A NOTE ON OFF-DIAGONAL SMALL ON-LINE RAMSEY NUMBERS FOR PATHS

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ABSTRACT. In this note we consider the on-line Ramsey numbers $\overline{\mathcal{R}}(P_n, P_m)$ for paths. Using a high performance computing clusters, we calculated the values for off-diagonal numbers for paths of lengths at most 8. Also, we were able to check that $\overline{\mathcal{R}}(P_9, P_9) = 17$, thus solving the problem raised in [5].

1. INTRODUCTION, DEFINITIONS, AND MAIN RESULTS

In this paper, we consider the on-line Ramsey numbers introduced by Kurek and Ruciński [4] and corresponding to them the on-line Ramsey game. (The game was considered earlier by Beck [1] but not in terms of the numbers; Friedgut et al. [2] also studied a variant of this game but in the context of the random graph theory.) Let G, H be a fixed graphs. The game between two players, called the Builder and the Painter, is played on an unbounded set of vertices. In each of her moves the Builder draws a new edge which is immediately coloured red or blue by the Painter. The goal of the Builder is to force the Painter to create a red copy of G or a blue copy of H; the goal of the Painter is the opposite, he is trying to avoid it for as long as possible. The payoff to the Painter is the number of moves until this happens. The Painter seeks the highest possible payoff. Since this is a two-person, full information game with no ties, one of the players must have a winning strategy. *The on-line Ramsey number* $\overline{\mathcal{R}}(G, H)$ is the smallest payoff over all possible

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strategies of the Builder, assuming the Painter uses an optimal strategy.

Similar to the classical Ramsey numbers (see a dynamic survey of Radziszowski [8] which includes all known nontrivial values and bounds for Ramsey numbers), it is hard to compute the exact value of $\overline{\mathcal{R}}(G,H)$ unless G, H are trivial. In this relatively new area of small on-line Ramsey numbers, very little is known.

Kurek and Ruciński considered in [4] the most interesting case where G and H are cliques, but besides the trivial $\overline{\mathcal{R}}(K_2,K_k) = \binom{k}{2}$, they were able to determine only one more value, namely $\overline{\mathcal{R}}(K_3,K_3) = 8$ (the upper bound can be shown by mimicking the proof of the upper bound for classical Ramsey number $R(K_3,K_3)$; the proof of the lower bound is elegant and definitely nontrivial). In [4], it has been shown that

$$\overline{\mathcal{R}}(K_k,K_k) \leq 2k\binom{2k-2}{k-1} \sim \frac{1}{2\sqrt{\pi}}\sqrt{k}4^k.$$

The author of this note, with computer support, showed that $\overline{\mathcal{R}}(K_3,K_4) = 17$, provided a general upper bound for $\overline{\mathcal{R}}(K_k,K_l)$, which gives a slightly better asymptotic upper bound of $\frac{3}{8\sqrt{\pi}}\frac{4^k}{\sqrt{k}}$ for diagonal numbers (see [6] for more details).

It is obvious that the real challenge is in computing the on-line Ramsey numbers for cliques, but surprisingly even paths, which can be considered as the simplest graphs to try, are difficult to study. Grytczuk et al. [3], dealing with many labourious subcases, determined the on-line Ramsey numbers for a few symmetric short paths ($\overline{\mathcal{R}}(P_2,P_2) = 1$, $\overline{\mathcal{R}}(P_3,P_3) = 3$, $\overline{\mathcal{R}}(P_4,P_4) = 5$, $\overline{\mathcal{R}}(P_5,P_5) = 7$, $\overline{\mathcal{R}}(P_6,P_6) = 10$). It is clear that $\overline{\mathcal{R}}(P_n,P_m) \geq n+m-3$ for $n,m \geq 2$ since the Painter may color safely the first $n-2$ edges red, and the next $m-2$ edges blue. Also it is not hard to prove that $\overline{\mathcal{R}}(P_n,P_m) \leq 2(n+m)-7$ for $n,m \geq 2$ (see [3] for more details) but it seems that determining the exact values for longer paths requires computer support. The author of this paper was able to determine some new values, namely $\overline{\mathcal{R}}(P_7,P_7) = 12$, $\overline{\mathcal{R}}(P_8,P_8) = 15$, and $\overline{\mathcal{R}}(P_9,P_9) \leq 17$ (see [5] for more details).

In this note, we determine missing values for off-diagonal on-line Ramsey numbers for paths of lengths at most 8 and we show that $\overline{\mathcal{R}}(P_9, P_9) = 17$, confirming the Conjecture 4.2 [5]. The results are presented in Table 1.

	2	3	4	5	6	7	8	9
2	1 [3]							
3	2	3 [3]						
4	3	4	5 [3]					
5	4	5	6	7 [3]				
6	5	7	8	9	10 [3]			
7	6	8	9	10	11	12 [5]		
8	7	9	11	12	13	14	15 [5]	
9	8	10	12	13	14	15	16	17

TABLE 1. Values of $\overline{\mathcal{R}}(P_n, P_m)$

For a few small numbers, we provide proofs in Section 2, but for larger values we have to be content with computer computations described in Section 3.

2. THEORETICAL RESULTS

As we already mentioned in the Introduction, $\overline{\mathcal{R}}(P_n, P_m) \geq n + m - 3$, for $n, m \geq 2$. This lower bound is clearly attained for $\overline{\mathcal{R}}(P_2, P_m)$ ($m \geq 2$), but also for $\overline{\mathcal{R}}(P_3, P_4)$, $\overline{\mathcal{R}}(P_3, P_5)$, and $\overline{\mathcal{R}}(P_4, P_5)$.

Proposition 2.1. $\overline{\mathcal{R}}(P_3, P_4) = 4$, $\overline{\mathcal{R}}(P_3, P_5) = 5$, and $\overline{\mathcal{R}}(P_4, P_5) = 6$.

Proof. Let us start with a proof that $\overline{\mathcal{R}}(P_3, P_4) = 4$. After presenting two edges of a path P_3, there are only two possible patterns (up to symmetry): *rb* and *bb*. Then the Builder creates a red path P_3 or a blue path P_4 in the next two moves, as depicted in Figure 1. (The final edge is drawn in two colours.)

In order to show that $\overline{\mathcal{R}}(P_3, P_5) = 5$ we consider the following strategy of the Builder: present two edges of P_3 and then extend

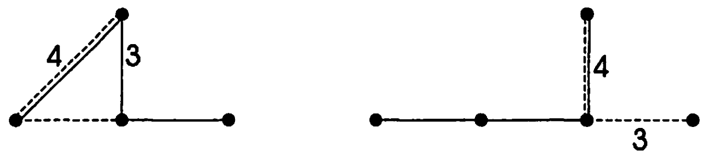

FIGURE 1. Forcing red P_3 or blue P_4

the path by adding an edge to vertex of degree 1. If a red colour has been used in the first two moves, then next edge is incident to the red one (that is, the Painter is forced to use blue to colour this edge). Thus, there are only three possible patterns that can appear after first three rounds: *bbb*, *bbr*, and *brb*. The Builder now is able to finish the game in the next two moves, as shown in Figure 2.

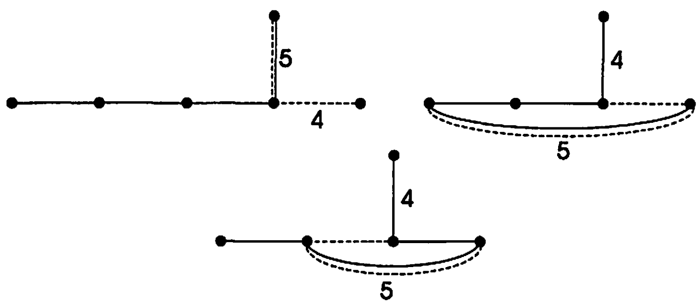

FIGURE 2. Forcing red P_3 or blue P_5

To prove that $\overline{\mathcal{R}}(P_4, P_5) = 6$ one have to analyze more subcases. Similarly as before, the Builder shows a path P_4 in the first three steps but she has to avoid the pattern *rbr* (otherwise, the Painter has a strategy to 'survive' to the end of sixth round). In order to do that, the Builder can use the same strategy as for the $\overline{\mathcal{R}}(P_3, P_5)$ case. Therefore, essentially one of the four possible color patterns appears: *bbb*, *bbr*, *brb*, and *rrb*. Then she obtains a red P_4 or a blue P_5 in the next three moves, as shown in Figure 3. (A circled number means that the Painter had a choice in that move, which led to a branching into subcases.)

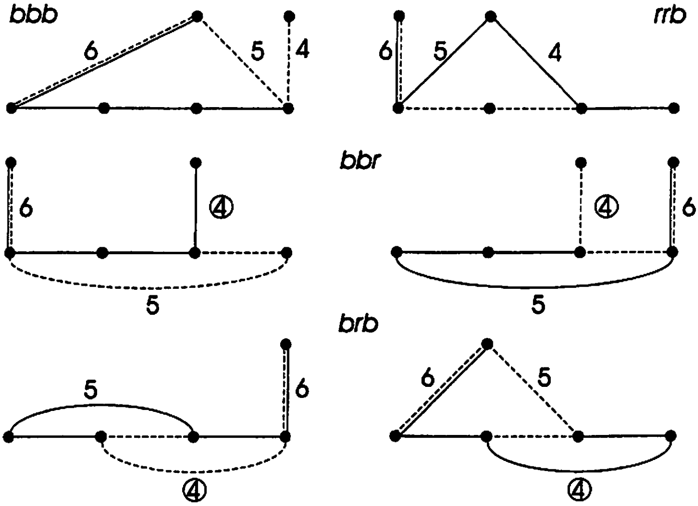

FIGURE 3. Forcing red P_4 or blue P_5

□

Our next on-line Ramsey number that we are going to establish is larger than the trivial lower bound we used so far.

Proposition 2.2. $\overline{\mathcal{R}}(P_3, P_6) = 7$.

Proof. For the lower bound, consider a natural strategy for the Painter: colour an edge red if it does not create a red copy of P_3, otherwise use blue. The game is finished when a blue copy of P_6 is created. The first edge is coloured red and if the Painter is able to use the color red in one of the next moves, we are done. Thus, the only way for the Builder to finish the game in the total of six rounds is to create in the next five rounds a blue P_6. This is possible only by using both ends of the red edge, making the last winning move impossible.

For the upper bound, suppose that the first edge is coloured red. Then the Builder can force the Painter to create a blue P_5 in the next four moves, as shown in Figure 4. Next edge extending the blue path must be coloured red but the Builder can finish a game in the very next move.

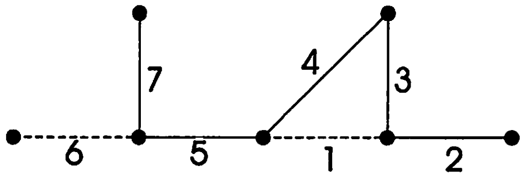

FIGURE 4. Forcing red P_3 or blue P_6

If the first edge is coloured blue, the Builder can continue extending a path until red is used and similar winning strategy can be applied. (It is even possible to prove that the Builder can win in the next five moves, that is, the Painter must use red in the first round, assuming he uses an optimal strategy.) □

3. COMPUTER COMPUTATIONS

We implemented and ran programs written in C/C++ using backtracking algorithms. (The programs can be downloaded from [7].) Backtracking is a refinement of the brute force approach, which systematically searches for a solution to a problem among all available options. Since it is not possible to examine all possibilities, we used many advanced validity criteria to determine which portion of the solution space needed to be searched. For example, one can look at the coloured graph in every round and try to estimate the number of red (and blue) edges needed to create desired structure. This knowledge can be used to avoid considering the whole branch in the searching tree. If the Painter can use red colour and 'survive' additional k rounds, then there is no point to check whether using blue colour forces him to finish the game earlier.

Using a set of clusters (see Section 4 for more details), we were able to run (independently) the program from different initial graphs with given colouring of edges. In the table below we present the numbers of nonisomorphic coloured graphs with k edges that have been found by computer. Since the game we play is nonsymmetric we have to consider more initial graphs than in the symmetric version (see [5] where the symmetric game for paths was considered). If the number of edges is odd, we have exactly two times more graphs to consider. For the even case, this number is a little bit smaller than double.

k	# of symmetric graphs	# of nonsymmetric graphs
1	1	2
2	4	6
3	12	24
4	51	93
5	203	406
6	1,004	1,959
7	5,117	10,234
8	29,153	58,013
9	176,778	353,556
10	1,150,164	2,298,303

TABLE 2. Number of nonisomorphic coloured graphs with k edges

Having results from computer computations starting from different initial graphs (even partial ones!) we are able to determine the exact value of the on-line Ramsey numbers. The relations between the partial results in different levels are complicated but can be found using a computer. The relations between levels 1 – 2, and 2 – 3 are described below. For simplicity, we present the symmetric case; the nonsymmetric one is studied in the same way.

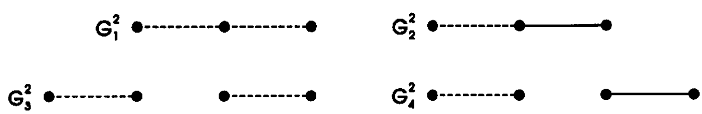

FIGURE 5. Coloured graphs with two edges

There is only one possible coloured graph G_1^1 with one edge (up to isomorphism). Graphs with two and three edges are presented in Figure 5 and Figure 6, respectively. Let x_i^m =

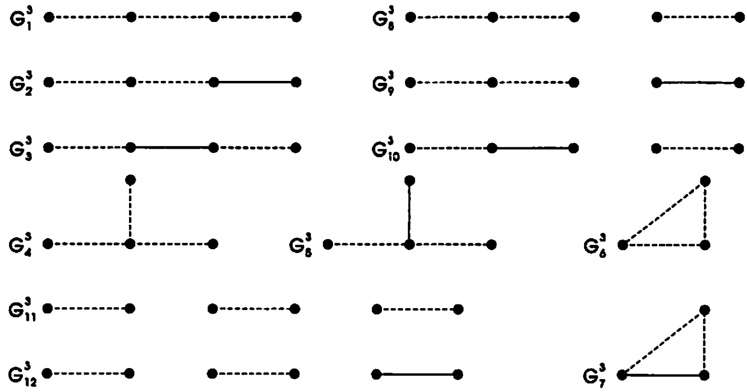

FIGURE 6. Coloured graphs with three edges

$x_i^m(G_i^m, k, l)$ denote the number of moves in a winning strategy of the Builder in the on-line Ramsey game, provided that after m moves a coloured graph is isomorphic to G_i^m. Using the notation

$$\begin{aligned} x_1 \vee x_2 &= \max\{x_1, x_2\} \\ x_1 \wedge x_2 \wedge \cdots \wedge x_k &= \min\{x_1, x_2, \ldots, x_k\}, \end{aligned}$$

it is not hard to see that

$$x_1^1 = (x_1^2 \vee x_2^2) \wedge (x_3^2 \vee x_4^2),$$

and

$$\begin{aligned} x_1^2 &= (x_1^3 \vee x_2^3) \wedge (x_8^3 \vee x_9^3) \wedge (x_4^3 \vee x_5^3) \wedge (x_6^3 \vee x_7^3) \\ x_2^2 &= (x_3^3 \vee x_2^3) \wedge x_{10}^3 \wedge x_5^3 \wedge x_7^3 \\ x_3^2 &= (x_1^3 \vee x_3^3) \wedge (x_8^3 \vee x_{10}^3) \wedge (x_{11}^3 \vee x_{12}^3) \\ x_4^2 &= x_2^3 \wedge (x_9^3 \vee x_{10}^3) \wedge x_{12}^3 . \end{aligned}$$

Each "$\vee$" sign corresponds to the Painter's move, "$\wedge$" corresponds to the Builder's one. He tries to play as long as possible, choosing the maximum value, but she would like to win as soon as possible.

We describe the approach to determine the value of $\overline{\mathcal{R}}(P_9, P_9)$ with a little bit more details below. The values for other cases are 'calculated' the same way and thus we present the results of computer computations in Tables 4 – 8 only.

Proposition 3.1. $\overline{\mathcal{R}}(P_9, P_9) = 17$

Proof. It follows from Theorem 2.3 [5] that $\overline{\mathcal{R}}(P_9, P_9) \leq 17$. In order to show that $\overline{\mathcal{R}}(P_9, P_9) > 16$ we examined 1, 150, 164 initial configurations with 10 edges. Exactly 1, 352 graphs contain a monochromatic P_9 so we put $x_i^{10} \leq 10$ for these graphs. For the rest, we run the program to check whether $x_i^{10} \leq 16$. The results are presented below.

	# of initial configurations
$x_i^{10} \leq 10$	1, 352
$11 \leq x_i^{10} \leq 16$	47, 011
$17 \leq x_i^{10} < \infty$	1, 101, 801
total	1, 150, 164

TABLE 3. Results for a game for symmetric paths of length 8

Next we verified that the Painter has a strategy to reach one of the 'good' configurations that allow him to survive the next six moves. □

$\overline{\mathcal{R}}(P_4, P_6) = 8$	
$x_i^1 = 8$	2
total	2

$\overline{\mathcal{R}}(P_5, P_6) = 9$	
$x_i^1 = 9$	2
total	2

$\overline{\mathcal{R}}(P_3, P_7) = 8$	
$x_i^1 = 8$	2
total	2

$\overline{\mathcal{R}}(P_4, P_7) = 9$	
$x_i^1 = 9$	2
total	2

TABLE 4. Results of computer computations I

4. ACKNOWLEDGMENT

This work was made possible by the facilities of

- the Shared Hierarchical Academic Research Computing Network SHARCNET (www.sharcnet.ca): 8,082 CPUs,

$\overline{\mathcal{R}}(P_5, P_7) = 10$	
$x_i^5 \leq 5$	16
$x_i^5 = 9$	36
$x_i^5 = 10$	255
$x_i^5 \geq 11$	99
total	406

$\overline{\mathcal{R}}(P_6, P_7) = 11$	
$x_i^6 \leq 6$	21
$x_i^6 = 10$	82
$x_i^6 = 11$	1,107
$x_i^6 \geq 12$	749
total	1,959

$\overline{\mathcal{R}}(P_3, P_8) = 9$	
$x_i^1 = 9$	2
total	2

TABLE 5. Results of computer computations II

$\overline{\mathcal{R}}(P_4, P_8) = 11$	
$x_i^5 \leq 5$	76
$x_i^5 = 9$	23
$x_i^5 = 10$	140
$x_i^5 = 11$	141
$x_i^5 \geq 12$	26
total	406

$\overline{\mathcal{R}}(P_5, P_8) = 12$	
$x_i^7 \leq 7$	1385
$x_i^7 = 10$	269
$x_i^7 = 11$	2,737
$x_i^7 = 12$	4,272
$x_i^7 \geq 13$	1,571
total	10,234

$\overline{\mathcal{R}}(P_6, P_8) = 13$	
$x_i^7 \leq 7$	283
$x_i^7 = 11$	162
$x_i^7 = 12$	3,358
$x_i^7 = 13$	4,823
$x_i^7 \geq 13$	156
$x_i^7 \geq 14$	1,452
total	10,234

TABLE 6. Results of computer computations III

$\overline{\mathcal{R}}(P_7, P_8) = 14$	
$x_i^8 \leq 8$	546
$x_i^8 = 12$	422
$x_i^8 = 13$	13,967
$x_i^8 = 14$	28,002
$x_i^8 \geq 14$	5,581
$x_i^8 \geq 15$	9,495
total	58,013

$\overline{\mathcal{R}}(P_3, P_9) = 10$	
$x_i^1 = 10$	2
total	2

$\overline{\mathcal{R}}(P_4, P_9) = 12$	
$x_i^8 \leq 8$	26,048
$x_i^8 = 10$	260
$x_i^8 = 11$	3,596
$x_i^8 = 12$	13,347
$x_i^8 \geq 13$	14,762
total	58,013

TABLE 7. Results of computer computations IV

- the Atlantic Computational Excellence Network ACEnet (www.ace-net.ca): 480 CPUs,

$\overline{\mathcal{R}}(P_5, P_9) = 13$	
$x_i^8 \leq 8$	11,450
$x_i^8 = 11$	459
$x_i^8 = 12$	6,868
$x_i^8 = 13$	21,234
$x_i^8 \geq 14$	18,002
total	58,013

$\overline{\mathcal{R}}(P_6, P_9) = 14$	
$x_i^8 \leq 8$	3,213
$x_i^8 = 12$	309
$x_i^8 = 13$	8,818
$x_i^8 = 14$	21,062
$x_i^8 \geq 14$	10,641
$x_i^8 \geq 15$	13,970
total	58,013

$\overline{\mathcal{R}}(P_7, P_9) = 15$	
$x_i^9 \leq 9$	7,465
$x_i^9 = 13$	935
$x_i^9 = 14$	39,389
$x_i^9 = 15$	134,652
$x_i^9 \geq 15$	71,456
$x_i^9 \geq 16$	99,659
total	353,556

TABLE 8. Results of computer computations V

- the Department of Combinatorics and Optimization, University of Waterloo (www.math.uwaterloo.ca/CandO_Dept): 80 CPUs.

In order to find new on-line Ramsey numbers we checked (independently) millions initial configurations. A running time of one serial program varied between a few seconds and 10 hour. We can estimate the total computational requirements to be around 250,000 CPU hours (28.5 years).

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