Global Defensive Alliances in the Join, Corona and Composition of Graphs[1]

Rolito G. Eballe
Mathematics Department
Central Mindanao University, Bukidnon, Philippines

Rodelito M. Aldema
Mathematics Department
Mindanao State University-Marawi, Philippines

Esamel M. Paluga
Mathematics Department
Caraga State University, Philippines

Ricky F. Rulete
Mathematics Department
University of Southeastern Philippines, Philippines

Ferdinand P. Jamil[2]
Mathematics Department
MSU-Iligan Institute of Technology
Email: ferdinand.jamil@g.msuiit.edu.ph

Abstract

By a defensive alliance in a graph G we mean any set S of vertices in G such that each vertex in S is adjacent to at least as many vertices inside S, including the vertex itself, as outside S. If, in addition, we require that every vertex outside a defensive alliance S is adjacent to at least one vertex in S, then S becomes a global defensive alliance. The minimum cardinality of a global defensive alliance is the global defensive alliance number of G. In this paper, we determine bounds for the global defensive alliance numbers in the join, corona and composition of graphs.

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[2]Corresponding author

1 Introduction

Defensive alliances in graphs, together with the other two forms (offensive and powerful alliances), first appeared in a paper by S.M. Hedetniemi, S.T. Hedetniemi and P. Kristiansen (see[1]). In the same paper, mathematical properties of these three alliances are being studied, and the alliance numbers in a class of graphs, including cycles, wheels, grids and complete graphs, are investigated. J. A. Rodriguez-Velasquez and J. M. Sigarreta in [2], and with I. G. Yero in [3], determined tight bounds for the defensive alliance number of line graphs in terms of degree sequences and algebraic connectivity. Tight bounds are also obtained for complement graphs in [5]. Sigarreta, Bermudo and Fernau also proved in [5] the NP-completeness of the decision problem underlying the defensive alliance number.

The present paper focuses on global defensive alliances, a study being initiated by T.W. Haynes, S.T. Heditniemi and M.A. Henning [6]. In the referred paper, bounds or exact values for the associated invariant have been determined for special graphs including trees, complete graphs and bipartite graphs. Further investigation on trees is done in [7] while various mathematical properties are studied in [8].

Several applications of alliances in graphs are listed in [1]. An application of global defensive alliances in computing networks is also being mentioned in [6].

In this paper, we consider undirected graphs with no loops. Let $G = (V, E)$ be a graph of order $|V|$ and size $|E|$. For purposes of emphasis, we also write $V = V(G)$ and $E = E(G)$. For a given vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u : uv \in E\}$ of all vertices adjacent to v; every vertex $u \in N(v)$ is called a *neighbor* of v. The *degree* of a vertex v in G is $deg_G(v) = |N(v)|$, the minimum degree of a vertex $v \in V$ is denoted $\delta(G)$, and the maximum degree of a vertex in V is denoted $\Delta(G)$. The *closed neighborhood* of a vertex v is the set $N[v] = N(v) \cup \{v\}$.

For a set $S \subseteq V$ of vertices, the *open neighborhood* is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* is the set $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$ [10]. The *distance* $d_G(u, v)$ between two vertices $u, v \in V$ equals the minimum length of a path joining u and v; any such minimum length path is called a *geodesic*.

A nonempty set $S \subseteq V$ is called a *defensive alliance* if for every vertex $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V \backslash S)|$ A defensive alliance S is a *global defensive alliance* if it is also a dominating set. The *global defensive alliance number* of a graph G, denoted $\gamma_a(G)$, equals the minimum cardinality of a global defensive alliance in G; any global defensive alliance S of cardinality $\gamma_a(G)$ is called a γ_a*-set*. In this paper we determine bounds for $\gamma_a(G)$ for joins, coronas and compositions of graphs.

Haynes et. al. in [9] presented, among others, the following results:

Theorem 1.1 [9] *If G is a graph of order n, then $\gamma_a(G) \geq \frac{\sqrt{4n+1}-1}{2}$.*

Theorem 1.2 [9] *If G is a graph of order n, then $\gamma_a(G) \leq n - \left\lceil \frac{\delta(G)}{2} \right\rceil$.*

Theorem 1.3 [9] *For any graph G of order n, $\gamma_a(G) = n$ if and only if $G = \overline{K}_n$.*

For complete graph K_n and complete bipartite graph $K_{r,s}$, the following formulas can also be found in [9]):

$$\gamma_a(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor \quad \text{and} \quad \gamma_a(K_{r,s}) = \begin{cases} 1 + \left\lfloor \frac{s}{2} \right\rfloor, & r = 1, \\ \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{s}{2} \right\rfloor, & r, s \geq 2. \end{cases}$$

A graph G is said to be a *complete multipartite graph* if $V(G)$ can be partitioned into nonempty subsets U_1, U_2, ..., U_n, called *partite sets* of G, such that $xy \in E(G)$ if and only if $xy \notin E(\langle U_k \rangle)$ for all $k = 1, 2, \ldots, n$, where $\langle U_k \rangle$ is the subgraph of G induced by U_k. In particular, if $n = 2$, then G is a complete bipartite. In what follows, we consider complete multipartite graphs having at least three nonsingleton partite sets. For covenience, we denote by $\Omega(G)$ the set of all partite sets U of G with $|U| \geq 2$.

Theorem 1.4 *Let G be a complete multipartite graph of order p and with $\Omega(G) = \{U_{r_1}, U_{r_2}, \ldots, U_{r_n}\}$, where $n \geq 3$. Let $s = \sum_{i=1}^{n} r_i$, and let m denote the number of odd integer subscripts r_i in $\Omega(G)$.*

i. If $m = 0$, then $\gamma_a(G) = \begin{cases} \frac{s}{2}, & p = s, s+1, \\ \left\lceil \frac{p-s}{2} \right\rceil + \frac{s}{2}, & p > s+1. \end{cases}$

ii. If $m = 1$, then $\gamma_a(G) = \begin{cases} \sum_{j=1}^{n} \left\lfloor \frac{r_j}{2} \right\rfloor, & p = s, \\ 1 + \left\lfloor \frac{p-s}{2} \right\rfloor + \sum_{j=1}^{n} \left\lfloor \frac{r_j}{2} \right\rfloor, & p > s. \end{cases}$

iii. If $m \geq 2$ and $r_1, r_2, \ldots, r_m$ are odd, then

$$\gamma_a(G) = \begin{cases} \sum_{j=1}^{\lceil \frac{m}{2} \rceil} \left\lceil \frac{r_j}{2} \right\rceil + \sum_{1+\lceil \frac{m}{2} \rceil}^{n} \left\lfloor \frac{r_j}{2} \right\rfloor, & p = s, \\ \left\lfloor \frac{p-s}{2} \right\rfloor + \sum_{j=1}^{\lceil \frac{m}{2} \rceil} \left\lceil \frac{r_j}{2} \right\rceil + \sum_{j=1+\lceil \frac{m}{2} \rceil}^{n} \left\lfloor \frac{r_j}{2} \right\rfloor, & p > s, \ m \text{ is odd}, \\ \left\lceil \frac{p-s}{2} \right\rceil + \sum_{j=1}^{\lceil \frac{m}{2} \rceil} \left\lceil \frac{r_j}{2} \right\rceil + \sum_{j=1+\lceil \frac{m}{2} \rceil}^{n} \left\lfloor \frac{r_j}{2} \right\rfloor, & p > s, \ m \text{ is even}. \end{cases}$$

Proof: First, we note that if $v \in U_{r_j}$, then $N[v] = \{v\} \cup (V(G) \setminus U_{r_j})$, and if $v \in V(G) \setminus (\cup_{j=1}^{n} U_{r_j})$, then $N[v] = V(G)$. To prove (i), we consider three cases: $p = s$, $p = s+1$, $p > s+1$. Suppose that $p = s$. For each $j = 1, 2, \ldots, n$, let $S_{r_j} \subseteq U_{r_j}$ with $|S_{r_j}| = \frac{1}{2} r_j$. Put $S = \cup_{j=1}^{n} S_{r_j}$. Clearly, S is a dominating set in G. Let $v \in S_{r_j}$. Then

$$|N[v] \cap S| - |N[v] \setminus S| = 1.$$

This means that S is a global defensive alliance of G. Hence, $\gamma_a(G) \leq |S| = \frac{1}{2}s$. Now, suppose that $W \subseteq V(G)$ be such that $|W| < |S|$. If $|W \cap U_{r_i}| < \frac{r_i}{2}$ for all $i = 1, 2, \ldots, n$, and if $w \in W \cap U_{r_j}$, then

$$|N[w] \cap W| < 1 + \sum_{i=1, i \neq j}^{n} \frac{r_i}{2} \leq \sum_{i=1, i \neq j}^{n} |U_{r_i} \setminus W| = |N[w] \setminus W|.$$

Suppose that for some j, $|W \cap U_{r_j}| \geq \frac{r_j}{2}$. Let $w \in W \cap U_{r_j}$. Then we have $|N[w] \cap W| \leq \frac{s}{2} - |W \cap U_{r_j}|$ so that

$$\begin{aligned} |N[w] \setminus W| &= s - |W| - (r_j - |W \cap U_{r_j})|) \\ &> s - \frac{s}{2} - (r_j - |W \cap U_{r_j}|) \\ &\geq \frac{s}{2} - |W \cap U_{r_j}| \\ &\geq |N[w] \cap W|. \end{aligned}$$

This is impossible for a defensive alliance. Thus $\gamma_a(G) = \frac{1}{2}s$.

Suppose that $p = s+1$. Let $S = \cup_{i=1}^{n} S_{r_i}$, where $S_{r_i} \subseteq U_{r_i}$ with $|S_{r_i}| = \frac{r_i}{2}$. Then S is a dominating set in G. Let $v \in S_{r_j}$. Then

$$|N[v] \cap S| = 1 + \frac{1}{2}(s - r_j) = |N[v] \setminus S|.$$

This means that S is a global defensive alliance in G. Thus $\gamma_a(G) \leq \frac{1}{2}s$. Suppose that $W \subseteq V(G)$ with $|W| < \frac{1}{2}s$. If $w \in W \setminus \cup_{i=1}^{n} U_{r_i}$, then

$$|N[w] \setminus W| = 1 + s - |W| > 1 + s - \frac{1}{2}s = 1 + \frac{1}{2}s > |W| = |N[w] \cap W|.$$

On the other hand, suppose that $W \subseteq \cup_{i=1}^{n} U_{r_i}$. Following the proof in first case, if $|W \cap U_{r_i}| < \frac{r_i}{2}$ for all $i = 1, 2, \ldots, n$, then for any $w \in W$, $|N[w] \setminus W| > |N[w] \cap W|$. Suppose that $|W \cap U_{r_j}| \geq \frac{r_j}{2}$ for some j, and let $w \in W \cap U_{r_j}$. Then $|N[w] \cap W| < 1 + \frac{s}{2} - |W \cap U_{r_j}|$, which yields

$$|N[w] \setminus W| = 1 + s - |W| - (r_j - |W \cap U_{r_j}|)$$

$$\begin{aligned} &> 1+s-\frac{s}{2}-(r_j-\left|W\cap U_{r_j}\right|) \\ &\geq 1+\frac{s}{2}-\left|W\cap U_{r_j}\right| \\ &> |N[w]\cap W|. \end{aligned}$$

This means that W cannot be a defensive alliance. Thus $\gamma_a(G)=\frac{1}{2}s$.

For the third case, suppose that $p>s+1$. Let $S=\cup_{j=1}^{n+1}S_{r_j}$, where for $j=1,2,\ldots,n$, $S_{r_j}\subseteq U_{r_j}$ with $\left|S_{r_j}\right|=\frac{1}{2}r_j$, and $S_{r_{n+1}}\subseteq V(G)\setminus(\cup_{j=1}^{n}U_{r_j})$ such that $\left|S_{r_{n+1}}\right|=\lceil\frac{p-s}{2}\rceil$. Then S is a dominating set in G. If $v\in S_{r_{n+1}}$, then

$$|N[v]\cap S|=|S|=\left\lceil\frac{p-s}{2}\right\rceil+\frac{1}{2}s\geq|N[v]\setminus S|.$$

If $v\in S_{r_j}$, $1\leq j\leq n$, then

$$|N[v]\cap S|-|N[v]\setminus S|\geq\left\lceil\frac{p-s}{2}\right\rceil-\left\lfloor\frac{p-s}{2}\right\rfloor+1\geq 1.$$

Thus, S is a global defensive alliance of G, and hence, $\gamma_a(G)\leq\lceil\frac{p-s}{2}\rceil+\frac{1}{2}s$. Let $W\subseteq V(G)$ be such that $|W|<\lceil\frac{p-s}{2}\rceil+\frac{1}{2}s$. Suppose that $W\setminus(\cup_{j=1}^{n}U_{r_j})\neq\emptyset$, and let $w\in W\setminus(\cup_{j=1}^{n}U_{r_j})$. Then

$$|N[w]\cap W|=|W|<\left\lceil\frac{p-s}{2}\right\rceil+\frac{1}{2}s\leq|N[w]\setminus W|.$$

Suppose, on the other hand, that $W\setminus(\cup_{j=1}^{n}U_{r_j})=\emptyset$. If $\left|W\cap U_{r_j}\right|\leq\frac{r_j}{2}$ for all $j=1,2,\ldots,n$, and $w\in W\cap U_{r_j}$, then

$$|N[w]\cap W|\leq 1+\sum_{i=1,i\neq j}^{n}\frac{r_i}{2},$$

while

$$|N[w]\setminus W|\geq(p-s)+\sum_{i=1,i\neq j}^{n}\frac{r_i}{2}.$$

Since $p-s>1$, we have $|N[w]\setminus W|>|N[w]\cap W|$. Finally, suppose that $\left|W\cap U_{r_j}\right|>\frac{r_j}{2}$ for some j. Let $w\in W\cap U_{r_j}$. Since $|N[w]\cap W|\leq\lceil\frac{p-s}{2}\rceil+\frac{s}{2}-\left|W\cap U_{r_j}\right|$, we have

$$\begin{aligned} |N[w]\setminus W| &= p-|W|-(r_j-\left|W\cap U_{r_j}\right|) \\ &\geq 1+p-(\left\lceil\frac{p-s}{2}\right\rceil+\frac{s}{2})-(r_j-\left|W\cap U_{r_j}\right|) \end{aligned}$$

$$\begin{aligned} &> 1+\left\lfloor\frac{p-s}{2}\right\rfloor+\frac{s}{2}-\left|W\cap U_{r_j}\right| \\ &\geq \left\lceil\frac{p-s}{2}\right\rceil+\frac{s}{2}-\left|W\cap U_{r_j}\right| \\ &\geq |N[w]\cap W|. \end{aligned}$$

This means that W, in any of the three options above, cannot be a defensive alliance in G. Therefore, $\gamma_a(G) = \lceil\frac{p-s}{2}\rceil + \frac{1}{2}s$.

To prove *(ii)*, first we consider the case where $p = s$. Assume r_1 is odd. Let $S = \cup_{i=1}^{n} S_{r_i}$, where $S_{r_i} \subseteq U_{r_i}$ with $|S_{r_i}| = \lfloor\frac{r_i}{2}\rfloor$. Then S is a dominating set in G. Let $v \in S$. Then

$$|N[v]\cap S| - |N[v]\setminus S| = \begin{cases} 1, & v \in S_{r_1}, \\ 0, & v \notin S_{r_1}. \end{cases}$$

This means that S is a global defensive alliance in G. Thus $\gamma_a(G) \leq \sum_{i=1}^{n}\lfloor\frac{r_i}{2}\rfloor$. Suppose that $W \subseteq V(G)$ with $|W| < \sum_{i=1}^{n}\lfloor\frac{r_i}{2}\rfloor$. If $|W\cap U_{r_i}| \leq \lfloor\frac{r_i}{2}\rfloor$ for all $i = 1, 2, \ldots, n$, and if $w \in W\cap U_{r_j}$, then

$$\begin{aligned} |N[w]\setminus W| &= \sum_{i=1, i\neq j}^{n} |U_{r_i}\setminus W| \\ &> \sum_{i=1, i\neq j}^{n} \left\lceil\frac{r_i}{2}\right\rceil \\ &= \begin{cases} 1+\sum_{i=1, i\neq j}^{n}\lfloor\frac{r_i}{2}\rfloor, & j\neq 1, \\ \sum_{i=2}^{n}\lfloor\frac{r_i}{2}\rfloor, & j = 1 \end{cases} \\ &\geq |N[w]\cap W|. \end{aligned}$$

Suppose that $\left|W\cap U_{r_j}\right| > \lfloor\frac{r_j}{2}\rfloor$, and thus $2\left|W\cap U_{r_j}\right| \geq r_j+1$, for some j. If $w \in W\cap U_{r_j}$, then

$$\begin{aligned} |N[w]\setminus W| - |N[w]\cap W| &= [p-|W|-(r_j-\left|W\cap U_{r_j}\right|)] \\ &\quad -[|W|-\left|W\cap U_{r_j}\right|+1] \\ &= p-2|W|+2\left|W\cap U_{r_j}\right|-1-r_j \\ &> 2\left|W\cap U_{r_j}\right|-r_j \\ &> 0. \end{aligned}$$

This means that W, in any of the two options above, is not a defensive alliance in G. Thus $\gamma_a(G) = \sum_{i=1}^{n}\lfloor\frac{r_i}{2}\rfloor$.

For the next case, suppose that $p > s$. Again, assume that r_1 is odd. For each $i = 2, 3, \ldots, n$, let $S_{r_i} \subseteq U_{r_i}$ with $|S_{r_i}| = \lfloor\frac{r_i}{2}\rfloor$. Let $S_{r_1} \subseteq U_{r_1}$ with

$|S_{r_1}| = \lceil \frac{r_1}{2} \rceil$, and let $S_{r_{n+1}} \subseteq V(G) \setminus (\cup_{i=1}^{n} U_{r_i})$ such that $|S_{r_{n+1}}| = \lfloor \frac{p-s}{2} \rfloor$. Let $v \in S = \cup_{i=1}^{n+1} S_{r_i}$. Then

$$|N[v] \cap S| - |N[v] \setminus S| = \begin{cases} 2 + \lfloor \frac{p-s}{2} \rfloor - \lceil \frac{p-s}{2} \rceil, & j \neq 1, n+1, \\ 1 + \lfloor \frac{p-s}{2} \rfloor - \lceil \frac{p-s}{2} \rceil, & j = 1, n+1. \end{cases}$$

Thus S is a global defensive alliance in G, and thus $\gamma_a(G) \leq 1 + \lfloor \frac{p-s}{2} \rfloor + \sum_{j=1}^{n} \lfloor \frac{r_j}{2} \rfloor$. Let $W \subseteq V(G)$ with $|W| < |S|$. Similarly as above, if $W \setminus (\cup_{i=1}^{n} U_{r_i}) \neq \emptyset$, and $w \in W \setminus (\cup_{i=1}^{n} U_{r_i})$, then $|N[w] \setminus W| > |N[w] \cap W|$. Suppose that $W \subseteq \cup_{i=1}^{n} U_{r_i}$. Following similar arguments, if $|W \cap U_{r_i}| \leq \lfloor \frac{r_i}{2} \rfloor$ for all i and $w \in W \cap U_{r_j}$, then $|N[w] \setminus W| > |N[w] \cap W|$. Suppose that $|W \cap U_{r_j}| > \lfloor \frac{r_j}{2} \rfloor$ for some j, and let $w \in W \cap U_{r_j}$. Then

$$\begin{aligned} |N[w] \setminus W| - |N[w] \cap W| &= [p - |W| - (r_j - |W \cap U_{r_j}|)] \\ &\quad -[|W| - |W \cap U_{r_j}| + 1] \\ &\geq p - 1 - 2\left\lfloor \frac{p-s}{2} \right\rfloor - 2\sum_{i=1}^{n} \left\lfloor \frac{r_i}{2} \right\rfloor \\ &\quad +2|W \cap U_{r_j}| - r_j \\ &\geq 1 + \left\lceil \frac{p-s}{2} \right\rceil - \left\lfloor \frac{p-s}{2} \right\rfloor \\ &> 0. \end{aligned}$$

This shows that $\gamma_a(G) = 1 + \lfloor \frac{p-s}{2} \rfloor + \sum_{j=1}^{n} \lfloor \frac{r_j}{2} \rfloor$.

The proof of (*iii*) is straightforwad, and is omitted. ■

2 On join of graphs

The *join* of two graphs G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), u \in V(H)\}$. In view of Theorem 1.2, we have the following lemmas.

Lemma 2.1 *If G and H are graphs of orders n and m, respectively, then*

$$\gamma_a(G + H) \leq m + n - min\left\{\left\lceil \frac{\delta(G) + m}{2} \right\rceil, \left\lceil \frac{\delta(H) + n}{2} \right\rceil\right\}.$$

In particular, if $H = K_m$, then $\gamma_a(G + H) \leq m + n - \left\lceil \frac{\delta(G)+m}{2} \right\rceil$.

The upper bound given in Lemma 2.1 is sharp. Consider, for example, the join $K_m + K_n$. Since $K_m + K_n = K_{m+n}$,

$$\gamma_a(K_m + K_n) = \left\lfloor \frac{m+n+1}{2} \right\rfloor = 1 + \left\lfloor \frac{m+n-1}{2} \right\rfloor = m + n - \left\lceil \frac{m+n-1}{2} \right\rceil.$$

Lemma 2.2 *Let $S \subseteq V(G)$ be a dominating set in a graph G. Then $N[v] \subseteq S$ for all $v \in S$ if and only if $S = V(G)$.*

Proof: Suppose that $S \neq V(G)$ is a dominating set in G, and let $v \in V(G) \setminus S$. Since $N[S] = V(G)$, there exists $u \in S$ such that $v \in N[u]$. That is, $v \in N[u] \setminus S$. This means that $N[u] \nsubseteq S$. The converse is obvious. ■

Lemma 2.3 *Let G be any graph and $m \geq 1$, and let $S \subseteq V(G + K_m)$. If $S \subseteq V(G)$ and S is a global defensive alliance in $G+K_m$, then S is a global defensive alliance in G.*

Proof: Since S is a dominating set in $G + K_m$, it is a dominating set in G. Let $v \in S$. Then

$$|(N[v] \cap V(G)) \cap S| = |N[v] \cap S| \geq |N[v] \setminus S| \geq |(N[v] \cap V(G)) \setminus S|.$$

Since v is arbitrary, the conclusion is established. ■

Lemma 2.4 *If S is a global defensive alliance in $G+K_m$ and $S \cap V(K_m) \neq \emptyset$, then*

$$|S| \geq \left\lceil \frac{m + |V(G)|}{2} \right\rceil.$$

Proof: Let $v \in S \cap V(K_m)$. Then $N[v] = V(G + K_m)$ so that

$$|S| = |N[v] \cap S| \geq |N[v] \setminus S| = |V(G + K_m) \setminus S|.$$

It follows that $|S| \geq \left\lceil \frac{m+|V(G)|}{2} \right\rceil$. ■

A dominating set S in a graph G is a said to be a *global strong defensive alliance* (see [6]) in G if for each $u \in S$, $|N[u] \cap S| > |N[u] \setminus S|$. The minimum cardinality of a global strong defensive alliance in G is called the *global strong defensive alliance number*, and is denoted by $\gamma_{\hat{a}}(G)$. In, particular, $\gamma_{\hat{a}}(K_n) = \lceil \frac{n+1}{2} \rceil$ [6]. Also, clearly, $\gamma_{\hat{a}}(\overline{K}_n) = n$.

Theorem 2.5 *Let G be a graph of order n. Then*

$$min \left\{ \gamma_{\hat{a}}(G), \left\lceil \frac{n+1}{2} \right\rceil \right\} \leq \gamma_a(G + K_1) \leq \gamma_{\hat{a}}(G), \qquad (1)$$

and these bounds are sharp.

Proof: Let $V(K_1) = \{v\}$. Let S be a global strong defensive alliance in G. Since $v \in N[S]$, S is a dominating set in $G + K_1$. Moreover, for each $u \in S$, $|N[u] \setminus S| = 1 + |(N[u] \cap V(G)) \setminus S|$, and thus, $|N[u] \cap S| \geq |N[u] \setminus S|$. This means that S is a global defensive alliance in $G + K_1$. This makes the right-hand inequality in (1).

Let S be a global defensive alliance in $G + K_1$. Suppose that $S \subseteq V(G)$, and let $u \in S$. Since $v \in N[u] \setminus S$, $|(N[u] \cap V(G)) \cap S| = |N[v] \cap S| \geq |N[u] \setminus S| > |(N[u] \cap V(G)) \setminus S|$. This means that S is a global strong defensive alliance in G. Thus $\gamma_{\hat{a}}(G) \leq |S|$. Suppose that $v \in S$. Then $|S| \geq \lceil \frac{n+1}{2} \rceil$, by Lemma 2.4. This establishes the left-hand inequality in (1).

That these bounds are sharp, can be seen as follows: If $G = \overline{K}_4$, then $\min\{\gamma_{\hat{a}}(G), \lceil \frac{4+1}{2} \rceil\} = \min\{4, 3\} = 3 = \gamma_a(K_{1,4}) = \gamma_a(G + K_1)\}$. On the other hand, if G is the cycle graph C_4 on $n = 4$ vertices, then $\gamma_{\hat{a}}(G) = 2 = \gamma_a(G + K_1)$. In this case, $\gamma_{\hat{a}}(G) < \lceil \frac{n+1}{2} \rceil$. ■

Corollary 2.6 *Let G be a graph of order n. If $\gamma_{\hat{a}}(G) \leq \lceil \frac{n+1}{2} \rceil$, then $\gamma_a(G + K_1) = \gamma_{\hat{a}}(G)$.*

Theorem 2.7 *Let G be a graph of order n, and let $m \geq 1$. Then*

$$\gamma_a(G + K_m) \geq min\left\{m + 1, \left\lceil \frac{m+n}{2} \right\rceil\right\}, \tag{2}$$

and this bound is sharp. If $m + 1 \leq n \leq m + 2$, then $\gamma_a(G + K_m) = m + 1$. Moreover, if G is noncomplete and $n \notin \{m+1, m+2\}$, then for equality to hold in (2), it is necessary that exactly one of the following holds:

a. $S \subseteq V(G)$ for all γ_a-sets S in $G + K_m$;

b. $S \cap V(K_m) \neq \emptyset$ for all γ_a-sets S in $G + K_m$.

Proof: Let S be a global defensive alliance in $G + K_m$. Suppose that $|S| < min\{m + 1, \lceil \frac{m+n}{2} \rceil\}$. In view of Lemma 2.4, $S \subseteq V(G)$. By Lemma 2.3, S is a global defensive alliance in G. Moreover, $|S| \geq m$. Thus $n \geq m$, which yields

$$|S| < \left\lceil \frac{n+n}{2} \right\rceil = n.$$

This means that $S \neq V(G)$. By Lemma 2.2, there exists $v \in S$ such that $N[v] \cap V(G) \not\subseteq S$. Thus

$$|N[v] \setminus S| \geq m + 1 > |S| \geq |N[v] \cap S|.$$

This is a contradiction to the definition of S being a defensive alliance, and inequality (2) is established.

To see that this bound is sharp, consider the following graphs: First, the graph $C_3K_1 + K_2$ as in Figure 2.1. For this graph, $\lceil \frac{m+n}{2} \rceil = 4$, while $m + 1 = 3$. In view of inequality (2), $\gamma_a(C_3K_1 + K_2) \geq 3$. But since $\{v_1, v_2, v_3\}$ is a global defensive alliance in $C_3K_1+K_2$, $\gamma_a(C_3K_1+K_2) \leq 3$. Therefore, $\gamma_a(C_3K_1+K_2) = 3$. Second, consider the graph P_n+K_m, where

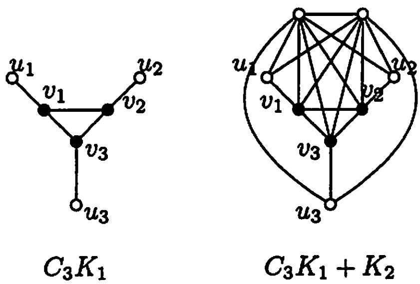

Figure 2.1

$n = 3$ and $m = 4$. Then $\lceil \frac{m+n}{2} \rceil < m+1$ so that inequality (2) implies that $\gamma_a(P_n + K_m) \geq \lceil \frac{m+n}{2} \rceil = 4$. But since $S = V(K_m)$ is a global defensive alliance in $P_n + K_m$, we have $\gamma_a(P_n + K_m) = 4$.

Let $m + 1 \leq n \leq m + 2$. Then $m + 1 = \lceil \frac{m+n}{2} \rceil$. By the first result, $\gamma_a(G + K_m) \geq m + 1$. And to attain equality, consider $S = V(K_m) \cup \{v\}$, where $v \in V(G)$. Clearly, S is a dominating set in $G+K_m$. Let $u \in S\setminus\{v\}$. Then

$$|N[u] \cap S| = |S| = m + 1 = \left\lceil \frac{m+n}{2} \right\rceil \geq |V(G + K_m) \setminus S| = |N[u] \setminus S|.$$

Note, further that

$$|N[v] \cap S| = m + 1 \geq n - 1 \geq |N[v] \setminus S|.$$

S, therefore, is a global defensive alliance in $G+K_m$ so that $\gamma_a(G+K_m) \leq m + 1$. Combining these two inequalities, $\gamma_a(G + K_m) = m + 1$.

Finally, to prove the last statement, suppose that G is noncomplete and $\gamma_a(G + K_m) = min\{m + 1, \lceil \frac{m+n}{2} \rceil\}$. Note that the hypothesis implies that $m + 1 \neq \lceil \frac{m+n}{2} \rceil$. Suppose that $m + 1 < \lceil \frac{m+n}{2} \rceil$, and let S be a γ_a-set in $G + K_m$. If $S \cap V(K_m) \neq \emptyset$, then $m + 1 = |S| \geq \lceil \frac{m+n}{2} \rceil$, a contradiction. Hence, $S \subseteq V(G)$, and so follows statement (a). Now, suppose that $\lceil \frac{m+n}{2} \rceil < m+1$, and let S be a γ_a-set in $G+K_m$. Suppose that

$S \subseteq V(G)$. We consider two cases: $S = V(G)$ or $S \neq V(G)$. If $S = V(G)$, then since G is not complete, there exists $v \in S$ such that $N[v] \cap S \neq S$. This means that $n > |N[v] \cap S| \geq |N[v] \setminus S| = m$, and consequently, $n \geq m+1$. This case, therefore, yields $\lceil \frac{m+n}{2} \rceil \geq \lceil \frac{m+m+1}{2} \rceil = m+1$, a contradiction. On the other hand, if $S \neq V(G)$, then

$$\left\lceil \frac{m+n}{2} \right\rceil = |S| < n = \left\lceil \frac{n+n}{2} \right\rceil.$$

This means that $m < n$. Writing $m+1 = \lceil \frac{m+m+1}{2} \rceil \leq \lceil \frac{m+n}{2} \rceil$, we also get a contradiction. Therefore, $S \cap V(K_m) \neq \emptyset$. Statement (b) follows immediately. ■

Let $G = H + K_m$, where H is a connected noncomplete graph of order n. If $m+n = 4$, then $H = P_3$ and $m = 1$. Theorem 1.1 gives the inequality $\frac{\sqrt{17}-1}{2} \approx 1.56 \leq \gamma_a(G)$, while Theorem 2.7 gives the estimate $2 \leq \gamma_a(G)$. Suppose that $m+n \geq 5$. Since

$$4(m+n)+1 < (m+n-1)(m+n)+(m+n) = (m+n)^2,$$

we have

$$\frac{\sqrt{4(m+n)+1}-1}{2} < \frac{m+n}{2} \leq \left\lceil \frac{m+n}{2} \right\rceil.$$

Thus, if $\lceil \frac{m+n}{2} \rceil \leq m+1$, then the lower bound given in Theorem 2.7 is a better estimate for the join than the one given in Theorem 1.1.

Corollary 2.8 *Let G be a graph of order n. If $m \geq n$, then*

$$\left\lceil \frac{m+n}{2} \right\rceil \leq \gamma_a(G+K_m) \leq m.$$

In particular, if $n = m-1$ or $n = m$, then $\gamma_a(G+K_m) = m$.

Proof: The first inequality follows from Theorem 2.7, while the second inequality follows from the fact that $V(K_m)$ is a global defensive alliance in $G+K_m$. ■

Corollary 2.9 *Let G be a graph of order n. If $m \geq n$, then*

$$\gamma_a(G) \leq \gamma_a(G+K_m). \quad (3)$$

Moreover, equality holds in (3) if and only if $G = \overline{K}_m$.

Proof: If $m \geq n$, then $\lceil \frac{m+n}{2} \rceil \geq n \geq \gamma_a(G)$. The desired inequality follows from Corollary 2.8.

To prove the last statement, suppose that $\gamma_a(G) = \gamma_a(G + K_m)$, and let S be a global defensive alliance in G with $|S| = \gamma_a(G + K_m)$. Since $|S| = \gamma_a(G)$, $|S| \leq n$. We claim that $\gamma_a(G) = n$ so that by Theorem 1.3, $G = \overline{K}_n$. Note that by Lemma 2.4, if $S \cap V(K_m) \neq \emptyset$, then $n \geq |S| \geq \lceil \frac{m+n}{2} \rceil \geq n$. If $S \subseteq V(G)$, then $n \geq |S| \geq m \geq n$. Either case yields $\gamma_a(G) = n$. Further, suppose that $m > n$. Since $V(G) = V(\overline{K}_n)$ is not a global defensive alliance in $G+K_m$, Lemma 2.4 implies that $\gamma_a(G+K_m) \geq \lceil \frac{m+n}{2} \rceil > n \geq \gamma_a(G)$. This is a contradiction. Hence $m = n$ and $G = \overline{K}_m$. Conversely, suppose that $G = \overline{K}_m$. By Theorem 1.3, $\gamma_a(G) = m$. Since $V(K_m)$ is a global defensive alliance in $G+K_m$, $\gamma_a(G+K_m) \leq m = \gamma_a(G)$. This, together with inequality in (3), implies that $\gamma_a(G+K_m) = \gamma_a(G)$. ■

The inequality in (3) may not be attained if $m < n$. Note, for example, that $\gamma_a(\overline{K}_6 + K_3) = 5 < 6 = \gamma_a(\overline{K}_6)$. Indeed, it is not always true that if H is a subgraph of a graph G, then $\gamma_a(H) \leq \gamma_a(G)$, and Corollary 2.9 provides a condition under which the desired inequality holds for some induced subgraph H of a connected graph G.

Equality in (3) can also be attained with $m < n$. To see this, we revisit the graph $C_3K_1 + K_2$ in Figure 2.1. Note that since $\{v_1, v_2, v_3\}$ is a global defensive alliance in C_3K_1, we have $\gamma_a(C_3K_1) \leq 3$. On the other hand, since $u_j \notin N[u_i] \cup N[v_i]$ for $i \neq j$, we have $\gamma_a(C_3K_1) \geq 3$. This makes $\gamma_a(C_3K_1) = 3$. Indeed, for this graph, $\gamma_a(C_3K_1 + K_2) = \gamma_a(C_3K_1)$.

Theorem 2.10 *Let G be a graph of order n, and let $m \geq 1$. If $\Delta(G) \leq m$, then*

$$\gamma_a(G + K_m) = \left\lceil \frac{m+n}{2} \right\rceil.$$

Proof: Suppose that $\gamma_a(G + K_m) < \lceil \frac{m+n}{2} \rceil$, and let S be a global defensive alliance in $G + K_m$ with $|S| = \gamma_a(G + K_m)$. In view of Lemma 2.4, $S \subseteq V(G)$. Either $S = V(G)$ or $S \neq V(G)$. Suppose that $S = V(G)$. Then the above assumption implies that $n < \lceil \frac{m+n}{2} \rceil$, which yields $n < m$. However, note that for each $v \in S$, $m = |N[v] \setminus S| \leq |N[v] \cap S| \leq |S| = n$. This is impossible. Now, suppose that $S \neq V(G)$. By Lemma 2.2 there exists $v \in S$ such that $N[v] \cap V(G) \not\subseteq S$. Thus, we have

$$|N[v] \setminus S| \geq m + 1 > deg_G(v) \geq |N[v] \cap S|.$$

This is a contradiction. Therefore, $\gamma_a(G + K_m) \geq \lceil \frac{m+n}{2} \rceil$.

Suppose that $m \geq n$. Let $l = \lceil \frac{m+n}{2} \rceil$. Then $l \leq m$. Let $S \subseteq V(K_m)$ such that $|S| = l$. Clearly, S is a dominating set in $G + K_m$. Moreover, for each $v \in S$

$$|N[v] \cap S| = |S| = l \geq |V(G + K_m) \setminus S| = |N[v] \setminus S|.$$

This means that S is a global defensive alliance in $G + K_m$. Suppose that $m < n$. Then $l = \lceil \frac{m+n}{2} \rceil - m \geq 1$. Let $S^* \subseteq V(G)$ with $|S^*| = l$. Consider the dominating set $S = S^* \cup V(K_m)$ in $G + K_m$. If $v \in V(K_m)$, then

$$|N[v] \cap S| = |S| = \left\lceil \frac{m+n}{2} \right\rceil \geq |N[v] \setminus S|.$$

If $v \in S^*$, then

$$|N[v] \cap S| \geq m + 1 > deg_G(v) \geq |N[v] \setminus S|.$$

This shows that S is a global defensive alliance in $G+K_m$ with $|S| = \lceil \frac{m+n}{2} \rceil$. Therefore, $\gamma_a(G + K_m) \leq \lceil \frac{m+n}{2} \rceil$. ■

Example 2.11 $\gamma_a(G + K_m) = \lceil \frac{m+n}{2} \rceil$ *for* $m \geq 2$, *where* G *is a path or a cycle of order* $n \geq 4$, *or any* m*-regular connected noncomplete graph of order* n.

Corollary 2.12 *For every pair of positive integers* m *and* n, *with* $m \geq 2$ *and* $n \geq 3$, *there exists a pair of connected noncomplete graphs* (G, G^*) *such that* $|V(G)| = m+n = |V(G^*)|$ *with* $\gamma_a(G) = \lceil \frac{m+n}{2} \rceil$ *and* $\gamma_a(G^*) = \lfloor \frac{m+n}{2} \rfloor$.

Proof: In view of Example 2.11, one may take $G = P_n + K_m$. Also, if $m + n$ is even, then we may take $G^* = P_n + K_m$. Suppose that $m + n$ is odd. If $m + n = 5$, then we may consider the graph G^* as in Figure 2.2. Note that the set $\{u, v\}$ is a γ_a-set in G^*. Thus $\gamma_a(G^*) = \lfloor \frac{5}{2} \rfloor$.

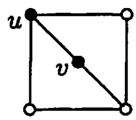

Figure 2.2: Graph G^* with $\gamma_a(G^*) = \lfloor \frac{5}{2} \rfloor$

Suppose that $m + n \geq 7$. Suppose further that $m + n = 3 + 2k$, $k \geq 2$. Take G^* to be the complete multipartite graph of order $m + n$ and with $\Omega(G^*) = \{U_3, U_{r_1}, U_{r_2}, \ldots, U_{r_k}\}$, where $r_1 = r_2 = \cdots = r_k = 2$. By Theorem 1.4, $\gamma_a(G^*) = \lfloor \frac{3}{2} \rfloor + k = \lfloor \frac{m+n}{2} \rfloor$. ■

Theorem 2.13 *Let G and H be graphs of orders n and m, respectively. If $\Delta(G) < m-1$ and $\Delta(H) < n-1$, then*

$$\gamma_a(G+H) \geq max\left\{\left\lceil\frac{1+\delta(H)+n}{2}\right\rceil, \left\lceil\frac{1+\delta(G)+m}{2}\right\rceil\right\}, \tag{4}$$

and this bound is sharp.

Proof: Let S be a global defensive alliance in $G+H$. If $S \subseteq V(G)$ and $v \in S$, then $\Delta(G) \geq deg_G(v) \geq |N[v] \cap S| - 1 \geq m$, a contradiction. Thus, $S \cap V(H) \neq \emptyset$. Similarly, $S \cap V(G) \neq \emptyset$. Take $v \in S \cap V(G)$. Then $|N[v]| = m+1+deg_G(v)$. Consequently, $|S| \geq \left\lceil\frac{m+1+\delta(G)}{2}\right\rceil$. Similarly, $|S| \geq \left\lceil\frac{n+1+\delta(H)}{2}\right\rceil$. The conclusion follows from the arbitrary nature of S.

Consider the complete bipartite graph $K_{2,2} = \overline{K}_2 + \overline{K}_2$. Note that $\gamma_a(K_{2,2}) = 2$, which coincides with the corresponding estimate given in (4). The given lower bound, therefore, is sharp. ■

A graph G is said to be a P_k*-graph* if G has P_k as an induced subgraph.

Theorem 2.14 *Let G and H be graphs of orders $n \geq 4$ and $m \geq 4$, respectively, such that $\Delta(G) = 2 = \Delta(H)$. If G and H are $P_{\lceil\frac{n}{2}\rceil}$-graph and $P_{\lceil\frac{m}{2}\rceil}$-graph, respectively, then*

$$\gamma_a(G+H) = \left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor.$$

Proof: Let $k_1 = \lfloor\frac{n}{2}\rfloor$ and $k_2 = \lfloor\frac{m}{2}\rfloor$, and suppose that P_{k_1} and P_{k_2} are induced subgraphs of G and H, respectively. Let $S = V(P_{k_1}) \cup V(P_{k_2})$. Clearly, S is a dominating set in $G+H$. If $v \in V(P_{k_1})$, then $|N[v] \cap S| \geq \lfloor\frac{m}{2}\rfloor + 2 \geq \lceil\frac{m}{2}\rceil + 1 \geq |N[v] \setminus S|$. Similarly, if $v \in V(P_{k_2})$, then $|N[v] \cap S| \geq |N[v] \setminus S|$. Therefore, S is a global defensive alliance in $G+H$, and hence $\gamma_a(G+H) \leq \lfloor\frac{n}{2}\rfloor + \lfloor\frac{m}{2}\rfloor$.

Suppose that $\gamma_a(G+H) < \lfloor\frac{n}{2}\rfloor + \lfloor\frac{m}{2}\rfloor$, and let $S \subseteq V(G+H)$ with $|S| = \gamma_a(G+H)$. Then $|S \cap V(G)| < \lfloor\frac{n}{2}\rfloor$ or $|S \cap V(H)| < \lfloor\frac{m}{2}\rfloor$. Suppose that $|S \cap V(G)| < \lfloor\frac{n}{2}\rfloor$. Note that either $S \cap V(H) = \emptyset$ or $S \cap V(H) \neq \emptyset$. Suppose that $S \cap V(H) = \emptyset$, i.e., suppose that $S \subseteq V(G)$. Since $S \neq V(G)$, there exists $v \in S$ such that $|N[v] \cap S| \leq 2$. Since $m \geq 4$, $|N[v] \cap S| < m \leq |N[v] \setminus S|$, implying that S cannot be a defensive alliance in $G+H$. Now, suppose that $S \cap V(H) \neq \emptyset$. If $S \cap V(H) = V(H)$, then $|S \cap V(G)| + m < \lfloor\frac{n}{2}\rfloor + \lfloor\frac{m}{2}\rfloor$. Or equivalently, $|S \cap V(G)| < \lfloor\frac{n}{2}\rfloor - \lceil\frac{m}{2}\rceil$.

Let $v \in V(H)$. Since $m \geq 4$,

$$\begin{aligned} |N[v] \cap S| &\leq 3 + |S \cap V(G)| \\ &< 3 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lceil \frac{m}{2} \right\rceil \\ &\leq 3 - 2 + \left\lfloor \frac{n}{2} \right\rfloor \\ &\leq |V(G) \setminus S| \\ &= |N[v] \setminus S|, \end{aligned}$$

a contradiciton. If $\emptyset \neq S \cap V(H) \neq V(H)$, then there exists $v \in S \cap V(H)$ such that $|(N[v] \cap V(H)) \cap S| \leq 2$. Since $m \geq 4$, we have $|N[v] \cap S| \leq 2 + |S \cap V(G)| < 2 + \lfloor \frac{n}{2} \rfloor - \lceil \frac{m}{2} \rceil \leq \lfloor \frac{n}{2} \rfloor < |N[v] \setminus S|$, a contradiction. The other case where $|S \cap V(H)| < \lfloor \frac{m}{2} \rfloor$ also leads to contradictions. Therefore, $\gamma_a(G + H) \geq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$, and the conclusion follows. ■

Example 2.15 *For positive integers $m, n \geq 4$, $\gamma_a(P_n + P_m) = \gamma_a(P_m + C_n) = \gamma_a(C_m + C_n) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$.*

3 On corona of graphs

The *corona* $G \circ H$ of graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H. It is customary to denote by H_v that copy of H whose vertices are adjoined with the vertex v of G. In effect, $G \circ H$ is composed of the subgraphs $H_v + v$ together with the edges of G. Clearly, $V(G \circ H) = \bigcup_{v \in V(G)} V(H_v + v)$.

Lemma 3.1 *If S is a global defensive allinace in $G \circ H$, then $S \cap V(H_v + v) \neq \emptyset$.*

Proof: If $S \cap V(H_v + v) = \emptyset$ and $w \in V(H_v)$, then $w \notin N[S]$. This is impossible for a dominating set S in $G \circ H$. ■

Lemma 3.2 *In the corona $G \circ H$, if $S_v \subseteq V(H_v)$ is a global defensive alliance in $H_v + v$, then $\cup_{v \in V(G)} S_v$ is a global defensive alliance in $G \circ H$.*

Proof: This immediately follows from the fact that if $S = \cup_{v \in V(G)} S_v$, then $N[u] \cap S = N[u] \cap S_v$ and $N[u] \setminus S = N[u] \setminus S_v$ for all $u \in S_v$. ■

Lemma 3.3 *If G is a graph of order n, then $n \leq \gamma_a(G \circ H) \leq n\gamma_{\hat{a}}(H)$ for any graph H.*

Proof: In view of Lemma 3.1, $n \leq \gamma_a(G \circ H)$. The right-hand inequality follows from Theorem 2.5. ■

Clearly, $\gamma_a(G \circ K_1) = |V(G)|$ for any graph G. On the other hand, $\gamma_a(P_m \circ C_4) = 2m = m\gamma_{\hat{a}}(C_4)$. These examples show that the bounds in Lemma 3.3 are sharp.

Proposition 3.4 *For any graphs G and H, if $deg_G(v) \geq |V(H)| - 1$ for all $v \in V(G)$, then $\gamma_a(G \circ H) = |V(G)|$.*

Proof: By Lemma 3.3, $\gamma_a(G \circ H) \geq |V(G)|$. Now, clearly $S = V(G)$ is a dominating set in $G \circ H$. Further, if $v \in S$, then

$$|N[v] \cap S| = 1 + deg_G(v) \geq |V(H)| = |N[v] \setminus S|.$$

Thus $\gamma_a(G \circ H) \leq |V(G)|$. ■

Theorem 3.5 *Let G be any graph and $m \geq 1$. Let S be a γ_a-set in $G \circ K_m$, and let $v \in V(G)$. If $deg_G(v) < m - 1$, then $S_v = S \cap V((K_m)_v + v)$ is a γ_a-set in $(K_m)_v + v$.*

Proof: For convenience, put $H = K_m$. First, we show that S_v is a global defensive alliance in $H_v + v$. If $S_v = \{v\}$, then

$$|N[v] \cap S| \leq 1 + deg_G(v) < m \leq |N[v] \setminus S|,$$

a contradiction. Hence, $S_v \cap V(H_v) \neq \emptyset$. Let $u \in S_v \cap V(H_v)$. Note that $N[u] = V(H_v + v)$. Since $u \in S$,

$$|N[u] \cap S_v| = |N[u] \cap S| \geq |N[u] \setminus S| = |N[u] \setminus S_v|.$$

Moreover, since $H_v + v$ is complete, $|N[v] \cap S_v| = |N[u] \cap S_v|$ and $|N[v] \setminus S_v| = |N[u] \setminus S_v|$ for all $u \in V(H_v)$. This shows that S_v is a global defensive alliance in $H_v + v$. The minimality follows from Lemma 3.2. ■

Corollary 3.6 *For any graph G and $m \geq 1$,*

$$\gamma_a(G \circ K_m) = |V(G) \setminus T| + \left\lceil \frac{m+1}{2} \right\rceil |T|,$$

where $T = \{v \in V(G) : deg_G(v) < m - 1\}$.

Proof: Put $H = K_m$, and let $T = \{v \in V(G) : deg_G(v) < m-1\}$. For each $v \in T$, let $S_v \subseteq V(H_v + v)$ such that $v \in S_v$ and $|S_v| = \lceil \frac{m+1}{2} \rceil$. Let $S = (V(G) \setminus T) \cup (\cup_{v \in T} S_v)$. If $v \in V(G) \setminus T$, then

$$|N[v] \cap S| = 1 + deg_G(v) \geq m = |N[v] \setminus S|.$$

If $u \in S_v$, $v \in T$, then

$$|N[u] \cap S| \geq |S_v| = \left\lceil \frac{m+1}{2} \right\rceil \geq |N[u] \setminus S|.$$

Thus S is a global defensive alliance in $G \circ K_m$. Hence, $\gamma_a(G \circ H) \leq |V(G) \setminus T| + \lceil \frac{m+1}{2} \rceil |T|$.

Let S be a γ_a-set in $G \circ H$. By Theorem 3.5, for each $v \in T$, $S_v = S \cap V(H_v + v)$ is a γ_a-set in $H_v + v$. Since $H_v + v$ is complete, $|S_v| = \lceil \frac{m+1}{2} \rceil$. In view of Lemma 3.1,

$$\begin{aligned} \gamma_a(G \circ H) = |S| &= \sum_{v \in T} |S_v| + |S \setminus \cup_{v \in T} S_v| \\ &\geq \sum_{v \in T} |S_v| + |V(G) \setminus T| \\ &= \left\lceil \frac{m+1}{2} \right\rceil |T| + |V(G) \setminus T|. \end{aligned}$$

The desired equality follows immediately. ∎

Corollary 3.7 *If* $m \geq 2$ *and* $\Delta(G) < m-1$, *then*

$$\gamma_a(G \circ K_m) = |V(G)| \left\lceil \frac{m+1}{2} \right\rceil.$$

Example 3.8 $\gamma_a(P_n \circ K_m) = \gamma_a(C_n \circ K_m) = n \lceil \frac{m+1}{2} \rceil$ *for* $m \geq 4$.

4 On Composition of Graphs

The *composition* or *lexicographic product* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in V(H)$. In this section, we consider only composition $G[H]$, where G is connected and $H = K_m$. The reader may find it interesting to investigate other possible cases. It should be noted that the symbol $N[(u,v)]$ refers to the closed neighborhood of (u,v) in $G[H]$, while $N[u]$ refers to the closed neighborhood of u in G.

Given $S \subseteq V(G[H])$, the *G-projection* S_f of S is the set of all first components of S. That is,

$$S_f = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}.$$

Lemma 4.1 *In the composition $G[K_m]$, where G is connected and $m \geq 1$, for every $u \in V(G)$, $N[(u,v)] = N[u] \times V(H)$ for all $v \in V(H)$.*

Theorem 4.2 *Let G be connected and $m \geq 1$, and let $S \subseteq V(G[K_m])$. If S is a global defensive alliance in $G[K_m]$, then S_f is a global defensive alliance in G.*

Proof: Let $x \in V(G) \setminus S_f$, and let $y \in V(K_m)$. There exists $(u, v) \in S$ such that $(x, y) \in N[(u, v)]$. Since $u \neq x$, $d_G(u, x) = 1$. That is, $x \in N[u] \subseteq N[S_f]$. This means that S_f is a dominating set in G.

Suppose that $u \in S_f$ with $|N[u] \cap S_f| < |N[u] \setminus S_f|$. Let $v \in V(K_m)$ such that $(u, v) \in S$. Since $(N[u] \setminus S_f) \times V(K_m) \subseteq N[(u, v)] \setminus S$, together with Lemma 4.1 we have

$$\begin{aligned}
|N[(u,v)] \cap S| &= |(N[u] \times V(K_m)) \cap S| \\
&\leq |(N[u] \cap S_f) \times V(K_m)| \\
&= |V(K_m)|\,|N[u] \cap S_f| \\
&< |V(K_m)|\,|N[u] \setminus S_f| \\
&\leq |N[(u,v)] \setminus S|.
\end{aligned}$$

Since S is a defensive alliance and $(u, v) \in S$, this is impossible. This proves that S_f is global defensive alliance in G. ■

For a global defensive alliance A in a connected graph G, we define

$$A^\circ = \{u \in A : \exists v \in A \text{ with } u \in N(v)\}.$$

Theorem 4.3 *Let G be a connected graph and $m \geq 1$. Let $S \subseteq V(G[K_m])$. If S is a global defensive alliance in $G[K_m]$, then there exists a global defensive alliance A in G such that $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$, where $T_f = A^\circ$. In this case, $|T| \geq \lfloor \frac{m}{2} \rfloor |A^\circ|$.*

Proof: Let $S \subseteq V(G[K_m])$ be a global defensive alliance in $G[K_m]$, and put $A = S_f$. By Theorem 4.2, A is a global defensive alliance in G. Define $T = \{(u, v) \in S : u \in A^\circ\}$. Then $T_f = A^\circ$, and $S \subseteq [(A \setminus A^\circ) \times V(K_m)] \cup T$. Let $x \in A \setminus A^\circ$, and suppose that there exists $y \in V(K_m)$ such that $(x, y) \notin S$. Since $x \notin A^\circ$, $N[(x, v)] \cap S = \{(x, u) :$

$u \in V(K_m)\} \cap S$ and $|N[(x,v)] \setminus S| \geq m$ for all $v \in V(K_m)$. Take $v \in V(K_m)$ such that $(x,v) \in S$. Since $(x,y) \notin S$, $|N[(x,v)] \cap S| \leq m-1 < |N[(x,v)] \setminus S|$. This contradiction implies that $(A \setminus A^\circ) \times V(K_m) \subseteq S$. Therefore, $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$.

Now, suppose that $|T| < \lfloor \frac{m}{2} \rfloor |A^\circ|$, where $T_u = T \cap (\{u\} \times V(K_m))$. Contradiction is easily attained if $|T_u| < \lfloor \frac{m}{2} \rfloor$ for all $u \in A^\circ$. But the hypothesis implies that there exists $u \in A^\circ$ such that

$$\sum_{v \in N[u] \cap A^\circ} |T_v| < |N[u] \cap A^\circ| \left\lfloor \frac{m}{2} \right\rfloor \leq \sum_{v \in N[u] \cap A^\circ} (m - |T_v|).$$

Choose $x \in V(K_m)$ such that $(u,x) \in T$. Then, using the above result,

$$\begin{aligned} |N[(u,x)] \cap T| &= |(N[u] \times V(K_m)) \cap T| \\ &= \sum_{v \in N[u] \cap A^\circ} |T_v| \\ &< \sum_{v \in N[u] \cap A^\circ} (m - |T_v|) \\ &\leq |(N[u] \times V(K_m)) \setminus T| \\ &= |N[(u,x)] \setminus T|. \end{aligned}$$

This is a contradiction. Therefore, $|T| \geq \lfloor \frac{m}{2} \rfloor |A^\circ|$. ∎

Corollary 4.4 *For all connected graphs G and $m \geq 1$,*

$$\gamma_a(G[K_m]) \geq min\{m\,|A| - \left\lceil \frac{m}{2} \right\rceil |A^\circ| : A \text{ is a global defensive alliance in } G\}.$$

The lower bound given in Corollary 4.4 is sharp. Verify that $\gamma_a(K_2[K_2]) = 2$. If A is a global defensive alliance in K_2 and $|A| = 1$, then $A^\circ = \emptyset$ and, with $m = 2$, $m\,|A| - \lceil \frac{m}{2} \rceil |A^\circ| = 2$. If A is a global defensive alliance in K_2 and $|A| = 2$, then $A^\circ = A$ and, with $m = 2$, $m\,|A| - \lceil \frac{m}{2} \rceil |A^\circ| = 2$.

Theorem 4.5 *Let G be a connected graph and $m \geq 1$. Let $A \subseteq V(G)$ be a global defensive alliance in G, and for each $u \in A^\circ$, let $T_u \subseteq \{u\} \times V(K_m)$ with the following properties:*

1. *If $N[u] \subseteq A^\circ$, then $|T_u| \geq \lceil \frac{m}{2} \rceil$; and*
2. *If $N[u] \setminus A^\circ \neq \emptyset$, then $|T_v| = m$ for all $v \in N[u] \cap A^\circ$.*

If $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$*, where* $T = \cup_{u \in A^\circ} T_u$*, then* S *is a global defensive alliance in* $G[K_m]$.

Proof: In view of Lemma 4.1, since A is a dominating set in G, S is a dominating set in $G[K_m]$. If $(x,y) \in (A \setminus A^\circ) \times V(K_m)$, then $|N[(x,y)] \cap S| = m = |N[(x,y)] \setminus S|$. Let $(x,y) \in T$ such that $N[x] \subseteq A^\circ$. Then

$$\begin{aligned} |N[(x,y)] \cap S| &= |(N[x] \times V(K_m)) \cap T| \\ &\geq |N[x]| \cdot \left\lceil \frac{m}{2} \right\rceil \\ &\geq |(N[x] \times V(K_m)) \setminus T| \\ &= |N[(x,y)] \setminus S| \end{aligned}$$

Now, let $(x,y) \in T$ such that $N[x] \setminus A^\circ \neq \emptyset$.

$$\begin{aligned} |N[(x,y)] \cap S| &= m\,|N[x] \cap A^\circ| + m\,|N[x] \setminus A^\circ| \\ &\geq m\,|N[x] \setminus A^\circ| \\ &= |N[(x,y)] \setminus S| . \end{aligned}$$

Therefore S is a global defensive alliance in $G[K_m]$. ■

Corollary 4.6 *For any connected graph* G *and* $m \geq 1$*, if* $A \subseteq V(G)$ *is a global defensive alliance in* G*, then* $A \times V(K_m)$ *is a global defensive alliance in* $G[K_m]$.

Corollary 4.7 *For all connected graphs* G *and* $m \geq 1$, $\gamma_a(G[K_m]) \leq m\gamma_a(G)$.

The bound in Corollary 4.7 is sharp. Consider, for example, the graph $G[K_2]$ as in Figure 4.1. The set $\{x, y\}$ is a γ_a-set in G so that $\gamma_a(G) = 2$.

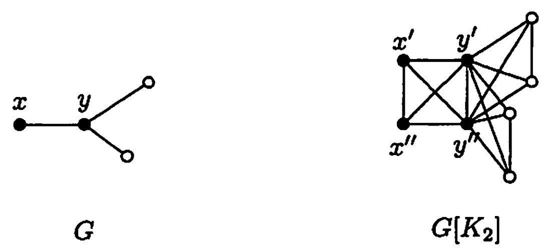

Figure 4.1

Similarly, $\{x', y', x'', y''\}$ is a γ_a-set in $G[K_2]$. Thus $\gamma_a(G[K_2]) = 4 = 2\gamma_a(G)$.

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