An operation based on complete graphs with an application to the line graphs of trees

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Abstract. Given a disjoint union of some complete graphs, one can define a graph by choosing one vertex from each complete graph and making all of these vertices adjacent. This observation leads us to define a new operation on certain graphs. We compute characteristic polynomial of the resulting graphs and indicate a method for computing determinant of this matrix for obtaining characteristic polynomial of new graphs. We show that line graphs of trees can be obtained by performing this operation on some graphs and, as an application, we compute the characteristic polynomials of line graphs of trees.

Keywords: Characteristic polynomial, Line graph, Tree.

1 Introduction

The interactions between algebra and combinatorics is a fruitful subject of study, as shown by the increasing amount of literature on the subject that has appeared in the last two decades. In particular, a considerable effort has been devoted to the use of algebraic techniques in the study of graphs as, for instance, the achievement of bounds for (some of) their parameters, also in the study of metric parameters, such as the mean distance, diameter, radius, topological index in terms of their adjacency or Laplacian spectra [1, 2, 4, 5, 6]. In quantum chemistry, the skeletons of certain non-saturated hydrocarbons are represented by graphs. The stability of the molecule as well as other chemically important properties are closely related to the graph spectrum. In particular, the calculation of the characteristic polynomial of a molecular graph plays an important role in this theory [7].

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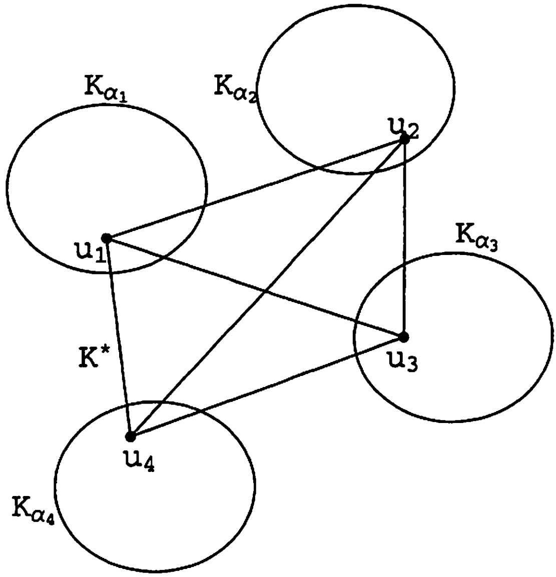

Figure 1: $K(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$; K^* is the complete graph on $V^1 = \{u_1, u_2, u_3, u_4\}$.

Many graphs can be obtained by combining two or more graphs. For example join, composition and cartesian product are well-known operations and several graphs can be obtained by these operations from simple graphs. We define an operation on complete graphs as follows.

Denote by K_n, the complete graph on n vertices. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers. We define a new graph $K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ by replacing the vertices of K_n by graphs K_{α_i}, $1 \leq i \leq n$. More precisely, for $1 \leq i \leq n$, we choose an arbitrary vertex from K_{α_i}, and make all of these vertices adjacent. Since K_{α_i}, $1 \leq i \leq n$, is a complete graph the construction of $K := K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is independent of the choice of vertices. More formally for $i = 1, 2, \ldots, n$, let V_i and E_i be the set of vertices and edges of K_{α_i}, respectively. We may assume that $V_i \cap V_j = \emptyset$, for $i \neq j$. Let $V = \bigcup_{i=1}^{n} V_i$ and $E^1 = \bigcup_{i=1}^{n} E_i$. For all $i = 1, 2, \ldots, n$, choose an arbitrary vertex u_i from V_i and put $V^1 = \{u_1, u_2, \ldots, u_n\}$. Let $E^2 = \{u_i u_j \mid 1 \leq i \neq j \leq n\}$ and $K^* = (V^1, E^2)$. The graph $K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ has the set of vertices $V = \bigcup_{i=1}^{n} V_i$ and the set of edges $E = E^1 \cup E^2$ (see Figure 1).

In section 2 we obtain the characteristic polynomial of $K(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then based on this operation we define a new operation on graphs and compute the characteristic polynomials of resulting graphs. In section 3 we show that line graphs of trees can be obtained by performing this operation on some certain graphs and, as an application, we compute the characteristic polynomials of line graphs of trees.

2 Main results

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers. Throughout the paper we assume that $K := K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and K^* denote the complete graph (V^1, E^2). We keep the above notations for later use. In this section we compute the characteristic polynomial of K. The distance matrix of K can easily obtained by definition. The distances of distinct vertices $u \in K_{\alpha_i}$ and $v \in K_{\alpha_j}$ of graph K is

$$d(u,v) = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \text{ and } [u \in V^1, \quad v \notin V^1] \text{ or } [u \notin V^1, \quad v \in V^1] \\ 3 & \text{if } i \neq j \text{ and } u \notin V^1, \quad v \notin V^1. \end{cases}$$

Now let $E_{m,n}$ be an $m \times n$ matrix whose $(1,1)$ entry is 1 and other entries are zero. If A_{α_i} denote the adjacency matrix of complete graph K_{α_i}, $1 \leq i \leq n$, then by suitable labelling for vertices of K (vertices of K_{α_i} have consecutive labels) the adjacency matrix of K is

$$A = \begin{bmatrix} A_{\alpha_1} & E_{\alpha_1,\alpha_2} & E_{\alpha_1,\alpha_3} & \cdots & E_{\alpha_1,\alpha_n} \\ E_{\alpha_2,\alpha_1} & A_{\alpha_2} & E_{\alpha_2,\alpha_3} & \cdots & E_{\alpha_2,\alpha_n} \\ E_{\alpha_3,\alpha_1} & E_{\alpha_3,\alpha_2} & A_{\alpha_3} & \cdots & E_{\alpha_3,\alpha_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{\alpha_n,\alpha_1} & E_{\alpha_n,\alpha_2} & E_{\alpha_n,\alpha_3} & \cdots & A_{\alpha_n} \end{bmatrix}.$$

Let $B_{\alpha_i} = A_{\alpha_i} - xI_{\alpha_i}$, where x is an indeterminant variable and I_{α_i} is the identity $\alpha_i \times \alpha_i$ square matrix. Denote by $|A|$ the determinant of the square matrix A. Then the characteristic polynomial of K can be computed by the following determinant:

$$\det(A - xI) = \begin{vmatrix} B_{\alpha_1} & E_{\alpha_1,\alpha_2} & E_{\alpha_1,\alpha_3} & \cdots & E_{\alpha_1,\alpha_n} \\ E_{\alpha_2,\alpha_1} & B_{\alpha_2} & E_{\alpha_2,\alpha_3} & \cdots & E_{\alpha_2,\alpha_n} \\ E_{\alpha_3,\alpha_1} & E_{\alpha_3,\alpha_2} & B_{\alpha_3} & \cdots & E_{\alpha_3,\alpha_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{\alpha_n,\alpha_1} & E_{\alpha_n,\alpha_2} & E_{\alpha_n,\alpha_3} & \cdots & B_{\alpha_n} \end{vmatrix}.$$

To compute the characteristic polynomial of K, first we prove the following two lemmas. Throughout the paper $\bar{A}$ denote the matrix obtained by deletion of the first row and column of the matrix A and $\bar{A} := [1]$ if $A = [a_{1,1}]$.

Lemma 2.1 Let $A_{n_1}, A_{n_2}, \ldots, A_{n_s}$ be $n_i \times n_i$ square matrices. If

$$X = \begin{bmatrix} A_{n_1} & E_{n_1,n_2} & E_{n_1,n_3} & \cdots & E_{n_1,n_s} \\ E_{n_2,n_1} & A_{n_2} & E_{n_2,n_3} & \cdots & E_{n_2,n_s} \\ E_{n_3,n_1} & E_{n_3,n_2} & A_{n_3} & \cdots & E_{n_3,n_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{n_s,n_1} & E_{n_s,n_2} & E_{n_s,n_3} & \cdots & A_{n_s} \end{bmatrix},$$

then

$$\det(X) = \begin{vmatrix} |A_{n_1}| & |\bar{A}_{n_1}| & |\bar{A}_{n_1}| & \cdots & |\bar{A}_{n_1}| \\ |\bar{A}_{n_2}| & |A_{n_2}| & |\bar{A}_{n_2}| & \cdots & |\bar{A}_{n_2}| \\ |\bar{A}_{n_3}| & |\bar{A}_{n_3}| & |A_{n_3}| & \cdots & |\bar{A}_{n_3}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & \cdots & |A_{n_s}| \end{vmatrix}.$$

Proof: We prove the lemma by induction on s. Suppose that the lemma is true for all positive integers which are smaller than s. Let A'_{ij} denote the cofactor of (i,j)th entry of A, $\bar{A}_j$ denote the matrix obtained by deletion jth column of A and $\bar{A}_{ij}$ denote the matrix obtained by deletion ith row and jth column of A. If $A_{n_1} = [a_{ij}]$ and $m_i = n_1 + n_2 + \cdots + n_i$ then by the expanding $\det(X)$ with respect to the first row of X we have

$$\det(X) = \sum_{r=1}^{n_1} a_{1r} X'_{1r} + \sum_{r=2}^{s} X'_{1,m_{r-1}+1} \tag{1}$$

By definition of the matrix X and induction hypothesis, for $1 \le r \le n_1$, we have

$$\begin{aligned} X'_{1r} &= (-1)^{r+1} \begin{vmatrix} (\bar{A}_{n_1})_{1r} & 0 & 0 & \cdots & 0 \\ (\bar{E}_{n_2,n_1})_r & A_{n_2} & E_{n_2,n_3} & \cdots & E_{n_2,n_s} \\ (\bar{E}_{n_3,n_1})_r & E_{n_3,n_2} & A_{n_3} & \cdots & E_{n_3,n_s} \\ \vdots & \vdots & \vdots & \ddots & \\ (\bar{E}_{n_s,n_1})_r & E_{n_s,n_2} & E_{n_s,n_3} & \cdots & A_{n_s} \end{vmatrix} \\ &= (A_{n_1})'_{1r} \begin{vmatrix} A_{n_2} & E_{n_2,n_3} & \cdots & E_{n_2,n_s} \\ E_{n_3,n_2} & A_{n_3} & \cdots & E_{n_3,n_s} \\ \vdots & \vdots & \ddots & \\ E_{n_s,n_2} & E_{n_s,n_3} & \cdots & A_{n_s} \end{vmatrix} \\ &= (A_{n_1})'_{1r} \begin{vmatrix} |A_{n_2}| & |\bar{A}_{n_2}| & |\bar{A}_{n_2}| & \cdots & |\bar{A}_{n_2}| \\ |\bar{A}_{n_3}| & |A_{n_3}| & |\bar{A}_{n_3}| & \cdots & |\bar{A}_{n_3}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & \cdots & |A_{n_s}| \end{vmatrix} \\ &= (A_{n_1})'_{1r} F \end{aligned} \tag{2}$$

where $F = [\alpha_{r,t}]$ is the square matrix of order $s-1$ such that

$$\alpha_{r,t} = \begin{cases} |A_{n_{r+1}}| & \text{if} \quad r = t \\ |\bar{A}_{n_{r+1}}| & \text{if} \quad r \neq t. \end{cases}$$

Now let $2 \leq i \leq s$ and $\hat{A}_{n_1}$ be the matrix obtained by deletion of the first row of A_{n_1}. Then we have

$$(-1)^{m_{i-1}+2} X'_{1,m_{i-1}+1} = \begin{vmatrix} \hat{A}_{n_1} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ E_{n_2,n_1} & A_{n_2} & E_{n_2,n_3} & \cdots & 0 & \cdots & E_{n_2,n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{n_i,n_1} & E_{n_i,n_2} & E_{n_i,n_3} & \cdots & (\bar{A}_{n_i})_1 & \cdots & E_{n_j,n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{n_s,n_1} & E_{n_s,n_2} & E_{n_s,n_3} & \cdots & 0 & \cdots & A_{n_s} \end{vmatrix}.$$

Let I be the first column of $\hat{A}_{n_1}$ and J be the first row of $(\bar{A}_{n_i})_1$. Then $(-1)^{m_{i-1}} X'_{1,m_{i-1}+1}$ can be written as

$$\begin{vmatrix} I & \bar{A}_{n_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ E_{n_2,1} & 0 & A_{n_2} & \cdots & 0 & E_{2,n_{i+1}} & \cdots & E_{n_2,n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{1,1} & 0 & E_{1,n_i} & \cdots & J & E_{1,n_{i+1}} & \cdots & E_{1,n_s} \\ 0 & 0 & 0 & \cdots & \bar{A}_{n_i} & 0 & \cdots & 0 \\ E_{n_{i+1},1} & 0 & E_{n_{i+1},n_2} & \cdots & 0 & E_{n_{i+1},n_{i+1}} & \cdots & E_{n_{i+1},n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{n_s,1} & 0 & E_{n_s,n_2} & \cdots & 0 & E_{n_s,n_{i+1}} & \cdots & A_{n_s} \end{vmatrix}$$

$$= (-1)^{m_{i-1}-1} \begin{vmatrix} \bar{A}_{n_1} & 0 & \cdots & [I,0] & 0 & \cdots & 0 \\ 0 & A_{n_2} & \cdots & E_{n_2,n_i} & E_{n_2,n_{i+1}} & \cdots & E_{n_2,n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & E_{1,n_2} & \cdots & [E_{1,1}, J] & E_{1,n_{i+1}} & \cdots & E_{1,n_s} \\ 0 & 0 & \cdots & [0, \bar{A}_{n_i}] & 0 & \cdots & 0 \\ 0 & E_{n_{i+1},n_2} & \cdots & E_{n_{i+1},n_i} & E_{n_{i+1},n_{i+1}} & \cdots & E_{n_{i+1},n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & E_{n_s,n_2} & \cdots & E_{n_s,n_i} & E_{n_s,n_{i+1}} & \cdots & A_{n_s} \end{vmatrix}$$

$$= (-1)^{m_{i-1}-1}|\bar{A}_{n_1}| \begin{vmatrix} A_{n_2} & \cdots & E_{n_2,n_i} & E_{n_2,n_i+1} & \cdots & E_{n_2,n_s} \\ E_{n_3,n_2} & A_{n_3} & \cdots & E_{n_3,n_i+1} & \cdots & E_{n_3,n_s} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{n_i,n_2} & \cdots & \begin{bmatrix} E_{1,1} & J \\ 0 & \bar{A}_{n_i} \end{bmatrix} & E_{n_i,n_i+1} & \cdots & E_{n_i,n_s} \\ E_{n_i+1,n_1} & \cdots & E_{n_i+1,n_i} & A_{n_i+1} & \cdots & E_{n_i+1,n_s} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{n_s,n_1} & \cdots & E_{n_s,n_i} & E_{n_s,n_i+1} & \cdots & A_{n_s} \end{vmatrix}.$$

Therefore by induction hypothesis $X'_{1,m_{i-1}+1}$ is equal to

$$-|\bar{A}_{n_1}||\bar{A}_{n_i}| \begin{vmatrix} A_{n_2} & \bar{A}_{n_2} & \cdots & \bar{A}_{n_2} & \bar{A}_{n_2} & \cdots & \bar{A}_{n_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{A}_{n_{i-1}} & \bar{A}_{n_{i-1}} & \cdots & A_{n_{i-1}} & \bar{A}_{n_{i-1}} & \cdots & \bar{A}_{n_{i-1}} \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \bar{A}_{n_{i+1}} & \bar{A}_{n_{i+1}} & \cdots & \bar{A}_{n_{i+1}} & A_{n_{i+1}} & \cdots & \bar{A}_{n_{i+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{n_s} & \bar{A}_{n_s} & \cdots & \bar{A}_{n_s} & \bar{A}_{n_s} & \cdots & A_{n_s} \end{vmatrix}.$$

Thus if $H_i = [\beta_{r,t}]$, $2 \le i \le s$, is an square matrix of order $s-1$ such that

$$\beta_{r,t} = \begin{cases} |\bar{A}_{n_{r+1}}| & \text{if } r+1 = i \\ |A_{n_{r+1}}| & \text{if } r+1 \neq i \text{ and } r = t \\ |\bar{A}_{n_{r+1}}| & \text{if } r+1 \neq i \text{ and } r \neq t, \end{cases}$$

then we have

$$X'_{1,m_{i-1}+1} = -|\bar{A}_{n_1}||H_i|. \tag{3}$$

Hence by (1), (2), and (3) we have

$$\begin{aligned} \det(X) &= |F| \sum_{r=1}^{n_1} a_{1r}(A'_{n_1})_{1r} - |\bar{A}_{n_1}| \sum_{i=2}^{s} |H_i| \\ &= |A_{n_1}||F| - |\bar{A}_{n_1}| \sum_{i=2}^{s} |H_i| \\ &= \begin{vmatrix} |A_{n_1}| & |\bar{A}_{n_1}| & |\bar{A}_{n_1}| & \cdots & |\bar{A}_{n_1}| \\ |\bar{A}_{n_2}| & |A_{n_2}| & |\bar{A}_{n_2}| & \cdots & |\bar{A}_{n_2}| \\ |\bar{A}_{n_3}| & |\bar{A}_{n_3}| & |A_{n_3}| & \cdots & |\bar{A}_{n_3}| \\ \vdots & \vdots & \vdots & \vdots & \\ |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & |\bar{A}_{n_s}| & \cdots & |A_{n_s}| \end{vmatrix}. \end{aligned}$$

In the last equation the expansion of the determinant with respect to the first row from left to right is considered. This completes the proof. □

In order to compute the determinant obtained in above Lemma we need the following Lemma which is well-know (see for example [3]) but we state our proof here.

Lemma 2.2 Let F_n be the set of all permutations on the set $X_n := \{1, 2, 3, \ldots, n\}$ which have no fixed points. Then the difference between the number of odd permutations and the number of even permutations in F_n is $(-1)^{n-1}(n-1)$, that is $|\{\sigma \in F_n \mid \sigma \text{ is even}\}| - |\{\sigma \in F_n \mid \sigma \text{ is odd}\}| = (-1)^{n-1}(n-1)$.

Proof: We prove the Lemma by induction on n. Suppose that the Lemma is true for all positive integers which are smaller than n. Let $d(n) := |\{\sigma \in F_n \mid \sigma \text{ is even}\}| - |\{\sigma \in F_n \mid \sigma \text{ is odd}\}|$. Suppose that n is even. If $\sigma \in F_n$, then we can write σ as a product of disjoint cycles, such that n belongs to the last cycle. Now write each cycle in the decomposition of σ as a product of transpositions so that $\sigma = \tau(m, n)$, where $m \in X_{n-1}$ and τ is a permutation on X_{n-1}. We distinguish two cases:

(a) τ moves m and belongs to F_{n-1}.

(b) τ does not move m and moves all elements of $X_{n-1} - \{m\}$.

By induction hypothesis, $d(n-1) = (-1)^{n-2}(n-2) = 2-n$. Thus the difference between the number of even permutations of the form (a) and the number of odd permutations of the form (a) in F_n is $n-2$, that is if $T_m = \{\sigma \in F_n \mid \sigma = \tau(m, n)$ and τ satisfies (a), then

$$|\{\sigma \in T_m \mid \sigma \text{ is even }\}| - |\{\sigma \in T_m \mid \sigma \text{ is odd}\}| = n-2.$$

Now put $T = \{\sigma \in F_n \mid \sigma = \tau(m, n) \text{ and } \tau \text{ satisfies (a)}\}$. Then

$$|\{\sigma \in T \mid \sigma \text{ is even }\}| - |\{\sigma \in T \mid \sigma \text{ is odd}\}| = (n-1)(n-2).$$

By induction hypothesis $d(n-2) = (-1)^{n-3}(n-3) = n-3$. Thus the difference between the number of even permutations of the form (b) and the number of odd permutations of the form (b) in F_n is $3-n$, that is if $R_m = \{\sigma \in F_n \mid \sigma = \tau(m, n)$ and τ satisfies (b), then

$$|\{\sigma \in R_m \mid \sigma \text{ is even }\}| - |\{\sigma \in R_m \mid \sigma \text{ is odd}\}| = n-2.$$

Now put $R = \{\sigma \in F_n \mid \sigma = \tau(m, n) \text{ and } \tau \text{ satisfies (b)}\}$. Then

$$|\{\sigma \in R \mid \sigma \text{ is even }\}| - |\{\sigma \in R \mid \sigma \text{ is odd}\}| = (n-1)(n-2).$$

Therefore $d(n) = (n-1)(n-2) + (n-1)(3-n)) = n-1$ and the proof is complete, when n is even. If n is odd the argument is similar. □

In following lemma we expand the determinant founded in Lemma 2.1 using permutations in S_n. In fact in lemma below we find the determinant of an square matrix whose non-diagonal entries on each row are equal.

Lemma 2.3 Let r_i, r'_i, $i = 1, 2, \ldots, n$ be $2n$ arbitrary numbers. For any $2 \leq k \leq n$ we have $N_1, N_2, N_3, \ldots, N_p$ subsets with k elements of $\{1, 2, 3, \ldots, n\}$, where $p = \binom{n}{k}$. Define elements $x_{1j}^{(k)}$, $x_{2j}^{(k)}$, ..., $x_{nj}^{(k)}$, $1 \leq j \leq p$ as follows

$$x_{ij}^{(k)} = \begin{cases} r_i & \text{if } i \notin N_j \\ r'_i & \text{if } i \in N_j. \end{cases}$$

If

$$X = \begin{bmatrix} r_1 & r'_1 & r'_1 & \cdots & r'_1 \\ r'_2 & r_2 & r'_2 & \cdots & r'_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r'_n & r'_n & r'_n & \cdots & r_n \end{bmatrix},$$

then

$$\det(X) = r_1 r_2 r_3 \cdots r_n + \sum_{k=2}^{n} \sum_{j=1}^{\binom{n}{k}} (-1)^{k-1}(k-1) x_{1j}^{(k)} x_{2j}^{(k)} \cdots x_{nj}^{(k)}.$$

Proof: By definition of the determinant of square matrices, we have

$$\det(X) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)},$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ, which is 1 if σ is even and -1 if σ is odd, and $X_{i,j}$ is the (i,j)th entry of the matrix X. If $S_n[i]$ denotes the members of S_n which move i elements of $\{1, 2, 3, \ldots, n\}$, then we can write

$$\det(X) = X_{1,1} X_{2,2} \cdots X_{n,n} + \sum_{k=2}^{n} \sum_{\sigma \in S_n[k]} \operatorname{sgn}(\sigma) X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)}.$$

By definition of X, for $1 \leq i \leq n$, we have

$$X_{i,\sigma(i)} = \begin{cases} r'_i & \text{if } \sigma(i) \neq i \\ r_i & \text{if } \sigma(i) = i. \end{cases}$$

Now let S^* be the set of restrictions of permutations σ in $S_n[k]$ to the set of points moved by σ, that is $S^* = \{\sigma|_N : \sigma \in S_n[k],\ N = \{i : \sigma(i) \neq i\}\}$. Thus if $\sigma \in S_n[k]$ and $N = \{i : \sigma(i) \neq i\}$, then $X_{i,\sigma(i)} = r'_i$, for $i \in N$ and $X_{i,\sigma(i)} = r_i$ for $i \notin N$. Using the notation of Lemma 2.2 for S^*, we have $d(k) = (-1)^{k-1}(k-1)$. Thus

$$\sum_{\sigma \in S_n[k]} \operatorname{sgn}(\sigma) X_{1,\sigma(1)} X_{2,\sigma(2)} \cdots X_{n,\sigma(n)} = (-1)^{k-1}(k-1) \sum_{j=1}^{\binom{n}{k}} x_{1j}^{(k)} x_{2j}^{(k)} \cdots x_{nj}^{(k)}.$$

Therefore

$$\det(X) = r_1 r_2 r_3 \cdots r_n + \sum_{k=2}^{n} \sum_{j=1}^{\binom{n}{k}} (-1)^{k-1}(k-1) x_{1j}^{(k)} x_{2j}^{(k)} \cdots x_{nj}^{(k)}.$$

So the lemma is proved. □

Now suppose K_n is a complete graph of order n and $P(K_n)$ denote the characteristic polynomial of K_n, so $P(K_n) = -(x-(n-1))(x+1)^{n-1}$. Therefore the characteristic polynomial of K can be computed by using Lemmas 2.1 and 2.2.

Theorem 2.3 Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be positive integers and let P_i, $1 \le i \le n$, be the characteristic polynomial of K_{α_i}. For any $2 \le k \le n$ we have $N_1, N_2, N_3, \dots, N_p$ subsets with k elements of $\{1, 2, 3, \dots, n\}$, where $p = \binom{n}{k}$. Define $Q_{1j}^{(k)}$, $Q_{2j}^{(k)}$, …, $Q_{nj}^{(k)}$, $1 \le j \le p$ as follows

$$Q_{ij}^{(k)} = \begin{cases} P_i & \text{if } i \notin N_j \\ P_{i-1} & \text{if } i \in N_j. \end{cases}$$

Then the characteristic polynomial of $K = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ is given by

$$P_1 P_2 P_3 \cdots P_n + \sum_{k=2}^{n} \sum_{j=1}^{\binom{n}{k}} (-1)^{k-1}(k-1) Q_{1j}^{(k)} Q_{2j}^{(k)} \cdots Q_{nj}^{(k)}.$$

Proof: Suppose for $1 \le i \le n$, A_i denote adjacency matrix of complete graph K_{α_i} and $B_i = A_i - xI_{n,n}$. If $P_i = \det(B_i)$ denote the characteristic polynomial of K_{α_i}, then by Lemmas 2.1 and 2.3 we have

$$\begin{aligned} P(G) &= \begin{vmatrix} B_1 & E_{\alpha_1,\alpha_2} & E_{\alpha_1,\alpha_3} & \cdots & E_{\alpha_1,\alpha_n} \\ E_{\alpha_2,\alpha_1} & B_2 & E_{\alpha_2,\alpha_3} & \cdots & E_{\alpha_2,\alpha_n} \\ E_{\alpha_3,\alpha_1} & E_{\alpha_3,\alpha_2} & B_3 & \cdots & E_{\alpha_3,\alpha_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{\alpha_n,\alpha_1} & E_{\alpha_n,\alpha_2} & E_{\alpha_n,\alpha_3} & \cdots & B_n \end{vmatrix} \\ &= \begin{vmatrix} P_{\alpha_1} & P_{\alpha_1-1} & P_{\alpha_1-1} & \cdots & P_{\alpha_1-1} \\ P_{\alpha_2-1} & P_{\alpha_2} & P_{\alpha_2-1} & \cdots & P_{\alpha_2-1} \\ P_{\alpha_3-1} & P_{\alpha_3-1} & P_{\alpha_3} & \cdots & P_{\alpha_3-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{\alpha_n-1} & P_{\alpha_n-1} & P_{\alpha_n-1} & \cdots & P_{\alpha_n} \end{vmatrix} \\ &= P_1 P_2 P_3 \cdots P_n + \sum_{k=2}^{n} \sum_{j=1}^{\binom{n}{k}} (-1)^{k-1}(k-1) Q_{1j}^{(k)} Q_{2j}^{(k)} \cdots Q_{nj}^{(k)}. \end{aligned}$$

Therefore the Theorem is proved. □

Let us denote $K(\alpha_1, \alpha_2, \ldots, \alpha_n)$ by $K(\beta)$, where $\beta = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let $\beta_1 = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n_1})$, $\beta_2 = (\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n_2})$, ..., $\beta_{s_1} = (\alpha_{s_1 1}, \alpha_{s_1 2}, \ldots, \alpha_{s_1 n_{s_1}})$, where α_{ij}, $i = 1, 2, \ldots, s_1$, $j = 1, 2, \ldots, n_i$ are positive integers. Suppose that for all $i = 1, 2, \ldots, s_1$, there is a j such that $\alpha_{ij} = 1$, that is $K(\beta_i)^*$, which is a complete graph of order n_i, has a vertex of degree $n_i - 1$. We define a new graph $G^1 = K(\beta_1, \beta_2, \ldots, \beta_{s_1})$ as follows. Choose any vertex u_i, $i = 1, 2, \ldots, s_1$, of $K(\beta_i)^*$ of degree $n_i - 1$. The graph G^1 is obtained by adding the edges $u_i u_j$, where $1 \le i, j \le s_1$ to the union of $K(\beta_1)$, ..., $K(\beta_{s_1})$. We denote by $(G^1)^*$ the complete graph of order s_1 on the vertices $\{u_1, u_2, \ldots, u_{s_1}\}$. We write $G^1 = K(\gamma)$, where $\gamma = (\beta_1, \beta_2, \ldots, \beta_{s_1})$.

Now suppose we have vectors $\beta_{i1}, \beta_{i2}, \ldots, \beta_{i,m_i}$ of above type, $1 \le i \le s_2$. Put $\gamma_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{i,m_i})$, $1 \le i \le s_2$. We have defined the graphs $G_i^1 = K(\gamma_i)$. Suppose that $K(\gamma_i)^*$ has a vertex of degree $m_i - 1$ for all $i = 1, 2, \ldots, s_2$. We can define $G^2 = K(\gamma_1, \gamma_2, \ldots, \gamma_{s_2})$ as follows. Choose any vertex v_i, $i = 1, 2, \ldots, s_2$, from $K(\gamma_i)^*$ which have degree $m_i - 1$. Then add the edges $v_i v_j$, where $1 \le i, j \le s_2$ to the union of $K(\gamma_1), \ldots, K(\gamma_{s_2})$. We can continue this process (under certain conditions as above) to construct a graph G^k, for any positive integer k. (see Figure 1, with K_{α_i} replaced by G_i^1, the resulting graph is G^2.)

Suppose we have used vectors $\delta_1, \delta_2, \ldots, \delta_{s_k}$ for constructing the graph G^k. By suitable labelling we can determine the adjacency matrix of the graph G^k. For this purpose in labelling the vertices of the graph the vertices of $K(\delta_i)$ must have consecutive labels. If A_{n_i} is the adjacency matrix of graph $K(\delta_i)$, then the adjacency matrix of G^k is given by

$$\begin{bmatrix} A_{n_1} & E_{n_1,n_2} & E_{n_1,n_3} & \cdots & E_{n_1,n_{s_k}} \\ E_{n_2,n_1} & A_{n_2} & E_{n_2,n_3} & \cdots & E_{n_2,n_{s_k}} \\ E_{n_3,n_1} & E_{n_3,n_2} & A_{n_3} & \cdots & E_{n_3,n_{s_k}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{n_{s_k},n_1} & E_{n_{s_k},n_2} & E_{n_{s_k},n_3} & \cdots & A_{n_{s_k}} \end{bmatrix}.$$

Therefore characteristic polynomial of the graph G^k can be obtained by theorem 2.3. If $B_{n_i} = A_{n_i} - xI$ and $\bar{B}_{n_i}$ is the $(1,1)$ cofactor of B_{n_i}, then characteristic polynomial of G^k is

$$P(G^k) = \begin{vmatrix} |B_{n_1}| & |\bar{B}_{n_1}| & |\bar{B}_{n_1}| & \cdots & |\bar{B}_{n_1}| \\ |\bar{B}_{n_2}| & |B_{n_2}| & |\bar{B}_{n_2}| & \cdots & |\bar{B}_{n_2}| \\ |\bar{B}_{n_3}| & |\bar{B}_{n_3}| & |B_{n_3}| & \cdots & |\bar{B}_{n_3}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |\bar{B}_{n_{s_k}}| & |\bar{B}_{n_{s_k}}| & |\bar{B}_{n_{s_k}}| & \cdots & |B_{n_{s_k}}| \end{vmatrix}.$$

Now suppose that $\delta_i = (\delta_{i1}, \delta_{i2}, \ldots, \delta_{is_{k-1}})$, $i = 1, 2, \ldots, s_k$. Let $D_{n_{ir}}$ be the adjacency matrix of $G_i^{k-1} = K(\delta_{ir})$, $r = 1, 2, \ldots, s_{k-1}$. If $C_{n_{ir}} = D_{n_{ir}} - xI$, then

$$
\begin{aligned}
|B_{n_i}| &= \begin{vmatrix} C_{n_{i1}} & E_{n_{i1},n_{i2}} & \cdots & E_{n_{i1},n_{is_{k-1}}} \\ E_{n_{i2},n_{i1}} & C_{n_{i2}} & \cdots & E_{n_{i2},n_{is_{k-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n_{is_{k-1}},n_{i1}} & E_{n_{is_{k-1}},n_{i2}} & \cdots & C_{n_{is_{k-1}}} \end{vmatrix} \\
&= \begin{vmatrix} |C_{n_{i1}}| & |\bar{C}_{n_{i1}}| & |\bar{C}_{n_{i1}}| & \cdots & |\bar{C}_{n_{i1}}| \\ |\bar{C}_{n_{i2}}| & |C_{n_{i2}}| & |\bar{C}_{n_{i2}}| & \cdots & |\bar{C}_{n_{i2}}| \\ |\bar{C}_{n_{i3}}| & |\bar{C}_{n_{i3}}| & |C_{n_{i3}}| & \cdots & |\bar{C}_{n_{i3}}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |\bar{C}_{n_{is_{k-1}}}| & |\bar{C}_{n_{is_{k-1}}}| & |\bar{C}_{n_{is_{k-1}}}| & \cdots & |C_{n_{is_{k-1}}}| \end{vmatrix}.
\end{aligned}
$$

By definition of $\bar{B}_{n_i}$ we have

$$
\begin{aligned}
|\bar{B}_{n_i}| &= \begin{vmatrix} \bar{C}_{n_{i1}} & E_{n_{i1}-1,n_{i2}} & \cdots & E_{n_{i1}-1,n_{is_{k-1}}} \\ E_{n_{i2},n_{i1}-1} & C_{n_{i2}} & \cdots & E_{n_{i2},n_{is_{k-1}}} \\ \vdots & \vdots & \ddots & \vdots \\ E_{n_{is_{k-1}},n_{i1}-1} & E_{n_{is_{k-1}},n_{i2}} & \cdots & C_{n_{is_{k-1}}} \end{vmatrix} \\
&= \begin{vmatrix} |\bar{C}_{n_{i1}}| & |\overline{\bar{C}_{n_{i1}}}| & |\overline{\bar{C}_{n_{i1}}}| & \cdots & |\overline{\bar{C}_{n_{i1}}}| \\ |\bar{C}_{n_{i2}}| & |C_{n_{i2}}| & |\bar{C}_{n_{i2}}| & \cdots & |\bar{C}_{n_{i2}}| \\ |\bar{C}_{n_{i3}}| & |\bar{C}_{n_{i3}}| & |C_{n_{i3}}| & \cdots & |\bar{C}_{n_{i3}}| \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |\bar{C}_{n_{is_{k-1}}}| & |\bar{C}_{n_{is_{k-1}}}| & |\bar{C}_{n_{is_{k-1}}}| & \cdots & |\bar{C}_{n_{is_{k-1}}}| \end{vmatrix}.
\end{aligned}
$$

Similarly to compute $|C_i|$, $|\bar{C}_i|$ and $|\overline{\bar{C}_{n_{i1}}}|$ in above determinants we can consider the components of $G_i^{k-1} = K(\delta_{ir})$ which are graphs of type G^{k-2}. By performing this process k times we obtain determinants whose entries are characteristic polynomials of complete graphs (components of graphs of type G^1). Therefore we can compute the characteristic polynomial of G^k using Lemma 2.3.

3 Line graphs

In this section we introduce an application of operation on graphs mentioned above. For this purpose we show that line graph of a tree can be written as $K(\beta_1, \beta_2, \ldots, \beta_s)$, for some suitable vectors $\beta_1, \beta_2, \ldots, \beta_s$. Let

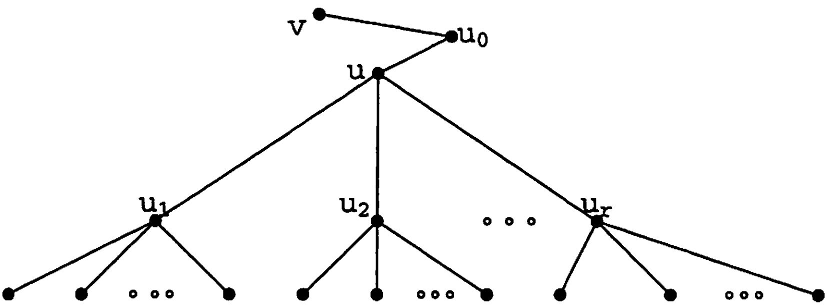

Figure 2: A subtree G_0 of a tree G, whose distances between its root and pendant vertices are equal to 2 (note that v is not a vertex of G_0).

$V(G)$ and $E(G)$ denote the set of vertices and edges of a graph G, respectively. Recall that the line graph of G, which is denote by $L(G)$, is a graph such that $V(L(G))$ is the set of edges of G and two vertices of $L(G)$ are adjacent if and only if corresponding edges in the graph G have a common vertex. Note that the line graph of an star graph is a complete graph.

Now we use induction on the number of vertices of a tree G to show that $G = K(\beta_1, \beta_2, \dots, \beta_s)$, for some vectors $\beta_1, \beta_2, \dots, \beta_s$. Let $u_1, u_2, \dots, u_r$ be all vertices which are adjacent to a same vertex u and all of their adjacent vertices are leaves. (see Figure 2). Thus u_i, $i = 1, 2, \dots, r$, are adjacent to pendant vertices of G. Let G_0 be the subtree whose edges are all edges which are adjacent to u and u_i, $i = 1, 2, \dots, r$. Put $d_i = \deg_{G_0}(u_i)$, $i = 1, 2, \dots, r$. For each $0 \leq i \leq r$ the subtree whose edges are all edges which are adjacent to u_i is star graph, and the line graph of this subtree is $K(d_i)$. Then clearly $L(G_0) = K(\beta)$, where $\beta = (1, d_1, \dots, d_r)$. Now let G_1 be the subtree of G obtained by deletion of G_0 from G. Then by induction hypothesis $L(G_1) = K(\beta_1, \dots, \beta_t)$, for some vectors $\beta_1, \dots, \beta_t$. Let x, y be vertices of $L(G_0)$ and $L(G_1)$ corresponding to the edges uu_0 and u_0v in G_0 and G_1, respectively. By adding edge xy as a bridge between $L(G_0)$ and $L(G_1)$, the line graph G is constructed as $L(G) = (\beta, \theta)$, where $\theta = (\beta_1, \dots, \beta_t)$ and the result follows.

Example 3.1 Let G be a tree with edges $\{1, 2, 3, \dots, 10\}$ as shown in Figure 3. The line graph of G is given by $L(G) = K((1, 1, 2), (1, 1, 1, 3))$. The subgraph G_1 of $L(G)$ induced by the vertices $\{1, 2, 3, 4\}$ is $K(1, 1, 2)$. In fact we have a complete graph on two vertices $\{3, 4\}$ and two complete graphs on a single vertex $\{1\}$, $\{2\}$, respectively. Then joining the vertices 3, 1, and 2 we obtain $G_1 = K(1, 1, 2)$. The subgraph G_2 of $L(G)$ induced by the vertices $\{5, 6, 7, 8, 9, 10\}$ is $K(1, 1, 1, 3)$. In fact we have a complete graph on three vertices $\{8, 9, 10\}$ and complete graphs on a single vertex

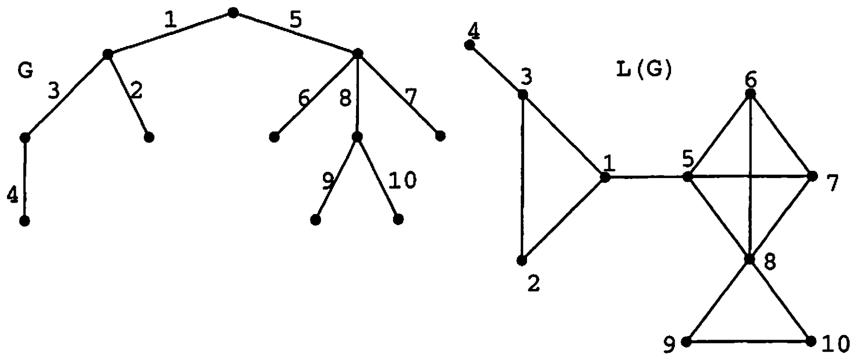

Figure 3: G is a tree with line graph $L(G)$

$\{5\}$, $\{6\}$, $\{7\}$, respectively. Then joining the vertices 5, 6, 7 and 8 we obtain $G_2 = K(1,1,1,3)$. Now joining vertices 1 and 2 in $G_1 \cup G_2$ we obtain the graph $L(G) = K((1,1,2),(1,1,1,3))$.

By Theorem 2.3 we can compute $P(L(G))$, the characteristic polynomial of $L(G)$. First we compute the characteristic polynomial of $G_1 = K(1,1,2)$ and $G_2 = K(1,1,1,3)$. We have $P_3 = P(K_3) = -(x+1)^2(x-2)$, $P_2 = P(K_2) = x^2 - 1$, $P_1 = P(K_1) = -x$ and $P_0 = P(K_0) = 1$. Thus

$$\begin{aligned}
P(L(G_1)) &= (P_1P_1P_2) - (P_0P_0P_2 + P_1P_0P_1 + P_0P_1P_1) + 2(P_0P_0P_1)\\
&= (-x)(-x)(x^2-1) - ((x^2-1) + 2(-x(-x)) + 2(-x)\\
&= x^4 - 4x^2 - 2x + 1\\
\bar{P}(L(G_1)) &= (P_1P_2) - (P_0P_1) = -x(x^2-1) - (-x) = x^3 + 2x
\end{aligned}$$

$$\begin{aligned}
P(L(G_2)) &= (P_1P_1P_1P_3) - (P_0P_0P_1P_3 + P_0P_1P_0P_3 + P_0P_1P_1P_2 +\\
&\qquad P_1P_0P_0P_3 + P_1P_0P_1P_2 + P_1P_1P_0P_2) +\\
&\qquad 2(P_0P_0P_0P_3 + P_1P_0P_0P_2 + P_0P_1P_0P_2 + P_0P_0P_1P_2) -\\
&\qquad\qquad 3(P_0P_0P_0P_2)\\
&= (-x)(-x)(-x)(-x^3+3x+2) - (3(-x)(-x^3+3x+2)) +\\
&\qquad 3(-x)(-x)(x^2-1)) + 2((-x^3+3x+2) +\\
&\qquad\qquad 3(-x(x^2-1))) - 3(x^2-1)\\
&= x^6 - 9x^4 - 10x^3 + 9x^2 + 18x + 7\\
\bar{P}(L(G_1)) &= (P_1P_1P_3) - (P_0P_0P_3 + P_1P_0P_2 + P_0P_1P_2) + 2(P_0P_0P_2)\\
&= (-x)(-x)(-x^3+3x+2) -\\
&\qquad ((-x^3+3x+2) + 2(-x(x^2-1)) + 2(x^2-1)\\
&= -x^5 + 6x^3 + 4x^2 - 5x - 4
\end{aligned}$$

Therefore characteristic polynomial of $L(G)$ is computed as:

$$P(L(G)) = P(G_1)P(G_2) - \bar{P}(G_1)\bar{P}(G_2)$$
$$= x^{10} - 14x^8 - 12x^7 + 54x^6 + 80x^5 - 35x^4 - 112x^3 - 45x^2 + 12x + 7$$

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