A Degree Condition for Graphs to Have (g, f)-Factors *†

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Abstract

Let G be a graph of order n, and let a and b be integers such that $1 \le a < b$, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. Then G has a (g, f)-factor if the minimum degree $\delta(G) \ge \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, $n > \frac{(a+b)(a+b-1)}{a+1}$ and $\max\{d_G(x), d_G(y)\} \ge \frac{(b-1)n}{a+b}$ for any two nonadjacent vertices x and y in G. Furthermore, it is showed that the result in this paper is best possible in some sense.

Keywords: graph, factor, (g, f)-factor, degree condition

AMS(2000) Subject Classification: 05C70

1 Introduction

The graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of

*Receive date: 2007-06-12

†This research was supported by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (10KJB110003) and Jiangsu University of Science and Technology (2010SL101J, 2009SL154J) and Sichuan Provincial Educational Department (08zb068), and was sponsored by Qing Lan Project of Jiangsu Province.

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vertices and the set of edges, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x)$, and $G[S]$ is the subgraph of G induced by S. The minimum degree of vertices in G is denoted by $\delta(G)$. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T. For a subset $S \subseteq V(G)$, We denote by $G - S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S.

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for each $x \in V(G)$. A (g,f)-factor of graph G is a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$ (Where of course d_F denotes the degree in F). If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f)-factor of G is called an $[a,b]$-factor of G. If $g(x) = f(x) = k$ for each $x \in V(G)$, then a (g,f)-factor of G is called a k-factor of G. The other terminologies and notations not given in this paper can be found in [1].

Many authors have investigated (g,f)-factors [2-5], factorizations [6]. The following results on k-factors and $[a,b]$-factors are known.

Theorem 1 [7] *Let k be a positive integer, and G be a graph of order n with $n \geq 4k-5$, kn even, and $\delta(G) \geq k$. Then G has a k-factor if the degree sum of each pair of nonadjacent vertices is at least n.*

Theorem 2 [8] *Let $k \geq 3$ be an integer and G be a connected graph of order n with $n \geq 4k-3$, kn even, and $\delta(G) \geq k$. If for each pair of nonadjacent vertices x, y of $V(G)$*

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2},$$

then G has a k-factor.

Theorem 3 [9] *Let G be a graph of order n, and let a and b be integers such that $1 \leq a < b$. Then G has an $[a,b]$-factor if $\delta(G) \geq a$, $n \geq 2a + b + \frac{a^2-a}{b}$ and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{an}{a+b}$$

for any two nonadjacent vertices x and y in G.

Theorem 4 [10] *Let aand b be integers such that $1 \leq a < b$, and let G be a graph of order n with $n \geq \frac{2(a+b)(a+b-1)}{b}$, and $\delta(G) \geq a$. If*

$$|N_G(x) \cup N_G(y)| \geq \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of G, then G has an $[a,b]$-factor.

Theorem 5 [11] *Let a and b be integers such that $2 \le a < b$, and let G be a graph of order n with $n \ge 6a+b$. Put $\lambda = \frac{a-1}{b}$. For any subset $X \subset V(G)$, we suppose*

$$N_G(X) = V(G) \qquad if\ |X| \ge \lfloor \frac{n}{1+\lambda} \rfloor;$$

or

$$|N_G(X)| \ge (1+\lambda)|X| \qquad if\ |X| < \lfloor \frac{n}{1+\lambda} \rfloor.$$

Then G has an $[a,b]$-factor.

Theorem 6 [12] *Let $1 \le a < b$ be integers and G a graph of order $n \ge \frac{2(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)}$. Suppose that $\delta(G) \ge a$ and*

$$|N_G(x) \cup N_G(y)| \ge \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of $V(G)$ such that $N_G(x) \cap N_G(y) \ne \varnothing$. Then G has an $[a,b]$-factor.

Theorem 7 [12] *Let $1 \le a < b$ be integers and G a graph of order $n \ge \frac{(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)}$. Suppose that $\delta(G) \ge a$ and*

$$\max\{d_G(x), d_G(y)\} \ge \frac{an}{a+b}$$

for any vertices x and y of G with $d(x,y) = 2$. Then G has an $[a,b]$-factor.

2 The Proof of Main Theorem

In this paper, we mainly prove the following theorem about the existence of a (g,f)-factor, which is an extension of Theorem 1 and Theorem 2 and Theorem 3. We extend Theorem 1 and Theorem 2 and Theorem 3 to (g,f)-factors.

Theorem 8 *Let G be a graph of order n, and let a and b be integers with $1 \le a < b$, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. Then G has a (g,f)-factor if $\delta(G) \ge \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, $n > \frac{(a+b)(a+b-1)}{a+1}$ and*

$$\max\{d_G(x), d_G(y)\} \ge \frac{(b-1)n}{a+b}$$

for any two nonadjacent vertices x and y in G.

In order to prove our main theorem, we depend heavily on the following theorem, which is a special case of *Lovász's* (g, f)-factor theorem.

Theorem 9 [13] *Let G be a graph, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0$$

for all disjoint subsets S and T of $V(G)$.

The Proof of Theorem 8. Suppose that G satisfies the conditions of Theorem 8, but has no (g, f)-factor. Then, by Theorem 9, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \leq -1. \tag{1}$$

We choose subsets S and T such that $|T|$ is minimum.

We first prove the following claims.

Claim 1. $d_{G-S}(x) < g(x) \leq b - 1$ for each $x \in T$.

Proof. Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \geq g(x)$. Then the subsets S and $T - \{x\}$ satisfy (1), which contradicts the choice of T. Therefore,

$$d_{G-S}(x) < g(x) \leq b - 1 \tag{2}$$

for each $x \in T$.

Claim 2. $|T| \geq a + 2$.

Proof. If $|T| \leq a + 1$, then by (1) and since $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq \frac{(b-1)^2-(a+1)(b-a-2)}{a+1} \geq b - 1$ for each $x \in T$ we obtain

$$\begin{aligned} -1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq |T||S| + d_{G-S}(T) - (b-1)|T| \\ &= \sum_{x \in T}(|S| + d_{G-S}(x) - (b-1)) \geq 0, \end{aligned}$$

which is a contradiction. So $|T| \geq a + 2$.

Since $T \neq \varnothing$, let $h_1 = \min\{d_{G-S}(x)|x \in T\}$, and let $x_1 \in T$ be a vertex such that $d_{G-S}(x_1) = h_1$. According to (2), we get

$$0 \leq h_1 \leq b - 2.$$

In the following, We shall consider two cases and derive a contradiction in each case.

Case 1. $T = N_T[x_1]$.

In view of Claim 2 and $|T| = |N_T[x_1]| \le d_{G-S}(x_1) + 1 = h_1 + 1 \le b - 1$, we have

$$h_1 \ge a + 1 \tag{3}$$

and

$$b \ge a + 3. \tag{4}$$

According to (1), (3), (4), $|T| \le b - 1$, $|S| + h_1 = |S| + d_{G-S}(x_1) \ge d_G(x_1) \ge \delta(G) \ge \frac{(b-1)^2-(a+1)(b-a-2)}{a+1}$, and the definition of h_1, we obtain

$$\begin{aligned} -1 &\ge \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\ge (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\ge (a+1)|S| + h_1|T| - (b-1)|T| \\ &= (a+1)|S| - (b - h_1 - 1)|T| \\ &\ge (a+1)(\frac{(b-1)^2 - (a+1)(b-a-2)}{a+1} \\ &\quad -h_1) - (b - h_1 - 1)(b-1) \\ &\ge (b-a-2)h_1 - (a+1)(b-a-2) \ge 0. \end{aligned}$$

This is a contradiction.

Case 2. $T \ne N_T[x_1]$.

It is clear that $T \setminus N_T[x_1] \ne \varnothing$. Then we define

$$h_2 = \min\{d_{G-S}(x) | x \in T \setminus N_T[x_1]\},$$

and let $x_2 \in T \setminus N_T[x_1]$ be a vertex such that $d_{G-S}(x_2) = h_2$. Note that $0 \le h_1 \le h_2 \le b-2$ hold.

Obviously, two vertex x_1 and x_2 are not adjacent. In view of the condition of the theorem, we get that

$$\max\{d_G(x_1), d_G(x_2)\} \ge \frac{(b-1)n}{a+b}. \tag{5}$$

Claim 3. $|S| + h_2 \ge \frac{(b-1)n}{a+b}$.

Proof. If $|S| + h_2 < \frac{(b-1)n}{a+b}$, then we get $|S| + h_1 \le |S| + h_2 < \frac{(b-1)n}{a+b}$, and this implies $d_G(x_1) < \frac{(b-1)n}{a+b}$ and $d_G(x_2) < \frac{(b-1)n}{a+b}$. This contradicts (5).

By Claim 3, we obtain

$$|S| \geq \frac{(b-1)n}{a+b} - h_2. \tag{6}$$

Case 2.1. $h_2 = 0$.

Clearly, $h_1 = 0$. According to (1), (6), and $|S| + |T| \leq n$, we get that

$$\begin{aligned} -1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq (a+1)|S| - (b-1)|T| \\ &\geq (a+1)|S| - (b-1)(n-|S|) \\ &= (a+b)|S| - (b-1)n \\ &\geq (a+b)\frac{(b-1)n}{a+b} - (b-1)n = 0, \end{aligned}$$

which is a contradiction.

Case 2.2. $1 \leq h_2 \leq b-2$.

By (1), (6), and $|S| + |T| \leq n$, and $|N_T[x_1]| \leq h_1 + 1$, we obtain

$$\begin{aligned} -1 &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq (a+1)|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - (b-1)|T| \\ &= (a+1)|S| + (h_1 - h_2)|N_T[x_1]| - (b - h_2 - 1)|T| \\ &\geq (a+1)|S| + (h_1 - h_2)|N_T[x_1]| - (b - h_2 - 1)(n - |S|) \\ &= (a + b - h_2)|S| + (h_1 - h_2)|N_T[x_1]| - (b - h_2 - 1)n \\ &\geq (a + b - h_2)(\frac{(b-1)n}{a+b} - h_2) + (h_1 - h_2)(h_1 + 1) - (b - h_2 - 1)n. \end{aligned}$$

Let $F(h_1, h_2) = (a+b-h_2)(\frac{(b-1)n}{a+b} - h_2) + (h_1 - h_2)(h_1 + 1) - (b - h_2 - 1)n$. Then we have

$$-1 \geq F(h_1, h_2). \tag{7}$$

If $2 \leq h_2 \leq b-2$, we have

$$\begin{aligned} F'_{h_2}(h_1, h_2) &= 2h_2 - \frac{(b-1)n}{a+b} - (a+b) - (h_1 + 1) + n \\ &= 2h_2 + \frac{(a+1)n}{a+b} - (a+b) - (h_1 + 1) \\ &> 2h_2 - h_1 - 2 \qquad (\textit{Since } n > \frac{(a+b)(a+b-1)}{a+1}) \\ &\geq h_2 - h_1 \geq 0. \end{aligned}$$

Since $h_1 \leq h_2$, then we have

$$F(h_1, h_2) \geq F(h_1, h_1). \tag{8}$$

In view of (7) and (8), we obtain

$$\begin{aligned}
-1 &\geq F(h_1, h_1) = (a+b-h_1)(\frac{(b-1)n}{a+b} - h_1) - (b-h_1-1)n \\
&\geq h_1^2 + (\frac{(a+1)n}{a+b} - (a+b))h_1 \\
&> h_1^2 - h_1 \geq 0, \qquad (\textit{Since } h_1 \geq 0 \textit{ is an integer})
\end{aligned}$$

that is a contradiction.

If $h_1 = h_2 = 1$, then we get

$$\begin{aligned}
F(h_1, h_2) &= (a+b-1)(\frac{(b-1)n}{a+b} - 1) - (b-2)n \\
&= \frac{(a+1)n}{a+b} - (a+b) + 1 \\
&> 0, \qquad (\textit{Since } n > \frac{(a+b)(a+b-1)}{a+1})
\end{aligned}$$

which contradicts (7).

If $h_1 = 0$, $h_2 = 1$, then we have

$$\begin{aligned}
F(h_1, h_2) &= (a+b-1)(\frac{(b-1)n}{a+b} - 1) - 1 - (b-2)n \\
&= \frac{(a+1)n}{a+b} - (a+b) \\
&> -1, \qquad (\textit{Since } n > \frac{(a+b)(a+b-1)}{a+1})
\end{aligned}$$

which contradicts (7).

From the argument above, we deduce the contradictions. Hence, G has a (g, f)-factor.
Completing the proof of Theorem 8.

Remark 1. Let us show that the condition $\max\{d_G(x), d_G(y)\} \geq \frac{(b-1)n}{a+b}$ in Theorem 8 can not be replaced by $\max\{d_G(x), d_G(y)\} \geq \frac{(b-1)n}{a+b} - 1$. Assume $b = a+1$, and define $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. Let $G = (A, B)$ be a complete bipartite graph such that $|A| = at$ and $|B| = bt + 1$, where t is any positive integer. Then it follows that $n = |A| + |B| = (a+b)t + 1$ and

$$\frac{(b-1)n}{a+b} > \max\{d_G(x), d_G(y)\} = at = (b-1)t > \frac{(b-1)n}{a+b} - 1$$

for any subset $\{x, y\}$ of B. However, G has no $[a, b]$-factor since $b|A| < a|B|$, i.e., G has no (g, f)-factor. In this sense, the condition $\max\{d_G(x), d_G(y)\} \geq \frac{(b-1)n}{a+b}$ is the best possible.

Remark 2. As $b > a + 1$, I guess the condition $\max\{d_G(x), d_G(y)\} \geq \frac{(b-1)n}{a+b}$ in Theorem 8 can be improved. Furthermore, the problem is worth investigating.

Acknowledgement. We would like to thank the referee for his detailed suggestions which substantially improved the presentation of this paper.

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