HYPERBOLIC ELLIPSES AND MÖBIUS TRANSFORMATIONS

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ABSTRACT. In this paper, we determine the images of hyperbolic ellipses under the Möbius and harmonic Möbius transformations.

1. INTRODUCTION

In [4] and [5], Coffman and Frantz determined the images of non-circular ellipses under the Möbius transformations. They proved that the only Möbius transformations which take ellipses to ellipses are the similarity transformations (see [1] for more details about Möbius transformations).

In [2], Chuaqui, Duren and Osgood introduced the harmonic Möbius transformations as a generalization of Möbius transformations to harmonic mappings. A harmonic Möbius transformation is a harmonic mapping of the form

$$f = h + \alpha\overline{h}, \tag{1.1}$$

where h is a Möbius transformation and α is a complex constant with $|\alpha| < 1$. A harmonic Möbius transformation $f = h + \alpha\overline{h}$ is the composition of the Möbius transformation h with the linear map $z \rightarrow z + \alpha\overline{z}$ (see [2] and [3] for more details).

In [6], N. Yılmaz Özgür determined the images of non-circular ellipses under the harmonic Möbius transformations and proved that the only harmonic Möbius transformations which take ellipses to ellipses are the harmonic similarity transformations of the form $f = h + \alpha\overline{h}$ where h is a similarity transformation.

Motivated by the above studies, we determine the images of hyperbolic ellipses under the Möbius and harmonic Möbius transformations. We use the Poincare disk model for the hyperbolic plane. The Poincare disk model is $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (see [7] for more details). We obtain that the images of hyperbolic ellipses under the Möbius and harmonic Möbius transformations are hyperbolic ellipses.

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2. Hyperbolic Ellipses and Möbius Transformations

Recall that the distance function from the origin to the point z is

$$d_D(z,0) = \ln\left(\frac{1+|z|}{1-|z|}\right),$$

(see [8]). If z and w be any two points in D, then the distance function between these points is the following:

$$d_D(z,w) = \ln\left(\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}\right).$$

A *hyperbolic ellipse* E may be defined as the set of all points P in $\mathbb{D}$, the sum of whose distances from two fixed points, F_1 and F_2, is constant. In other words, if F_1 and F_2 are fixed points in $\mathbb{D}$, then a hyperbolic ellipse is the collection of points

$$\{P : d_D(P,F_1) + d_D(P,F_2) = 2a\},$$

where a is a constant. The fixed points F_1 and F_2 are called the foci of the ellipse. The equation of the hyperbolic ellipse E can be written as

$$\begin{aligned}
&d_D(P,F_1) + d_D(P,F_2) = 2a \\
&\Rightarrow \ln\left(\frac{|1-z\overline{\mu}|+|z-\mu|}{|1-z\overline{\mu}|-|z-\mu|}\right) + \ln\left(\frac{|1-z(-\overline{\mu})|+|z-(-\mu)|}{|1-z(-\overline{\mu})|-|z-(-\mu)|}\right) = 2a \\
&\Rightarrow \ln\left(\frac{|1-z\overline{\mu}|+|z-\mu|}{|1-z\overline{\mu}|-|z-\mu|}\right) + \ln\left(\frac{|1+z\overline{\mu}|+|z+\mu|}{|1+z\overline{\mu}|-|z+\mu|}\right) = 2a \\
&\Rightarrow \frac{[|1-z\overline{\mu}|+|z-\mu|]\,[|1+z\overline{\mu}|+|z+\mu|]}{[|1-z\overline{\mu}|-|z-\mu|]\,[|1+z\overline{\mu}|-|z+\mu|]} = e^{2a}. \qquad (2.1)
\end{aligned}$$

A Möbius transformation T has the form

$$T(z) = \frac{az+b}{cz+d}; \quad a,b,c,d \in \mathbb{C} \text{ and } ad-bc \neq 0. \qquad (2.2)$$

The set of all Möbius transformations is a group under composition. The Möbius transformations with $c=0$ form the subgroup of *similarities.* Such transformations have the form

$$S(z) = Az + B; \quad A,B \in \mathbb{C},\ A \neq 0. \qquad (2.3)$$

The transformation $J(z) = \dfrac{1}{z}$ is called an *inversion.* Every Möbius transformation T of the form (2.2) is a composition of finitely many similarities and inversions (see [1]).

Now we can give the following lemmas.

Lemma 2.1. *The image of any hyperbolic ellipse under a similarity transformation $S(z) = Az+B$; $A,B \in \mathbb{C}$, $A \neq 0$ is a hyperbolic ellipse.*

Proof. If we apply the linear maps $z_1 = Az + B$, $\mu_1 = A\mu + B$ to the equation of the hyperbolic ellipse defined by (2.1), then we have

$$\frac{\left[\left|1-(Az+B)\left(\overline{A\mu+B}\right)\right|+|A|\,|z-\mu|\right]}{\left[\left|1-(Az+B)\left(\overline{A\mu+B}\right)\right|-|A|\,|z-\mu|\right]} \times \frac{\left[\left|1+(Az+B)\left(\overline{A\mu+B}\right)\right|+|A(z+\mu)+2B|\right]}{\left[\left|1+(Az+B)\left(\overline{A\mu+B}\right)\right|-|A(z+\mu)+2B|\right]} = e^{2a}$$

and so

$$\frac{\left[\left|1-|B|^2-\left(|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}\right)z\overline{\mu}\right|+|A|\,|z-\mu|\right]}{\left[\left|1-|B|^2-\left(|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}\right)z\overline{\mu}\right|-|A|\,|z-\mu|\right]} \times \frac{\left[\left|1+|B|^2+\left(|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}\right)z\overline{\mu}\right|+|A(z+\mu)+2B|\right]}{\left[\left|1+|B|^2+\left(|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}\right)z\overline{\mu}\right|-|A(z+\mu)+2B|\right]} = e^{2a}.$$

If $B = 0$, then we get

$$\frac{\left[|1-z_1\overline{\mu_1}|+|z_1-\mu_1|\right]\left[|1+z_1\overline{\mu_1}|+|z_1+\mu_1|\right]}{\left[|1-z_1\overline{\mu_1}|-|z_1-\mu_1|\right]\left[|1+z_1\overline{\mu_1}|-|z_1+\mu_1|\right]} = e^{2a},$$

where $|A|\,z = z_1$ and $|A|\,\mu = \mu_1$. This last equation is the equation of a hyperbolic ellipse. In the other cases we obtain

$$\frac{\left[\left|1-\left(\frac{|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}}{1-|B|^2}\right)z\overline{\mu}\right|+|A|\,|z-\mu|\right]}{\left[\left|1-\left(\frac{|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}}{1-|B|^2}\right)z\overline{\mu}\right|-|A|\,|z-\mu|\right]} \times \frac{\left[\left|1+\left(\frac{|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}}{1+|B|^2}\right)z\overline{\mu}\right|+|A(z+\mu)+2B|\right]}{\left[\left|1+\left(\frac{|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}}{1+|B|^2}\right)z\overline{\mu}\right|-|A(z+\mu)+2B|\right]} = e^{2a},$$

where $(Az+B)\left(\overline{A\mu+B}\right) = \dfrac{|A|^2+\frac{A\overline{B}}{\overline{\mu}}+\frac{\overline{A}B}{z}}{1-|B|^2}$. If we put $z_1' = (Az+B)$ and $\mu_1' = A\mu+B$ the above equation shows the equation of a hyperbolic ellipse. □

Lemma 2.2. *The inversion map* $z \longrightarrow \dfrac{1}{z}$ *takes hyperbolic ellipses to hyperbolic ellipses in* $\mathbb{D}$.

Proof. The image of a hyperbolic ellipse with equation (2.1) under the inversion maps $z \longrightarrow \dfrac{1}{z}$ and $\mu \longrightarrow \dfrac{1}{\mu}$ is a hyperbolic ellipse. Indeed we have

$$\frac{\left[\left|1-\frac{1}{z\overline{\mu}}\right|+\left|\frac{1}{z}-\frac{1}{\mu}\right|\right]\left[\left|1+\frac{1}{z\overline{\mu}}\right|+\left|\frac{1}{z}+\frac{1}{\mu}\right|\right]}{\left[\left|1-\frac{1}{z\overline{\mu}}\right|-\left|\frac{1}{z}-\frac{1}{\mu}\right|\right]\left[\left|1+\frac{1}{z\overline{\mu}}\right|-\left|\frac{1}{z}+\frac{1}{\mu}\right|\right]}=e^{2a}$$

and hence

$$\frac{\left[\left|1-|(z\mu)|^{-2}\overline{z}\mu\right|+|z-\mu|\,|(z\mu)|^{-1}\right]\left[\left|1+|(z\mu)|^{-2}\overline{z}\mu\right|+|z+\mu|\,|(z\mu)|^{-1}\right]}{\left[\left|1-|(z\mu)|^{-2}\overline{z}\mu\right|-|z-\mu|\,|(z\mu)|^{-1}\right]\left[\left|1+|(z\mu)|^{-2}\overline{z}\mu\right|-|z+\mu|\,|(z\mu)|^{-1}\right]}=e^{2a}.$$

Let $|z\mu|\,z=z_2$ and $|z\mu|\,\mu=\mu_2$. Then we can write

$$\frac{[|1-\overline{z_2}\mu_2|+|z_2-\mu_2|]\,[|1+\overline{z_2}\mu_2|+|z_2+\mu_2|]}{\left[|1-\overline{z_2}\mu_2|-|z_2-\mu_2|^{-1}\right][|1+\overline{z_2}\mu_2|-|z_2+\mu_2|]}=e^{2a}.$$

□

Lemma 2.3. *The linear map $z \longrightarrow z+\alpha\overline{z}$ takes hyperbolic ellipses to hyperbolic ellipses in $\mathbb{D}$.*

Proof. First, if we apply the linear maps $z \longrightarrow \alpha\overline{z}$, $\mu \longrightarrow \alpha\overline{\mu}$ to the equation of the hyperbolic ellipse defined by (2.1) then we have

$$\frac{[|1-\alpha\overline{z}\overline{\alpha}\mu|+|\alpha\overline{z}-\alpha\overline{\mu}|]\,[|1+\alpha\overline{z}\overline{\alpha}\mu|+|\alpha\overline{z}+\alpha\overline{\mu}|]}{[|1-\alpha\overline{z}\overline{\alpha}\mu|-|\alpha\overline{z}-\alpha\overline{\mu}|]\,[|1+\alpha\overline{z}\overline{\alpha}\mu|-|\alpha\overline{z}+\alpha\overline{\mu}|]}=e^{2a}$$

and so

$$\frac{\left[\left|1-|\alpha|^2\overline{z}\mu\right|+|\alpha|\,|\overline{z}-\overline{\mu}|\right]\left[\left|1+|\alpha|^2\overline{z}\mu\right|+|\alpha|\,|\overline{z}+\overline{\mu}|\right]}{\left[\left|1-|\alpha|^2\overline{z}\mu\right|-|\alpha|\,|\overline{z}-\overline{\mu}|\right]\left[\left|1+|\alpha|^2\overline{z}\mu\right|-|\alpha|\,|\overline{z}+\overline{\mu}|\right]}=e^{2a}.$$

If we put $|\alpha|\,z=z_3$ and $|\alpha|\,\mu=\mu_3$, then the above equation becomes

$$\frac{[|1-\overline{z_3}\mu_3|+|z_3-\mu_3|]\,[|1+\overline{z_3}\mu_3|+|z_3+\mu_3|]}{\left[|1-\overline{z_3}\mu_3|-|z_3-\mu_3|^{-1}\right][|1+\overline{z_3}\mu_3|-|z_3+\mu_3|]}=e^{2a}$$

and this is the equation of a hyperbolic ellipse.

This linear map is the same as similarity transformation. Since any hyperbolic ellipse have been transformed to a hyperbolic ellipse under similarity transformation, the linear map $z \longrightarrow z+\alpha\overline{z}$ takes hyperbolic ellipses to hyperbolic ellipses in $\mathbb{D}$. □

Now we can give the following theorems.

Theorem 2.1. *The images of hyperbolic ellipses under the Möbius transformations are hyperbolic ellipses.*

Proof. The proof follows by Lemmas 2.1 and 2.2 using the fact that every Möbius transformation T of the form (2.2) is a composition of finitely many similarities and inversions. $\square$

Theorem 2.2. *The images of hyperbolic ellipses under the harmonic Möbius transformations are hyperbolic ellipses.*

Proof. The proof follows by Lemma 2.3 and Theorem 2.1 using the fact that every harmonic Möbius transformation f of the form (1.1) is a composition of a Möbius transformation of the form (2.2) with the linear map $z \longrightarrow z + \alpha\overline{z}$. $\square$

Example 2.1. $z = x + iy$, $\mu = \frac{1}{4} + \frac{i}{4}$. *The images of this hyperbolic ellipse under the harmonic Möbius transformations* $f_1(\mu) = \mu - \frac{i}{3}\overline{\mu}$ *and* $f_2(\mu) = \frac{-1}{\mu + \frac{1}{40}} + \frac{1}{40}\frac{-1}{\overline{\mu} + \frac{1}{40}}$ *are other hyperbolic ellipses.*

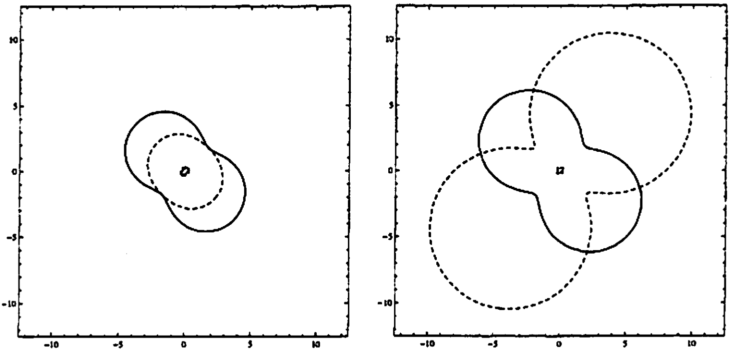

FIGURE 1. In the first and the second graphs, we see the original ellipse and its image under the transformations f_1 and f_2, respectively. Here, we have $\mu = 1/4 + i/4$ and $a = 0.995$. In both of the graphics, the solid ellipses are the original ones and the dashed ones are the transformed ellipses.

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