family of subgraphs of G. An *L-decomposition* of G is an edge-disjoint decomposition of G into positive integer α_i copies of H_i, where $i \in \{1, 2, \ldots, r\}$. Furthermore, if each H_i ($i \in \{1, 2, \ldots, r\}$) is isomorphic to a graph H, we say that G has an *H-decomposition*. The existence of a decomposition of the complete graph K_n into triangles is equivalent to the existence of a Steiner triple

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system of order n ($STS(n)$). It is not difficult to see that the necessary condition for such a decomposition to exist is that $n \equiv 1$ or $3 \pmod 6$. This condition was proved to be sufficient by Kirkman [10]. An automorphism of a STS (V, B) is a bijection $\alpha : V \to V$ such that $\{a, b, c\} \in B$ if and only if $\{\alpha(a), \alpha(b), \alpha(c)\} \in B$. A $STS(n)$ is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length n. In 1939, Peltesohn [12] proved that for all $n \equiv 1$ *or* $3 \pmod 6$ except $n = 9$, there exists a cyclic $STS(n)$.

A *C_k-packing* of a graph G is a set of edge-disjoint C_k's in G. A C_k-packing C of G is *maximum* if $|C| \geq |C'|$ for all other C_k-packings C' of G. The *leave* L of a C_k-packing C is the subgraph induced by the set of edges of G that do not occur in any C_k of the packing C. There is a paper by Hanani [7] in which the leaves for the maximum C_3-packings of K_n are summarized.

Path decomposition of graphs and cycle decomposition of graphs have both been studied extensively and so it is natural to consider the problem of L-decompositions of K_n, where L is a combination of paths and cycles. In [16], we proved that necessary conditions for decomposing K_n into p copies of P_{k+1} and q copies of C_l are that $pk + ql = e(K_n)$, $p \neq 1$ if n is odd, and $p \geq \frac{n}{2}$ if n is even. Besides, when $k = l$ and k is even, we gave some sufficient conditions for such a decomposition to exist. Especially, we obtained necessary and sufficient conditions for the existence of a decomposition of K_n into p copies of P_5 and q copies of C_4. In this paper we use the same arguments in the proof for the existence of a cyclic $STS(n)$ and the results on the leaves for the maximum C_3-packings of K_n to prove that those necessary conditions mentioned in [16] are also sufficient for the existence of a decomposition of K_n into p copies of P_4 and q copies of C_3 (see Theorem 3.5 and Theorem 4.8).

There are several papers concerned with a decomposition of a complete graph into triangles and some other subgraph or subgraphs. Bryant and Maenhaut [4] proved necessary and sufficient conditions for a decomposition of K_n into p triangles and q Hamilton cycles. Horak et al. [9] solved the problem of finding 2-factorizations of K_n into p triangle factors and q Hamilton cycles for several infinite classes of order n. Rees [13] proved necessary and sufficient conditions for a decomposition of K_n into p triangle factors and q 1-factors. Finally, there is a paper by Fu and Rodger [6] in which necessary and sufficient conditions for a decomposition of λK_n into p triangles and q 1-factors are proved.

2 Preliminaries

In this section we will introduce some known results concerned with path decomposition and cycle decomposition. Also, we give some useful lemmas and theorems which will be used for proving the main theorem. Let us first introduce two results on P_{k+1}-decomposition and C_k-decomposition of K_n as follows.

Theorem 2.1 (Tarsi [19]). *Let k and n be positive integers. K_n has a P_{k+1}-decomposition if and only if $n \geq k+1$ and $n(n-1) \equiv 0$ (mod $2k$).* □

Theorem 2.2 (Alspach, Gavlas and Šajna [1, 15]).

(1) *Let n and k be positive integers. K_n has a C_k-decomposition if and only if n is odd, $3 \leq k \leq n$, and $n(n-1) \equiv 0$ (mod $2k$).*

(2) *Let n and k be positive integers such that n is even and let I is an 1-factor of K_n. $K_n - I$ has a C_k-decomposition if and only if $3 \leq k \leq n$ and $n(n-2) \equiv 0$ (mod $2k$).* □

When n is odd, the following theorem gave a necessary condition for decomposing complete graphs K_n into paths and cycles.

Theorem 2.3 (Shyu [16, Theorem 2.2]). *Let n, l, and k be positive integers such that n is odd and $n \geq max\{l, k+1\}$. If K_n can be decomposed into p copies of P_{k+1} and q copies of C_l for nonnegative integers p and q, then $pk + ql = e(K_n)$ and $p \neq 1$.* □

When n is even, the following theorem gave a necessary condition for decomposing complete graphs K_n into paths and cycles.

Theorem 2.4 (Shyu [16, Theorem 2.4]). *Let n, l, and k be positive integers such that n is even and $n \geq max\{l, k+1\}$. If K_n can be decomposed into p copies of P_{k+1} and q copies of C_l for nonnegative integers p and q, then $pk + ql = e(K_n)$ and $p \geq \frac{n}{2}$.* □

For our discussion we need the following notations. Let $x_1x_2 \dots x_{k+1}$ denote the path P_{k+1} with vertices $x_1, x_2, \dots, x_{k+1}$ and edges $x_1x_2, x_2x_3, \dots, x_kx_{k+1}$ and let $(x_1, x_2, \dots, x_k)$ denote the cycle C_k with vertices $x_1, x_2, \dots, x_k$ and edges $x_1x_2, x_2x_3, \dots, x_{k-1}x_k, x_kx_1$.

The following two lemmas will give sufficient conditions for decomposing an edge-disjoint union of cycles into P_{k+1}'s and C_k's.

Lemma 2.5 *Let k and n be positive integers such that $k \geq 3$ and $n \geq 2$. Suppose that for $i \in \{1, 2, \dots, n\}$, $C^{(i)}$ denotes the cycle of length k $(x_{(i,1)}, x_{(i,2)}, \dots, x_{(i,k)})$. If $x_{(1,1)} = x_{(2,1)} = \dots = x_{(n,1)}$, $x_{(i+1,2)} \notin V(C^{(i)})$*

for $i \in \{1,2,\ldots,n-1\}$, *and* $x_{(1,2)} \notin V(C^{(i)})$ *for* $i \in \{2,3,\ldots,n\}$, *then* $\bigcup_{i=1}^{n} C^{(i)}$ *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_k *for each pair* p, q *of nonnegative integers such that* $p+q=n$ *and* $p \neq 1$.

Proof. The case, $p=0$, is trivial. For any positive integer p such that $2 \leq p \leq n$, by assumption, $\bigcup_{i=1}^{p} C^{(i)}$ can be decomposed into p paths of length k below: $x_{(1,2)}x_{(1,3)}\cdots x_{(1,k)}x_{(1,1)}x_{(2,2)}$, $x_{(2,2)}x_{(2,3)}\cdots x_{(2,k)}x_{(2,1)}x_{(3,2)}$, $\ldots$, $x_{(p,2)}x_{(p,3)}\cdots x_{(p,k)}x_{(p,1)}x_{(1,2)}$. Since $\bigcup_{i=1}^{n} C^{(i)}$ can be viewed as an edge-disjoint union of $\bigcup_{i=1}^{p} C^{(i)}$ and $\bigcup_{i=p+1}^{n} C^{(i)}$, it implies that $\bigcup_{i=1}^{n} C^{(i)}$ can be decomposed into p copies of P_{k+1} and $n-p$ copies of C_k. □

The above lemma is used in the following.

Lemma 2.6 *Let* $n_1 \leq n_2 \leq \cdots \leq n_m$, m *and* k *be positive integers such that* $m \geq 2$, $k \geq 3$, $n_1 \geq 2$ *and* $n_m \geq 3$. *Suppose that* $C^{(i,j)}$ *is a cycle of length* k *for* $i \in \{1,2,\ldots,m\}$ *and* $j \in \{1,2,\ldots,n_m\}$. *If for* $i \in \{1,2,\ldots,m\}$, *there exists a vertex* x_i *such that* $V(C^{(i,s)}) \cap V(C^{(i,t)}) = \{x_i\}$ *for* $1 \leq s < t \leq n_i$, *then* $\bigcup_{i=1}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ *can be decomposed into* p *copies of* P_{k+1} *and* q *copies of* C_k *for each pair* p, q *of nonnegative integers such that* $p+q=\sum_{i=1}^{m} n_i$ *and* $p \neq 1$.

Proof. The case, $p=0$, is trivial and so suppose that $2 \leq p \leq \sum_{i=1}^{m} n_i$. Let $n_0=0$. Now assume $\sum_{i=0}^{l} n_i < p \leq \sum_{i=1}^{l+1} n_i$ for some $l \in \{0,1,\ldots,m-1\}$. We consider three cases as follows.

If $p-\sum_{i=0}^{l} n_i \neq 1$, by Lemma 2.5, then $\bigcup_{i=1}^{l}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=0}^{l} n_i$ copies of P_{k+1}; $\bigcup_{j=1}^{n_{l+1}} C^{(l+1,j)}$ can be decomposed into $p-\sum_{i=0}^{l} n_i$ copies of P_{k+1} and $n_{l+1}-(p-\sum_{i=0}^{l} n_i)$ copies of C_k for $2 \leq p-\sum_{i=0}^{l} n_i \leq n_{l+1}$. By assumption, $\bigcup_{i=l+2}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=l+2}^{m} n_i$ copies of C_k (note that $\bigcup_{i=l+2}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ is a null graph when $l=m-1$).

If $p-\sum_{i=0}^{l} n_i = 1$ and $n_l > 2$ (note that in this case $l \geq 1$), by Lemma 2.5 again, then $\bigcup_{i=1}^{l-1}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=0}^{l-1} n_i$ copies of P_{k+1} (note that $\bigcup_{i=1}^{l-1}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ is a null graph when $l=1$); $\bigcup_{j=1}^{n_l} C^{(l,j)}$ can be decomposed into n_l-1 copies of P_{k+1} and one copy of C_k; $\bigcup_{j=1}^{n_{l+1}} C^{(l+1,j)}$ can be decomposed into 2 copies of P_{k+1} and $n_{l+1}-2$ copies of C_k. By assumption, $\bigcup_{i=l+2}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=l+2}^{m} n_i$ copies of C_k (note that $\bigcup_{i=l+2}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ is a null graph when $l=m-1$).

If $p-\sum_{i=0}^{l} n_i = 1$ and $n_l = 2$ (note that in this case $1 \leq l \leq m-1$ and $n_1=\cdots=n_l=2$), by Lemma 2.5 again, then $\bigcup_{i=1}^{l-1}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=0}^{l-1} n_i$ copies of P_{k+1} (note that $\bigcup_{i=1}^{l-1}(\bigcup_{j=1}^{n_i} C^{(i,j)})$

is a null graph when $l = 1$); $\bigcup_{j=1}^{n_m} C^{(m,j)}$ can be decomposed into 3 copies of P_{k+1} and $n_m - 3$ copies of C_k. By assumption, $\bigcup_{i=l}^{m-1}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into $\sum_{i=l}^{m-1} n_i$ copies of C_k.

In each case mentioned above, we have that $\bigcup_{i=1}^{m}(\bigcup_{j=1}^{n_i} C^{(i,j)})$ can be decomposed into p copies of P_{k+1} and q copies of C_k for each pair p, q of nonnegative integers such that $p + q = \sum_{i=1}^{m} n_i$ and $p \neq 1$. □

In the following theorem we will give the necessary and sufficient conditions for the existence of a decomposition of a complete tripartite graph with equal parts into p copies of P_4 and q copies of C_3.

Theorem 2.7 *Let p and q be nonnegative integers and let n be a positive integer such that $n \geq 3$. There exists a decomposition of $K_{n,n,n}$ into p copies of P_4 and q copies of C_3 if and only if $3(p + q) = e(K_{n,n,n})$ and $p \neq 1$.*

Proof. (Necessity) Condition $3(p + q) = e(K_{n,n,n})$ is trivial. On the contrary, suppose that $p = 1$. Let P denote the only path of length 3 in the decomposition. It follows that the end vertices of P have odd degree $2n-1$ in $K_{n,n,n} - E(P)$. Therefore, $K_{n,n,n} - E(P)$ can not be decomposed into cycles. We obtained a contradiction.

(Sufficiency) Suppose that (X, Y, Z) is the tripartition of $K_{n,n,n}$, where $X = \{x_0, x_1, \ldots, x_{n-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ and $Z = \{z_0, z_1, \ldots, z_{n-1}\}$. For i, $j \in \{0, 1, \ldots, n-1\}$, we use $C^{(i,j)}$ to denote the cycle of length 3 (x_i, y_j, z_{i+j-1}); the subscripts of z_i's are taken modulo n. It is not difficult to see that those n^2 copies of C_3 are edge disjoint. Since $3(p + q) = e(K_{n,n,n}) = \binom{3}{2}n^2$, we have $p+q = n^2$ and hence $\{C^{(i,j)} \mid i, j = 0, 1, \ldots, n-1\}$ is a C_3-decomposition of $K_{n,n,n}$. On the other hand, since for $i \in \{0, 1, \ldots, n-1\}$, $V(C^{(i,s)}) \cap V(C^{(i,t)}) = \{x_i\}$ for $0 \leq s < t \leq n-1$, by Lemma 2.6, $\bigcup_{i=0}^{n-1}(\bigcup_{j=0}^{n-1} C^{(i,j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q = n^2$ and $p \neq 1$. This completes the proof. □

3 The case when n is odd

In this section we will use the same argument in the proof for the existence of a cyclic $STS(n)$ to prove that when n is odd, K_n can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p + q) = e(K_n)$ and $p \neq 1$.

Before proving the main theorem, we need to show three special cases for decomposing complete graphs into P_4's and C_3's. We first show that K_7 can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p + q) = e(K_7)$ and $p \neq 1$.

Lemma 3.1 *If p and q are nonnegative integers such that $p+q=7$ and $p \neq 1$, then K_7 can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_7) = \{x_1, x_2, \ldots, x_7\}$. We exhibit that K_7 can be decomposed into p copies of P_4 and q copies of C_3, for each pair p, q of nonnegative integers such that $3(p+q) = \binom{7}{2}$ (i.e. $p+q=7$) and $p \neq 1$ as follows:
(1) $p=0$ and $q=7$: By Theorem 2.2 (1), we are done.
(2) $p=2$ and $q=5$: $x_5x_1x_3x_7$, $x_7x_2x_6x_5$, (x_1,x_2,x_4), (x_1,x_6,x_7), (x_2,x_3,x_5), (x_3,x_4,x_6), (x_4,x_5,x_7).
(3) $p=3$ and $q=4$: $x_1x_7x_6x_5$, $x_5x_1x_3x_7$, $x_7x_2x_6x_1$, (x_1,x_2,x_4), (x_2,x_3,x_5), (x_3,x_4,x_6), (x_4,x_5,x_7).
(4) $p=4$ and $q=3$: $x_1x_6x_2x_7$, $x_4x_5x_6x_7$, $x_4x_7x_1x_5$, $x_5x_7x_3x_1$, (x_1,x_2,x_4), (x_2,x_3,x_5), (x_3,x_4,x_6).
(5) $p=5$ and $q=2$: $x_2x_3x_4x_5$, $x_2x_5x_1x_7$, $x_3x_1x_6x_5$, $x_3x_5x_7x_4$, $x_4x_6x_2x_7$, (x_1,x_2,x_4), (x_3,x_6,x_7).
(6) $p=6$ and $q=1$: $x_2x_3x_4x_5$, $x_2x_5x_1x_7$, $x_3x_1x_6x_7$, $x_3x_5x_7x_4$, $x_4x_6x_2x_7$, $x_5x_6x_3x_7$, (x_1,x_2,x_4).
(7) $p=7$ and $q=0$: By Theorem 2.1, we are done. □

Next, we will show that K_6 can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = \binom{6}{2}$ and $p \geq 3$.

Lemma 3.2 *If p and q are nonnegative integers such that $p+q=5$ and $p \geq 3$, then K_6 can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_6) = \{x_1, x_2, \ldots, x_6\}$. We exhibit that K_6 can be decomposed into p copies of P_4 and q copies of C_3, for each pair p, q of nonnegative integers such that $3(p+q) = \binom{6}{2}$ (i.e. $p+q=5$) and $p \geq 3$ as follows:
(1) $p=3$ and $q=2$: $x_1x_6x_2x_5$, $x_2x_4x_3x_6$, $x_3x_5x_1x_4$, (x_1,x_2,x_3), (x_4,x_5,x_6).
(2) $p=4$ and $q=1$: $x_1x_6x_2x_5$, $x_2x_4x_6x_5$, $x_3x_5x_1x_4$, $x_5x_4x_3x_6$, (x_1,x_2,x_3).
(3) $p=5$ and $q=0$: By Theorem 2.1, we are done. □

Finally, we will show that K_9 can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_9)$ and $p \neq 1$.

Lemma 3.3 *If p and q are nonnegative integers such that $3(p+q) = e(K_9)$ and $p \neq 1$, then K_9 can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Since $p \neq 1$ and $p+q=12$, we get $0 \leq p \leq 12$ and $p \neq 1$. Let us first prove that the theorem holds for $0 \leq p \leq 9$ and $p \neq 1$. It is easily seen that K_9 can be viewed as an edges-disjoint union of one copy of $K_{3,3,3}$

and three copies of K_3 (note that $K_3 \cong C_3$). By Theorem 2.7, $K_{3,3,3}$ can be decomposed into p copies of P_3 and $9-p$ copies of C_3 for $0 \leq p \leq 9$ and $p \neq 1$ and hence K_9 can be decomposed into p copies of P_4 and $12-p$ copies of C_3 for $0 \leq p \leq 9$ and $p \neq 1$. For $p = 12$, by Theorem 2.1, we are done and so the remaining cases are $p = 10$ and $p = 11$. It is not difficult to see that K_9 can be viewed as an edges-disjoint union of K_6, $K_{6,3}$ and K_3 . By Lemma 3.2, K_6 can be decomposed into l copies of P_4 and $5-l$ copies of C_3 for $3 \leq l \leq 5$. It is easily seen that $K_{3,3}$ can be decomposed into 3 copies of P_4. Since $K_{6,3}$ can be viewed as an edge-disjoint union of two copies of $K_{3,3}$, we have that $K_{6,3}$ can be decomposed into 6 copies of P_4. Therefore, K_9 can be decomposed into p copies of P_4 and $12-p$ copies of C_3 for $9 \leq p \leq 11$. This completes the proof. □

Before going into more detail, we need the following notation and theorem for our discussion. The *label* of an edge ij of K_n with vertex set $\{0, 1, \ldots, n-1\}$ is the number $\min\{|j-i|, n-|j-i|\}$. The label of any edge is thus one of the numbers $1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. If n is odd, then there are n edges of label i for $i \in \{1, 2, \ldots, \frac{n-1}{2}\}$. Suppose that C is the cycle $(i_1, i_2, \ldots, i_k)$ in K_n. For an integer t, we use $C+t$ to denote the cycle $(i_1+t, i_2+t, \ldots, i_k+t)$, where (i_j+t)'s are taken modulo n. It is easily seen that the labels of $C+t$ and of C are the same.

Let n be a positive integer. A set M consisting of $2n$ integers is called a *Skolem set of order n* if it can be written in the form $M = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n\}$, where $b_i = a_i + i$ for $i \in \{1, 2, \ldots, n\}$. We introduce a theorem concerned with the existence of a Skolem set of order n below.

Theorem 3.4 (Skolem [17], O'Keefe [11], Rosa [14], Hilton [8], Colbourn and Mathon [5],) *Suppose that n is a positive integer.*
(1)*If* $n \equiv 0$ *or* $1 \pmod 4$, *then* $\{1, 2, \ldots, 2n\}$ *is a Skolem set.*
(2)*If* $n \equiv 2$ *or* $3 \pmod 4$, *then* $\{1, 2, \ldots, 2n-1, 2n+1\}$ *is a Skolem set.*
(3)*If* $n \equiv 1$ *or* $2 \pmod 4$ *and* $n \neq 1$, *then* $\{1, 2, \ldots, n, n+2, n+3, \ldots, 2n, 2n+2\}$ *is a Skolem set.*
(4)*If* $n \equiv 0$ *or* $3 \pmod 4$, *then* $\{1, 2, \ldots, n, n+2, n+3, \ldots, 2n+1\}$ *is a Skolem set.* □

In the proof of Theorem 3.5 the arguments concerned with the proof of the existence of a cyclic C_3-decomposition of K_n are essentially given by Peltesohn [12], Skolem [17], O'Keefe [11], Rosa [14], Hilton [8], Colbourn and Mathon [5] (see 7.31-37 [3]), but we include them again here for completeness.

Theorem 3.5 *Let p and q be nonnegative integers and let n be a positive odd integer. There exists a decomposition of K_n into p copies of P_4 and q copies of C_3 if and only if $3(p+q) = e(K_n)$ and $p \neq 1$.*

Proof. (Necessity) By Theorem 2.3, we are done.

(Sufficiency) By assumption, we have that $6|n(n-1)$ and so $n \geq 3$. Since n is odd, it implies that n will be either $6k+1$ or $6k+3$, where k is a nonnegative integer. The case, $n = 3$, is trivial. Thus we assume $n \geq 7$ (i.e. $k \geq 1$). Now we consider two cases below.

Case 1. $n = 6k+1$.
Suppose that $V(K_n) = \{0, 1, 2, \ldots, 6k\}$. In this case there are n edges of label i for $i \in \{1, 2, \ldots, 3k\}$. Note that by definition, the labels $3k$ and $3k+1$ are the same. By Theorem 3.4 (1) and (2), either $\{1, 2, \ldots, 2k\}$ or $\{1, 2, \ldots, 2k-1, 2k+1\}$ is a Skolem set. Write the Skolem set as $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$. It follows that the triples $\{i, a_i + k, b_i + k\}$ for $i \in \{1, 2, \ldots, k\}$ form a decomposition of the set $\{1, 2, \ldots, 3k\}$ or $\{1, 2, \ldots, 3k-1, 3k+1\}$. Let $C^{(i)}$ denote the cycle $(0, i, b_i + k)$. Since $b_i + k = (a_i + k) + i$, we have that $(0, i, b_i + k)$ consists of edges with the labels $i, a_i + k, b_i + k$ and so $\bigcup_{i=1}^{k} C^{(i)}$ consists of edges with the labels $1, 2, \ldots, 3k$ or $1, 2, \ldots, 3k-1, 3k+1$. It implies that K_n can be decomposed into n copies of $\bigcup_{i=1}^{k} C^{(i)}$ as follows: $\bigcup_{i=1}^{k} C^{(i)}$, $\bigcup_{i=1}^{k}(C^{(i)} + 1)$,..., $\bigcup_{i=1}^{k}(C^{(i)} + (n-1))$. For $k \geq 3$, since for $j \in \{0, 1, \ldots, n-1\}$, $V(C^{(s)} + j) \cap V(C^{(t)} + j) = \{j\}$ for $1 \leq s < t \leq k$, by Lemma 2.6, $\bigcup_{j=0}^{n-1}[\bigcup_{i=1}^{k}(C^{(i)} + j)]$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{6k+1})$ (i.e. $p + q = (6k+1)k$) and $p \neq 1$. For $k = 2$, since for $j \in \{0, 1, \ldots, n-1\}$, $(C^{(1)} + j) \cup (C^{(2)} + j)$ can only be decomposed two copies of P_4 or two copies of C_3, we have that $\bigcup_{j=0}^{n-1}[\bigcup_{i=1}^{k}(C^{(i)} + j)]$ can be decomposed into p copies of P_4 and $(6k+1)k - p$ copies of C_3 for nonnegative even integer p such that $0 \leq p \leq (6k+1)k$. On the other hand, since $(\bigcup_{i=1}^{2} C^{(i)}) \cup (\bigcup_{i=1}^{2}(C^{(i)} + 1)) = (0, 1, b_1 + 2) \cup (0, 2, b_2 + 2) \cup (1, 2, b_1 + 3) \cup (1, 3, b_2 + 3)$, by Lemma 2.5, $(0, 1, b_1 + 2) \cup (1, 2, b_1 + 3) \cup (1, 3, b_2 + 3)$ can be decomposed into three copies of P_4 and so $(0, 1, b_1 + 2) \cup (0, 2, b_2 + 2) \cup (1, 2, b_1 + 3) \cup (1, 3, b_2 + 3)$ can be decomposed into three copies of P_4 and one copy of C_3. It is implies that $\bigcup_{j=0}^{n-1}[\bigcup_{i=1}^{k}(C^{(i)} + j)]$ can be decomposed into p copies of P_4 and $(6k+1)k - p$ copies of C_3 for nonnegative odd integer p such that $3 \leq p \leq (6k+1)k$. Therefore, for $k \geq 2$, K_{6k+1} can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = (6k+1)k$ and $p \neq 1$. As to the remaining case, $k = 1$, by Lemma 3.1, we are done.

Case 2. $n = 6k+3$.

The proof is similar to Case 1. Suppose that $V(K_n) = \{0, 1, 2, \dots, 6k + 2\}$. In this case there are n edges of label i for $i \in \{1, 2, \dots, 3k + 1\}$. Note that by definition, the labels $3k + 1$ and $3k + 2$ are the same. By Theorem 3.4 (3) and (4), either $\{1, 2, \dots, k, k + 2, k + 3, \dots, 2k + 1\}$ or $\{1, 2, \dots, k, k + 2, k + 3, \dots, 2k, 2k + 2\}$ is a Skolem set. Write the Skolem set as $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$. It follows that the triples $\{i, a_i + k, b_i + k\}$ for $i \in \{1, 2, \dots, k\}$ form a decomposition of the set $\{1, 2, \dots, 2k, 2k+2, 2k+3, \dots, 3k+1\}$ or $\{1, 2, \dots, 2k, 2k+2, 2k+3, \dots, 3k, 3k+2\}$. Let $C^{(i)}$ denote the cycle $(0, i, b_i + k)$. Since $b_i + k = (a_i + k) + i$, we have that $(0, i, b_i + k)$ consists of edges with the labels $i, a_i + k, b_i + k$ and so $\bigcup_{i=1}^{k} C^{(i)}$ consists of edges with the labels $1, 2, \dots, 2k, 2k+2, 2k+3, \dots, 3k+1$ or $1, 2, \dots, 2k, 2k+2, 2k+3, \dots, 3k, 3k+2$. Let B denote the cycle $(0, 2k+1, 4k+2)$. It is easy to see that all edges in $(0, 2k + 1, 4k + 2)$ have label $2k + 1$. It implies that K_n can be decomposed into n copies of $\bigcup_{i=1}^{k} C^{(i)}$ and $2k+1$ copies of B as follows: $\bigcup_{i=1}^{k} C^{(i)}$, $\bigcup_{i=1}^{k}(C^{(i)} + 1)$, $\dots$, $\bigcup_{i=1}^{k}(C^{(i)} + (n-1))$, $\bigcup_{i=0}^{2k} (B + i)$. For $k \geq 2$, since for $j \in \{0, 1, \dots, n-1\}$, $V(C^{(s)} + j) \cap V(C^{(t)} + j) = \{j\}$ for $1 \leq s < t \leq k$ and $V(C^{(s)} + j) \cap (B + j) = \{j\}$ for $1 \leq s \leq k$, by Lemma 2.5, $[\bigcup_{i=1}^{k}(C_{(i)} + j)] \cup (B + j)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = k + 1$ and $p \neq 1$. If $k \geq 2$, by Lemma 2.6, then $[\bigcup_{j=0}^{n-1}[\bigcup_{i=1}^{k}(C^{(i)} + j)]] \cup [\bigcup_{i=0}^{2k} (B + i)]$ can be decomposed into p copies of P_4 and $(6k + 3)k + 2k + 1 - p$ copies of C_3 for $0 \leq p \leq (6k+3)k + 2k + 1$ and $p \neq 1$. Note that $(6k+3)k + 2k + 1 = (3k + 1)(2k + 1) = \frac{(6k+3)(6k+2)}{6}$. Therefore, K_{6k+3} can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = \frac{(6k+3)(6k+2)}{6}$ and $p \neq 1$. As to the remaining case, $k = 1$, by Lemma 3.3, we are done. □

4 The case when n is even

In this section we use the results on the leaves for the maximum C_3-packings of K_n to prove that when n is even, K_n can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p + q) = e(K_n)$ and $p \geq \frac{n}{2}$. Therefore, we first introduce a theorem concerned with the leaves for the maximum C_3-packings of K_n as follows.

Theorem 4.1 (Hanani, Stanton and Rogers [7, 18]) *Let n be a positive integer.*

(1) *If $n \equiv 1$ or $3 \pmod 6$, then K_n can be packed with 3-cycles which has empty leave.*

(2) *If $n \equiv 0$ or $2 \pmod 6$, then K_n can be packed with 3-cycles which has leave an 1-factor.*

(3) *If $n \equiv 4 \pmod 6$, then K_n can be packed with 3-cycles which has*

leave L, where L is a vertex-disjoint union of $K_{1,3}$ and a matching with $\frac{n}{2}-2$ edges.

(4) *If $n \equiv 5 \pmod 6$, then K_n can be packed with 3-cycles which has leave C_4.* □

By Theorem 2.1 and Theorem 2.4, we obtain a theorem below.

Theorem 4.2 *Let p and q be nonnegative integers and let k be a positive odd integer. There exists a decomposition of K_{k+1} into p copies of P_{k+1} and q copies of C_k if and only if $p = \frac{k+1}{2}$ and $q = 0$.*

Proof. (Necessity) By Theorem 2.4, we have that $p + q = \frac{(k+1)}{2}$ and $p \geq \frac{k+1}{2}$ and so $p = \frac{k+1}{2}$ and $q = 0$.

(Sufficiency) By Theorem 2.1, K_{k+1} can be decomposed into $\frac{(k+1)k}{2k} = \frac{k+1}{2}$ copies of P_{k+1}. □

For our discussion, we need to show the following five lemmas for decomposing graphs into P_4's and C_3's below.

Lemma 4.3 *Suppose that H is a complete tripartite subgraph $K_{3,3,3}$ of K_{12}. If p and q are nonnegative integers such that $p+q = 13$ and $p \geq 6$, then $K_{12} - E(H)$ can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Suppose that $V(K_{12}) = \{x_1, x_2, \ldots, x_{12}\}$. Let the tripartition of H be $(\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\})$. We first show that $K_{12} - E(H)$ can be decomposed into 6 copies of P_4 and 7 copies of C_3 as follows: $P^{(1)} : x_1x_3x_{12}x_9$, $P^{(2)} : x_4x_6x_{10}x_3$, $P^{(3)} : x_7x_9x_{11}x_6$, $P^{(4)} : x_2x_{12}x_1x_{11}$, $P^{(5)} : x_5x_{10}x_4x_{12}$, $P^{(6)} : x_8x_{11}x_7x_{10}$, $C^{(1)} : (x_1, x_2, x_{10})$, $C^{(2)} : (x_2, x_3, x_{11})$, $C^{(3)} : (x_4, x_5, x_{11})$, $C^{(4)} : (x_5, x_6, x_{12})$, $C^{(5)} : (x_7, x_8, x_{12})$, $C^{(6)} : (x_8, x_9, x_{10})$, $C^{(7)} : (x_{10}, x_{11}, x_{12})$.

Since $V(C^{(1)}) \cap V(C^{(6)}) = V(C^{(1)}) \cap V(C^{(7)}) = V(C^{(6)}) \cap V(C^{(7)}) = \{x_{10}\}$, $V(C^{(2)}) \cap V(C^{(3)}) = \{x_{11}\}$, and $V(C^{(4)}) \cap V(C^{(5)}) = \{x_{12}\}$, by Lemma 2.6, $\bigcup_{i=1}^{7} C^{(i)}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q = 7$ and $p \neq 1$. On the other hand, $P^{(2)} \cup C^{(1)}$ can be decomposed into two copies of P_4 below: $x_2x_1x_{10}x_3$ and $x_4x_6x_{10}x_2$. Therefore, $(\bigcup_{i=1}^{7} C^{(i)}) \cup (\bigcup_{j=1}^{6} P^{(j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q = 13$ and $p \geq 6$. □

Lemma 4.4 *If p and q are nonnegative integers such that $p+q = 15$ and $p \geq 5$, then K_{10} can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_{10}) = \{x_1, x_2, \ldots, x_{10}\}$. We first show that K_{10} can be decomposed into 5 copies of P_4 and 10 copies of C_3 as follows: $P^{(1)}$:

$x_1x_7x_{10}x_4$, $P^{(2)}$: $x_2x_{10}x_8x_5$, $P^{(3)}$: $x_3x_9x_7x_6$, $P^{(4)}$: $x_7x_8x_9x_{10}$, $P^{(5)}$: $x_8x_1x_4x_9$, $C^{(1)}$: (x_1, x_2, x_3), $C^{(2)}$: (x_4, x_5, x_6), $C^{(3)}$: (x_1, x_5, x_{10}), $C^{(4)}$: (x_1, x_6, x_9), $C^{(5)}$: (x_2, x_4, x_7), $C^{(6)}$: (x_2, x_5, x_9), $C^{(7)}$: (x_2, x_6, x_8), $C^{(8)}$: (x_3, x_4, x_8), $C^{(9)}$: (x_3, x_5, x_7), $C^{(10)}$: (x_3, x_6, x_{10}).

Since $V(C^{(1)}) \cap V(C^{(3)}) = \{x_1\}$, $V(C^{(2)}) \cap V(C^{(4)}) = \{x_6\}$, $V(C^{(5)}) \cap V(C^{(6)}) = V(C^{(5)}) \cap V(C^{(7)}) = V(C^{(6)}) \cap V(C^{(7)}) = \{x_2\}$ and $V(C^{(8)}) \cap V(C^{(9)}) = V(C^{(8)}) \cap V(C^{(10)}) = V(C^{(9)}) \cap V(C^{(10)}) = \{x_3\}$, by Lemma 2.6, $\bigcup_{i=1}^{10} C^{(i)}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = 10$ and $p \neq 1$. On the other hand, $P^{(5)} \cup C^{(1)}$ can be decomposed into two copies of P_4 below: $x_8x_1x_2x_3$ and $x_3x_1x_4x_9$. Therefore, $(\bigcup_{i=1}^{10} C^{(i)}) \cup (\bigcup_{j=1}^{5} P^{(j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = 15$ and $p \geq 5$. □

Lemma 4.5 *Suppose that H is a complete subgraph K_4 of K_{10} and K is a complete subgraph K_3 of K_{10} such that H and K are vertex disjoint. If p and q are nonnegative integers such that $p + q = 12$ and $p \geq 3$, then $K_{10} - E(H \cup K)$ can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_{10}) = \{x_1, x_2, \ldots, x_{10}\}$. Suppose that $V(H) = \{x_7, x_8, x_9, x_{10}\}$ and $V(K) = \{x_4, x_5, x_6\}$. We first show that $K_{10} - E(H \cup K)$ can be decomposed into 3 copies of P_4 and 9 copies of C_3 as follows: $P^{(1)}$: $x_1x_6x_{10}x_4$, $P^{(2)}$: $x_2x_3x_7x_6$, $P^{(3)}$: $x_3x_1x_7x_5$, $C^{(1)}$: (x_1, x_2, x_{10}), $C^{(2)}$: (x_1, x_4, x_8), $C^{(3)}$: (x_1, x_5, x_9), $C^{(4)}$: (x_2, x_4, x_7), $C^{(5)}$: (x_2, x_5, x_8), $C^{(6)}$: (x_2, x_6, x_9), $C^{(7)}$: (x_3, x_4, x_9), $C^{(8)}$: (x_3, x_5, x_{10}), $C^{(9)}$: (x_3, x_6, x_8).

Since $V(C^{(1)}) \cap V(C^{(2)}) = V(C^{(1)}) \cap V(C^{(3)}) = V(C^{(2)}) \cap V(C^{(3)}) = \{x_1\}$, $V(C^{(4)}) \cap V(C^{(5)}) = V(C^{(4)}) \cap V(C^{(6)}) = V(C^{(5)}) \cap V(C^{(6)}) = \{x_2\}$, and $V(C^{(7)}) \cap V(C^{(8)}) = V(C^{(7)}) \cap V(C^{(9)}) = V(C^{(8)}) \cap V(C^{(9)}) = \{x_3\}$, by Lemma 2.6, $\bigcup_{i=1}^{9} C^{(i)}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = 9$ and $p \neq 1$. On the other hand, $P^{(1)} \cup C^{(1)}$ can be decomposed into two copies of P_4 below: $x_1x_6x_{10}x_2$ and $x_2x_1x_{10}x_4$. Therefore, $(\bigcup_{i=1}^{9} C^{(i)}) \cup (\bigcup_{j=1}^{3} P^{(j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p + q = 12$ and $p \geq 3$. □

Lemma 4.6 *Suppose that H is a complete subgraph K_4 of K_{10}. If p and q are nonnegative integers such that $p + q = 13$ and $p \geq 3$, then $K_{10} - E(H)$ can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_{10}) = \{x_1, x_2, \ldots, x_{10}\}$. Suppose that $V(H) = \{x_7, x_8, x_9, x_{10}\}$. We first show that $K_{10} - E(H)$ can be decomposed into 3 copies of P_4 and 10 copies of C_3 as follows: $P^{(1)}$: $x_1x_6x_{10}x_4$, $P^{(2)}$: $x_2x_3x_7x_6$, $P^{(3)}$: $x_3x_1x_7x_5$, $C^{(1)}$: (x_1, x_2, x_{10}), $C^{(2)}$: (x_1, x_4, x_8), $C^{(3)}$: (x_1, x_5, x_9),

$C^{(4)}:(x_2,x_4,x_7)$, $C^{(5)}:(x_2,x_5,x_8)$, $C^{(6)}:(x_2,x_6,x_9)$, $C^{(7)}:(x_3,x_4,x_9)$, $C^{(8)}:(x_3,x_5,x_{10})$, $C^{(9)}:(x_3,x_6,x_8)$, $C^{(10)}:(x_4,x_5,x_6)$.

Since $V(C^{(1)})\cap V(C^{(2)})=V(C^{(1)})\cap V(C^{(3)})=V(C^{(2)})\cap V(C^{(3)})=\{x_1\}$, $V(C^{(4)})\cap V(C^{(5)})=V(C^{(4)})\cap V(C^{(6)})=V(C^{(5)})\cap V(C^{(6)})=\{x_2\}$, and $V(C^{(7)})\cap V(C^{(8)})=\{x_3\}$, $V(C^{(9)})\cap V(C^{(10)})=\{x_6\}$, by Lemma 2.6, $\bigcup_{i=1}^{10}C^{(i)}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q=10$ and $p\neq 1$. On the other hand, $P^{(1)}\cup C^{(1)}$ can be decomposed into two copies of P_4 below: $x_1x_6x_{10}x_2$ and $x_2x_1x_{10}x_4$. Therefore, $(\bigcup_{i=1}^{10}C^{(i)})\cup(\bigcup_{j=1}^{3}P^{(j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q=13$ and $p\geq 3$. □

Lemma 4.7 *Suppose that H is a complete bipartite subgraph $K_{3,3}$ of K_{13} and K is a complete bipartite subgraph $K_{3,4}$ of K_{13} such that H and K are vertex disjoint. If p and q are nonnegative integers such that $p+q=19$ and $p\geq 5$, then $K_{13}-E(H\cup K)$ can be decomposed into p copies of P_4 and q copies of C_3.*

Proof. Let $V(K_{13})=\{x_1,x_2,\dots,x_{13}\}$. Suppose that the bipartition of H is $(\{x_1,x_2,x_3\},\{x_4,x_5,x_6\})$ and the bipartition of K is $(\{x_7,x_8,x_9\},\{x_{10},x_{11},x_{12},x_{13}\})$. We first show that $K_{13}-E(H\cup K)$ can be decomposed into 5 copies of P_4 and 14 copies of C_3 as follows: $P^{(1)}:x_1x_9x_2x_{13}$, $P^{(2)}:x_2x_7x_5x_{10}$, $P^{(3)}:x_3x_9x_8x_6$, $P^{(4)}:x_4x_{13}x_1x_{12}$, $P^{(5)}:x_5x_8x_3x_{11}$, $C^{(1)}:(x_1,x_{10},x_{11})$, $C^{(2)}:(x_2,x_{11},x_{12})$, $C^{(3)}:(x_3,x_{12},x_{13})$, $C^{(4)}:(x_4,x_{10},x_{12})$, $C^{(5)}:(x_5,x_{11},x_{13})$, $C^{(6)}:(x_6,x_{10},x_{13})$, $C^{(7)}:(x_1,x_2,x_8)$, $C^{(8)}:(x_2,x_3,x_{10})$, $C^{(9)}:(x_3,x_1,x_7)$, $C^{(10)}:(x_4,x_5,x_9)$, $C^{(11)}:(x_5,x_6,x_{12})$, $C^{(12)}:(x_6,x_4,x_{11})$, $C^{(13)}:(x_4,x_7,x_8)$, $C^{(14)}:(x_6,x_9,x_7)$.

Since $V(C^{(1)})\cap V(C^{(7)})=\{x_1\}$, $V(C^{(2)})\cap V(C^{(8)})=\{x_2\}$, $V(C^{(3)})\cap V(C^{(9)})=\{x_3\}$, $V(C^{(4)})\cap V(C^{(10)})=V(C^{(4)})\cap V(C^{(13)})=V(C^{(10)})\cap V(C^{(13)})=\{x_4\}$, $V(C^{(5)}\cap V(C^{(11)})=\{x_5\}$, and $V(C^{(6)})\cap V(C^{(12)})=V(C^{(6)})\cap V(C^{(14)})=V(C^{(12)})\cap V(C^{(14)})=\{x_6\}$, by Lemma 2.6, $\bigcup_{i=1}^{14}C^{(i)}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q=14$ and $p\neq 1$. On the other hand, $P^{(1)}\cup C^{(7)}$ can be decomposed into two copies of P_4 below: $x_1x_8x_2x_9$ and $x_9x_1x_2x_{13}$. Therefore, $(\bigcup_{i=1}^{14}C^{(i)})\cup(\bigcup_{j=1}^{5}P^{(j)})$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $p+q=19$ and $p\geq 5$. □

Now we prove the main theorem of this section.

Theorem 4.8 *Let p and q be nonnegative integers and let n be a positive even integers. There exists a decomposition of K_n into p copies of P_4 and q copies of C_3 if and only if $3(p+q)=e(K_n)$ and $p\geq\frac{n}{2}$.*

Proof. (Necessity) By Theorem 2.4, we have that $3(p+q) = e(K_n)$ and $p \geq \frac{n}{2}$.

(Sufficiency) By assumption, we have that $6|n(n-1)$. Since n is even, it implies that n will be either $6k$ or $6k+4$, where k is a nonnegative integer. If $n = 4$, by Theorem 4.2, then we are done and so assume $k \geq 1$. Now we consider two cases below.

Case 1. $n = 6k$.
If $k = 1$ (i.e. $n = 6$), by Lemma 3.2, we are done and so assume $k \geq 2$. Suppose that $V(K_n) = \{x_1, x_2, \ldots, x_{6k}\}$. Let $X_i = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$ for $i \in \{1, 2, \ldots, 2k\}$. Let G be the complete graph K_{2k} with vertex set $\{X_1, X_2, \ldots, X_{2k}\}$. Suppose that I denotes the 1-factor with edge set $\{X_{2i-1}X_{2i}| \ i = 1, 2, \ldots, k\}$. Since $2k$ is even, it follows that $2k$ will be either $6t$, $6t+2$ or $6t+4$, where t is a nonnegative integer.

When $2k = 6t$ or $6t+2$, we have that $3|\frac{2k(2k-2)}{2}$. By Theorem 2.2 (2), $G - I (\cong K_{2k} - I)$ can be decomposed into $\frac{2k(2k-2)}{6}$ copies of C_3. On the other hand, for $i \in \{1, 2, \ldots, k\}$, the edge of I, $X_{2i-1}X_{2i}$, can be viewed as the complete graph K_6 with vertex set $\{x_{6i-5}, x_{6i-4}, \ldots, x_{6i}\}$ and the C_3, (X_i, X_j, X_m), can be viewed as the complete tripartite graph $K_{3,3,3}$ with tripartition $(\{x_{3i-2}, x_{3i-1}, x_{3i}\}, \{x_{3j-2}, x_{3j-1}, x_{3j}\}, \{x_{3m-2}, x_{3m-1}, x_{3m}\})$. It implies that K_{6k} can be decomposed into k copies of K_6 and $\frac{2k(2k-2)}{6}$ copies of $K_{3,3,3}$.

When $2k = 6t+4$, by Theorem 4.1 (3), $G - E(H)$ can be decomposed $\lfloor \frac{(6t+4)(6t+2)}{6} \rfloor$ copies of C_3, where H is the subgraph of K_{2k} induced by the set of edges $\{X_{2i-1}X_{2i}| \ i = 1, 2, \ldots, k-2\} \cup \{X_{2k-3}X_{2k}, X_{2k-2}X_{2k}, X_{2k-1}X_{2k}\}$. By the same argument mentioned above, for $i \in \{1, 2, \ldots, k-2\}$, the edge of H, $X_{2i-1}X_{2i}$, can be viewed as the complete graph K_6 with vertex set $\{x_{6i-5}, x_{6i-4}, \ldots, x_{6i}\}$; the C_3, (X_i, X_j, X_m), can be viewed as the complete tripartite graph $K_{3,3,3}$ with tripartition $(\{x_{3i-2}, x_{3i-1}, x_{3i}\}, \{x_{3j-2}, x_{3j-1}, x_{3j}\}, \{x_{3m-2}, x_{3m-1}, x_{3m}\})$; the star $K_{1,3}$ induced by the set of edges $\{X_{2k-3}X_{2k}, X_{2k-2}X_{2k}, X_{2k-1}X_{2k}\}$ can be viewed as the graph $K - E(M)$, where K is the complete graph K_{12} with vertex set $\{x_{6k-11}, x_{6k-10}, \ldots, x_{6k}\}$ and M is the complete tripartite graph $K_{3,3,3}$ with tripartition $(\{x_{6k-11}, x_{6k-10}, x_{6k-9}\}, \{x_{6k-8}, x_{6k-7}, x_{6k-6}\}, \{x_{6k-5}, x_{6k-4}, x_{6k-3}\})$. It implies that K_{6k} can be decomposed into $k-2$ copies of K_6, $\lfloor \frac{2k(2k-2)}{6} \rfloor$ copies of $K_{3,3,3}$, and $K - E(M)$.

By Lemma 3.2, K_6 can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_6)$ (i.e. $p+q = 5$) and $p \geq 3$; by Theorem 2.7, $K_{3,3,3}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{3,3,3})$ (i.e. $p+q = 9$) and $p \neq 1$; by Lemma 4.3, $K - E(M)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K - E(M))$ (i.e. $p+q = 13$) and $p \geq 6$. Therefore, K_{6k} can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{6k})$ and $p \geq 3k$.

Case 2. $n = 6k+4$.
If $k = 1$ (i.e. $n = 10$), by Lemma 4.4, we are done and so assume $k \geq 2$. Suppose that $V(K_n) = \{x_1, x_2, \ldots, x_{6k+4}\}$. Let $X_i = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$ for $i \in \{1, 2, \ldots, 2k\}$ and $X_{2k+1} = \{x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$. Let G be the complete graph K_{2k+1} with vertex set $\{X_1, X_2, \ldots, X_{2k+1}\}$. Since $2k+1$ is odd, it follows that $2k+1$ will be either $6t+1$, $6t+3$ or $6t+5$, where t is a nonnegative integer.

When $2k+1 = 6t+1$ or $6t+3$, we have that $3|\frac{(2k+1)2k}{2}$. By Theorem 2.2 (1), $G(\cong K_{2k+1})$ can be decomposed into $\frac{(2k+1)2k}{6}$ copies of C_3. Let D be an arbitrary C_3-decomposition of G. It is easy to see that the vertex X_{2k+1} is contained in k members of D. Without loss of generality we assume that those k copies of C_3 are denoted by $(X_{2i-1}, X_{2i}, X_{2k+1})$ for $i \in \{1, 2, \ldots, k\}$. On the other hand, (X_1, X_2, X_{2k+1}) can be viewed as the complete graph K_{10} with vertex set $\{x_1, x_2, x_3, x_4, x_5, x_6, x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$; for $i \in \{2, 3, \ldots, k\}$, $(X_{2i-1}, X_{2i}, X_{2k+1})$ can be viewed as the graph $K - E(M)$, where K is the complete graph K_{10} with vertex set $\{x_{6i-5}, x_{6i-4}, \ldots, x_{6i}, x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$ and M is the complete graph K_4 with vertex set $\{x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$. If $X_{2k+1} \notin V((X_i, X_j, X_m))$, then (X_i, X_j, X_m), can be viewed as the complete tripartite graph $K_{3,3,3}$ with tripartition $(\{x_{3i-2}, x_{3i-1}, x_{3i}\}, \{x_{3j-2}, x_{3j-1}, x_{3j}\}, \{x_{3m-2}, x_{3m-1}, x_{3m}\})$. It implies that K_{6k+4} can be decomposed into one copy of K_{10}, $k-1$ copies of $K_{10} - E(K_4)$, and $\frac{(2k+1)2k}{6} - k$ copies of $K_{3,3,3}$.

By Lemma 4.4, K_{10} can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{10})$ (i.e. $p+q = 15$) and $p \geq 5$; by Lemma 4.6, $K_{10} - E(K_4)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{10} - E(K_4))$ (i.e. $p+q = 13$) and $p \geq 3$; by Theorem 2.7, $K_{3,3,3}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{3,3,3})$ (i.e. $p+q = 9$) and $p \neq 1$. Therefore, K_{6k+4} can be decomposed into p copies of P_4 and q copies of C_3, for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{6k+4})$ and $p \geq 3k+2$.

When $2k+1 = 6t+5$, by Theorem 4.1 (4), $G - E(H)$ can be decomposed $\lfloor\frac{(6t+5)(6t+4)}{6}\rfloor - 1$ copies of C_3, where H is the C_4, $(X_{2k-1}, X_{2k-2}, X_{2k}, X_{2k+1})$. Let D^* be an arbitrary C_3-decomposition of $G - E(H)$. It is not difficult to see that the vertex X_{2k+1} is contained in $k-1$ members of D^*. Without loss of generality we assume that those $k-1$ copies of C_3 are denoted by $(X_{2i-1}, X_{2i}, X_{2k+1})$, for $i \in \{1, 2, \ldots, k-1\}$. On the other hand, for $i \in \{1,2,\ldots,k-2\}$, $(X_{2i-1}, X_{2i}, X_{2k+1})$ can be viewed as the graph $K^* - E(M^*)$, where K^* is the complete graph K_{10} with vertex set $\{x_{6i-5}, x_{6i-4}, \ldots, x_{6i}, x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$ and M^* is the complete graph K_4 with vertex set $\{x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$. The cycle, $(X_{2k-3}, X_{2k-2}, X_{2k+1})$, can be viewed as the graph $U - E(V \cup M^*)$, where U is the complete graph K_{10} with vertex set $\{x_{6k-11}, x_{6k-10}, \ldots, x_{6k-6}, x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\}$ and V is the complete graph K_3 with vertex set $\{x_{6k-8}, x_{6k-7}, x_{6k-6}\}$. If $X_{2k+1} \notin V((X_i, X_j, X_m))$, then (X_i, X_j, X_m) can be viewed as the complete tripartite graph $K_{3,3,3}$ with tripartition $(\{x_{3i-2}, x_{3i-1}, x_{3i}\}, \{x_{3j-2}, x_{3j-1}, x_{3j}\}, \{x_{3m-2}, x_{3m-1}, x_{3m}\})$. Finally, let A denote the complete graph K_{13} with vertex set $\{x_{6k-8}, x_{6k-7}, \ldots, x_{6k+4}\}$; B denote the complete bipartite graph $K_{3,3}$ with bipartition $(\{x_{6k-5}, x_{6k-4}, x_{6k-3}\}, \{x_{6k-2}, x_{6k-1}, x_{6k}\})$; and C denote the complete bipartite graph $K_{3,4}$ with bipartition $(\{x_{6k-8}, x_{6k-7}, x_{6k-6}\}, \{x_{6k+1}, x_{6k+2}, x_{6k+3}, x_{6k+4}\})$. The cycle, $(X_{2k-1}, X_{2k-2}, X_{2k}, X_{2k+1})$, can be viewed as the graph $A - E(B \cup C)$. It implies that K_{6k+4} can be decomposed into $A - E(B \cup C)$, $U - E(V \cup M^*)$, $k-2$ copies of $K_{10} - E(K_4)$, and $\lfloor \frac{(2k+1)2k}{6} \rfloor - k$ copies of $K_{3,3,3}$.

By Lemma 4.7, $A - E(B \cup C)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(A - E(B \cup C))$ (i.e. $p+q=19$) and $p \geq 5$; by Lemma 4.5, $U - E(V \cup M^*)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(U - E(V \cup M^*))$ (i.e. $p+q=12$) and $p \geq 3$; by Lemma 4.6, $K_{10} - E(K_4)$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{10} - E(K_4))$ (i.e. $p+q=13$) and $p \geq 3$; by Theorem 2.7, $K_{3,3,3}$ can be decomposed into p copies of P_4 and q copies of C_3 for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{3,3,3})$ (i.e. $p+q=9$) and $p \neq 1$. Therefore, K_{6k+4} can be decomposed into p copies of P_4 and q copies of C_3, for each pair p, q of nonnegative integers such that $3(p+q) = e(K_{6k+4})$ and $p \geq 3k+2$. □

References

[1] B. Alspach, H. Gavlas, Cycle decompositions of K_n and $K_n - I$, J. Combin. Theory Ser. B 81 (2001), 77-99.

[2] J. A. Bondy, U. S. R. Murty, Graph theory with applications, The Macmillan Press Ltd, New York, 1976.

[3] Jural Bosák, Decompositions of Graphs, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[4] D. E. Bryant, B. Maenhaut, Decompositions of complete graphs into triangles and Hamilton cycles, J. Combin. Des. 12 (2004), 221-232.

[5] M. J. Colbourn and R. A. Mathon, On cyclic Steiner 2-Designs, Topics on Steiner systems, Annales of Discrete Math. 7 (1980), 215-253.

[6] H. L. Fu, and C. A. Rodger, Group divisible with two associate classes: $n = 2$ or $m = 2$, J. Combin. Theory Ser. A 83 (1998), 94-117.

[7] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975), 255-369.

[8] A. J. W. Hilton, On Steiner and similar triple systems, Math. Scand. 24 (1969), 208-216.

[9] P. Horak, R. Nedela, and A. Rosa, The Hamilton-Waterloo problem: The case of Hamilton cycles and triangle-factors, Discrete Math. 284 (2004), 181-188.

[10] T. P. Kirkman, On a problem in combinations, Cambridge Dublin Math. J. 2 (1847), 191-204.

[11] E. S. O'Keefe, Verification of a conjecture of Th. Skolem, Math. Scand. 9 (1961), 80-82.

[12] R. Peltesohn, Eine Lösung der beiden Heffterschen Differenzenprobleme, Compositio Math. 6 (1939), 251-257.

[13] R. Rees, Uniformly resolvable pairwise balanced designs with block sizes two and three, J. Combin. Theory Ser. A 45 (1987), 207-225.

[14] A. Rosa, A note on cyclic Steiner Triple Systems, Mat.-Fyz. Čas. 17 (1966), 285-290.

[15] M. Šajna, Cycle decompositions III: complete graphs and fixed length cycles, J. Combin. Des. 10 (2002), 27-78.

[16] T. W. Shyu, Decomposition of complete graphs into paths and cycles, Ars Combin. 97 (2010), 257-270.

[17] T. Skolem, On certain distribution of integers in pairs with given differences, Math. Scand. 5 (1957), 57-68.

[18] R. G. Stanton, M. J. Rogers, Packings and covering by triples, Ars Combin. 13 (1982), 61-69.

[19] M. Tarsi, Decomposition of complete multigraph into simple paths: Nonbalanced Handcuffed designs, J. Combin. Theory Ser. A 34 (1983), 60-70.