ON SOME FINITE HYPERBOLIC SPACES

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Abstract

Let π be a finite projective plane of order n. Consider the substructure π_{n+2} obtained from π by removing $n+2$ lines (including all points on them) no three are concurrent. In this paper, firstly, it is shown that π_{n+2} is a $B-L$ plane and it is also homogeneous. Let $PG(3,n)$ be a finite projective 3−space of order n. The substructure obtained from $PG(3,n)$ by removing a tetrahedron that is four planes of $PG(3,n)$ no three of them are collinear is a finite hyperbolic 3−space (Olgun-Özgür [10]). Finally, we prove that any two hyperbolic planes with same parameters are isomorphic in this hyperbolic 3−space. These results are appeared in the second author's Msc thesis.

MSC: 05B25, 51A45, 51E20

1 Preliminary Definitions and Propositions

An incidence structure is an ordered triple of sets $(P, L, \circ)$, where $P \cap L = \emptyset$, $\circ \subset P \times L$. For X in P and l in L, $X \circ l$ is read "X is on l".

Definition 1.1 (Bumcrot [5]) *A linear space is an incidence structure $(P, L, \circ)$ satisfying axioms below:*

$L1$) Each two distinct points are on exactly one line.
$L2$) Each line is on at least two points.

If $S = (P, L, \circ)$ is a linear space, we define, as usual,

$$v = |P|, \quad b = |L|$$

where $||$ denotes cardinality. For each point X and line l of S, let

$$\begin{aligned} r(X) &= |\{l \in L : X \circ l\}| \\ k(l) &= |\{X \in P : X \circ l\}| \end{aligned}$$

If v is finite then we say S is finite. For finite S we define,

$$\begin{aligned} k_m &= \min\{k(l) : l \in L\} \\ k_M &= \max\{k(l) : l \in L\} \\ r_m &= \min\{r(X) : X \in P\} \\ r_M &= \max\{r(X) : X \in P\} \end{aligned}$$

S is regular and is said to have order (k, r) if $k_m = k_M = k$ and $r_m = r_M = r$.

Definition 1.2 (Batten [2]) *A projective plane is a linear space satisfying axioms below:*

P1) Any two lines meet.
P2) There exists a set of four points no three of which are collinear.

Definition 1.3 (Hughes-Piper [8]) *Let $(P, L, \circ)$ and $(P', L', \circ')$ be any two incidence structures. f is called a homomorphism from $(P, L, \circ)$ to $(P', L', \circ')$ if*

$$f : P \cup L \to P' \cup L'$$

satisfies properties below:

i) $f(P) \subseteq P'$
ii) $f(L) \subseteq L'$
iii) $\forall X \in P,\ l \in L$ *and* $X \circ l \Rightarrow f(X) \circ' f(l)$

If f is a bijection, then the homomorphism f is called isomorphism. $(P, L, \circ)$ and $(P', L', \circ')$ are called isomorphic if there is an isomorphsim from $(P, L, \circ)$ to $(P', L', \circ')$ and denoted by

$$(P, L, \circ) \cong (P', L', \circ')$$

Here if we take $(P, L, \circ)$ instead of $(P', L', \circ')$, then f is called a collineation or an automorphism of $(P, L, \circ)$.

Theorem 1.1 (Hughes-Piper [8]) *Let π be a Desarguesian projective plane. Then a group of collineations of π is transitive on quadruples of points no three of them are collinear.*

A three dimensional projective space is called a projective 3−space. It is well known that any plane of a projective 3-space is a Desarguesian projective plane.

Definition 1.4 (Graves [6]) *Let $H = (P, L, \circ)$ be a linear space. Then H is a hyperbolic plane ($B - L$ plane) if it satisfies axioms below:*

H1) Through each point X not on a line b there pass at least two lines not meeting b.

H2) There exists a set of four points no three of which are collinear.

H3) If a subset of π contains three points not on a line and contains all the lines through any pair of its points, then that subset contains all the points of π.

Theorem 1.2 (Bumcrot [5]) *Any finite linear space satisfying:*

i) $r_m \geq k_M + 2$
ii) $k_m(k_m - 1) \geq r_M$

is a hyperbolic plane.

Definition 1.5 *Let H be any hyperbolic plane. Then H is called homogeneous in the sense of Graves [6] if for each pair of points there is a collineation carrying the first in to the second.*

Definition 1.6 (Bose [4]) *A partial geometry (r, k, t) as a system of undefined points and lines and undefined relation "incidence" satisfying axioms A1 through A4 below:*

A1) Any two points are incident with no more than one line.

A2) Each point is incident with r lines.

A3) Each line is incident with k points.

A4) If a point P is not incident with the line l, there pass through P exactly t lines $(t \geq 1)$ intersecting l.

Now suppose that there exists a projective plane of order $n = 2h$ which posseses an oval consisting of $2h + 2$. One can classify the lines of such a plane into two categories. The first category consists of lines, henceforth called secants, including two points of the oval. The second category consists of lines, henceforth called nonintersectors, not including any points of the oval. If we remove the points of the oval from the lines of the first category then each of the two categories of lines separately forms a partial geometry. Each of these partial geometries include all the points of the orginal projective plane except of the points of the oval.

Any Desarguesian plane of order $n = 2^m$, m a positive integer, has a hyperoval or a dual hyperoval (Seiden [16]).

Any $B - L$ plane obtained by Seiden in [16] is called a partial $B - L$ plane.

Proposition 1.3 (Seiden [16]) *The dual of the partial geometry formed by the nonintersectors of a hyperoval in a projective plane of even order $2h$, $h > 3$, is a partial $B - L$ plane.*

Proposition 1.4 (Seiden [16]) *The partial $B - L$ planes obtained from Desarguesian planes are homogeneous in the sense of Graves [6], provided that the oval used for their construction consists of a conic and its center, i.e. the point of intersection of the tangents to the conic.*

R. C. Bose showed in [3] that, if the 2^m+1 points of a nondegenerate conic are represented by the equation

$$ax^2+by^2+cz^2+fyz+gxz+hxy=0$$

then the (2^m+2) −nd point of the oval, the center of the conic, can be represented by the coordinates (f,g,h). We note that three points of the conic together with its center determine the conic uniquely.

Also, an important property of Desarguesian planes to be used here is that their collineations are transitive on quadrilaterals and therefore on quadrangles.

2 Main Results

2.1 A Model of A Finite Homogeneous Hyperbolic Plane

Let π be projective plane of order n and M be lines set including lines of π no three are concurrent, $|M|=m$. Consider the structure π_m obtained from π by removing lines of M (including all points on them). It is known that π_m is a hyperbolic plane under certain conditions (Bumcrot [5]). In proposition 2.1, the following question of Bumcrot is answered:

" Is there a finite projective plane π and a set of lines M with $|M|=m$ such that π_m is a regular hyperbolic plane? "

Proposition 2.1 *Let π be a finite projective plane of even order $n\geq 8$. π_{n+2} obtained from π by removing $n+2$ lines (including all points on them) such that no three are concurent, i.e. a dual hyperoval, is an $(\frac{n}{2},n+1)$−regular hyperbolic plane.*

Proof. It is clear that π_{n+2} is a linear space. First, we will show that π_{n+2} is an $(\frac{n}{2},n+1)$−regular plane. There are $n+1$ lines passing through each point of π_{n+2}, which means that π_{n+2} is point regular. On the other hand, since each line of π_{n+2} is also a line of π, each line of π_{n+2} intersects with each line of dual hyperoval at $\frac{n+2}{2}$ distinct points. Thus π_{n+2} is

$$n+1-(\tfrac{n+2}{2}) \quad = \quad \tfrac{n}{2}$$

line regular. Now, we will show that π_{n+2} is a hyperbolic plane. We have

$$k_m=k_M=\frac{n}{2} \text{ ve } r_m=r_M=n+1$$

and $n\geq 8$. Therefore, we obtain

$$n+1\geq\frac{n}{2}+2 \text{ ve } \frac{n}{2}(\frac{n}{2}-1)\geq n+1$$

Then π_{n+2} is a hyperbolic plane by Theorem 1.2. ∎

Proposition 2.2 *The π_{n+2} obtained from a Desarguesian plane is homogeneous in the sense of Graves [6], provided that the dual hyperoval used for their construction consists of a dual conic and its axis.*

Proof. Consider the Seiden partial $B-L$ plane. Note that transitivity on this plane is seen on lines, not on points. Although, it is known that a linear space which is line transitive is also point transitive (Batten [2]). Our aim is to show that the homogenity of π_{n+2} by using the homogeneity of Seiden partial $B-L$ plane. Remember that, any Desarguesian projective plane of order $n = 2^m$, m a positive integer, has a hyperoval or a dual hyperoval. If the 2^m+1 lines of a nondegenerate dual conic are represented by the equation

$$ax^2 + by^2 + cz^2 + fyz + gxz + hxy = 0$$

then the 2^m+2–nd line of the dual hyperoval, the axis of the dual conic, can be represented by the coordinates $[f, g, h]$. Thus, it is observed that any such dual conic is uniquely determined by its three lines and the axis. It is clear that any collineation of π_{n+2} is a collineation of π which leaves the dual conic invariant. The collineation can also be characterised as a collineation mapping the first into the second of each pair of quadrilaterals which consist of any three lines of the conic and the axis.

Let x, y be any lines of π_{n+2}. If we obtain a collineation of π_{n+2} such that it carries x to y, then the proof is finished. To find such a collineation we start with axis of the dual conic denoted by a. Any line of the projective plane intersects with the axis a. Consider points $P = xa$ ve $P' = ya$. A line of π_{n+2} is also a line of π. On the other hand, any line of π_{n+2} is concurrent with any two lines of dual hyperoval. Thus, there exist other lines belonging to the dual conic, which pass through P and P' denote by l_3 and l_6 respectively. Let l_1, l_2, l_4, l_5 be any lines of the dual conic. There exist the points Q and Q' on x and y, respectively, such that the lines l_4, l_5, x are concurrent at Q and the lines l_1, l_2, y are concurrent at Q'. The collination mapping the quadrilateral formed by lines l_3, l_4, a and l_5 to that formed by l_1, l_2, a and l_6 carries the line x to the line y. Since x and y are arbitrary, the collineations are transitive on lines. It is also known that line transitivity implies point transitivity, thus, π_{n+2} is homogeneous and the proof is completed.

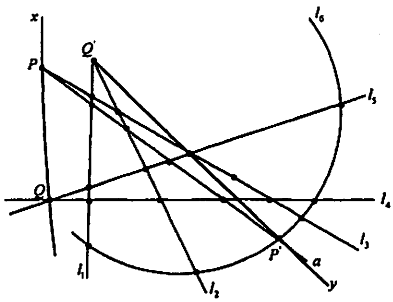

Figure 2.2.1

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2.2 A Model of A Finite Hyperbolic 3−Space

Proposition 2.3 (Olgun-Özgür [10]) *$S = PG(3,n) = (P, L, \circ)$ be a finite projective 3-space of order n, D is any set of some projective planes of S, satisfying:*

C. *The intersection of D and any projective plane in S does not belong to D contains non-concurrent three lines. Let P_1 and L_1 be points set and lines set belong to planes of D respectively. Then*

$$S_D = (P \setminus P_1,\ L \setminus L_1,\ \circ \cap (P \setminus P_1) \times (L \setminus L_1))$$

is a finite hyperbolic 3−spaces if

$$4 \leq d \leq n + \frac{1}{2}(1 - \sqrt{4n+5})$$

with $|D| = d$.

Proposition 2.4 *Any two hyperbolic planes of S_D in the meaning of Proposition 4.1 with the same parameters are isomorphic if and only if there exists an isomorphism from one to the other mapping the removed substructure of one to the removed substructure of the other.*

Proof. The proof is similar to *the* Lemma 3.11 in HUGHES-PIPER [8]. ∎

Proposition 2.5 *Any two hyperbolic planes with same parameters are isomorphic in the hyperbolic 3−space obtained from $PG(3,n)$ by removing a tetrahedron that is four planes of $PG(3,n)$ no three of them are collinear.*

Proof. When a tetrahedron is removed from $PG(3,n)$ (i.e. $d = 4$, in [10]), S_D has three different types of planes.

Type I: Planes passing through only one vertex of the tetrahedron. Each of these planes has four removed lines which three of them are concurrent.

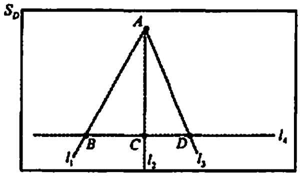

Figure 2.5.1

Type II: Planes passing through any edge of the tetrahedron. Each of these planes has three removed lines which are non-concurrent.

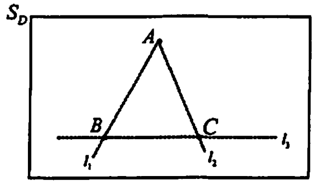

Figure 2.5.2

Type III: Planes not passing through any vertex of the tetrahedron. Each of these planes has four removed lines no three of them are concurrent.

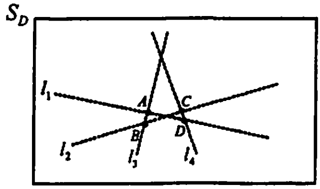

Figure 2.5.3

It is enough to give the proof for planes in type I. Because the proof in other cases are completely similar to type I. Let π_D and $\pi'_{D'}$ be any hyperbolic planes of S_D of type I. Let l_1, l_2, l_3, l_4 and $l_1'', l_2'', l_3'', l_4''$ be lines removed from π and π' respectively. There exists an isomorphism α from π to π' since $\pi \cong \pi'$. Let $\alpha(l_i) = l_i'$, $i = 1, 2, 3, 4$. So there exists a colineation β of π' such that $\beta(l_i') = l_i''$, $i = 1, 2, 3, 4$, since the collineations of π' is transitive on quadrilaterals. Thus, $\beta \circ \alpha$ is an isomorphism from π to π' such that $(\beta \circ \alpha)(l_i) = l_i''$, $i = 1, 2, 3, 4$.

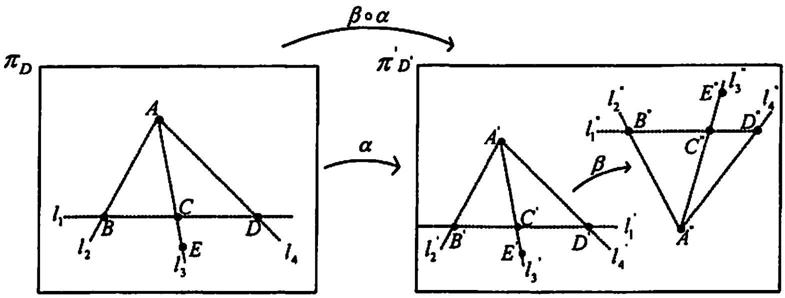

Figure 2.5.4

That is $\pi_D \cong \pi'_{D'}$. Thus, the proof is completed. ■

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