IDENTITIES FOR THE BERNOULLI AND EULER NUMBERS AND POLYNOMIALS

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ABSTRACT. In this paper, we investigate some interesting identities on the Euler numbers and polynomials arising from their generating functions and difference operators. Finally we give some properties of Bernoulli and Euler polynomials by using p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\cdot|_p$ is normally defined by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

(1) $$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [11])}.$$

As is well known, the Bernoulli polynomials are defined by the generating function as follows :

(2) $$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [14,15])},$$

with the usual convention about replacing B^n by B_n (see [1-16]). In the special case, $x = 0$, $B_n(0) = B_n$ are called the n-th Bernoulli numbers.
The Euler polynomials are also defined by the generating function as follows :

(3) $$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [11,12])}.$$

In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers.
In the sense of fermionic, the p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows :

(4) $$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [9])}.$$

From (1) and (2), we note that

(5) $$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

and

$$I(f_1) - I(f) = f'(0), \quad \text{(see [12-15])}. \tag{6}$$

By (2), (3), (5) and (6), we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu(x) = \frac{t}{e^t - 1}, \ and \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1}. \tag{7}$$

Thus, from (2), (3) and (7), we have

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n, \ and \int_{\mathbb{Z}_p} x^n d\mu(x) = B_n. \tag{8}$$

By (8), we easily see that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \ and \int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x). \tag{9}$$

In [5], we have the following identities for the Bernoulli polynomials: for $m, k \in \mathbb{N}$

$$\begin{aligned} &\sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right) \frac{B_{k+m+1-j}(x)}{k+m+1-j} \\ &= x^k (x-1)^m + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right) B_{k+m-j}(x) \\ &= x^k (x-1)^{m-1} \left((k+m)x - k \right), \end{aligned} \tag{11}$$

$$\sum_{j=0}^{[\frac{k-1}{2}]} \binom{k}{2j+1} \frac{B_{2k-2j}(x)}{2k-2j} = \frac{x^k (x-1)^k}{2} + \frac{(-1)^{k+1}}{(4k+2)\binom{2k}{k}}, \tag{12}$$

$$\begin{aligned} &\sum_{j=0}^{[\frac{k}{2}]} \left(\binom{k}{2j+1} + \binom{k+1}{2j+1} \right) \frac{B_{2k+1-2j}(x)}{2k+1-2j} \\ &= x^k (x-1)^k (x - \frac{1}{2}), \end{aligned} \tag{13}$$

where $[\cdot]$ is Gauss' symbol.
In this paper, we give some new identities for the Euler polynomials arising from their generating functions and difference operators. These identities are correspanding (10)-(13) in the case of Bernoulli polynomials. Finally, we give new identities on the Bernoulli and Euler polynomials by using p-adic integral on $\mathbb{Z}_p$.

2. On identities of Bernoulli and Euler numbers

Consider x as a fixed parameter and set

$$F_x = F_x(t) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{14}$$

Thus, by (14), we get

$$e^t F_x + F_x = 2e^{xt}. \tag{15}$$

Let us define difference operator D as $D = \frac{d}{dt}$. Then we see that

$$e^t(D+I)^k F_x + D^k F_x = 2x^k e^{xt}, \tag{16}$$

where $k \in \mathrm{N}$ and I is identity operator.
By multiplying e^{-t} on both sides in (16), we get

$$(D+I)^k F_x + e^{-t}D^k F_x = 2x^k e^{(x-1)t}. \tag{17}$$

Let us take difference operator $D^m (m \in \mathbb{N})$ on both sides of (17). Then we have

$$D^m(D+I)^k F_x + e^{-t}D^k(D-I)^m F_x = 2x^k(x-1)^m e^{(x-1)t}. \tag{18}$$

By multiplying e^t on both sides of (18), we get

$$e^t D^m(D+I)^k F_x + D^k(D-I)^m F_x = 2x^k(x-1)^m e^{xt} \tag{19}$$

Let G[0] (not G(0)) be the constant term in a Laurent series of G(t). Then, by (19), we get

(20)
$$\sum_{j=0}^{k}\binom{k}{j}\left(e^t D^{k+m-j}F_x(t)\right)[0] + \sum_{j=0}^{m}\binom{m}{j}(-1)^j\left(D^{k+m-j}F_x\right)[0] = 2x^k(x-1)^m.$$

By (14), we get

$$\left(D^N F_x(t)\right)[0] = E_N(x), \quad \left(e^t D^N F_x(t)\right)[0] = E_N(x). \tag{21}$$

From (20) and (21), we note that

$$\sum_{j=0}^{max\{k,m\}}\left(\binom{k}{j}E_{k+m-j}(x) + \binom{m}{j}(-1)^j E_{k+m-j}(x)\right) = 2x^k(x-1)^m. \tag{22}$$

By (3), we get the following recurrence formula:

$$E_0 = 1, \quad E_n(1) + E_n = 2\delta_{0,n}, \quad (n \in \mathbb{Z}_+ = \mathbb{Z} \cup \{0\}). \tag{23}$$

From (3), we note that

$$E_n(x) = (E+x)^n = \sum_{\ell=0}^{n}\binom{n}{\ell}E_\ell x^{n-\ell} = \sum_{\ell=0}^{n}\binom{n}{\ell}E_{n-\ell}x^\ell. \tag{24}$$

Thus, by (24), we get

$$\frac{dE_n(x)}{dx} = n\sum_{\ell=0}^{n-1}\binom{n-1}{l}E_\ell x^{n-1-\ell} = nE_{n-1}(x). \tag{25}$$

By (23) and (25), we get

$$\int_0^1 E_n(x)dx = \frac{1}{n+1}\left(E_{n+1}(1) - E_{n+1}\right) = \frac{-2E_{n+1}}{n+1}. \tag{26}$$

Let us take definite integral from 0 to 1 in (22). Then we have

$$\begin{aligned} &-2\sum_{j=0}^{max\{k,m\}} \frac{E_{k+m-j+1}}{k+m-j+1}\left(\binom{k}{j} + (-1)^j\binom{m}{j}\right) \\ &= 2(-1)^m B(k+1, m+1), \end{aligned} \tag{27}$$

where $B(\alpha, \beta)$ is beta function which is defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad (\alpha > 0, \beta > 0).$$

Thus, by (27), we get

$$\begin{aligned} &\sum_{j=0}^{max\{k,m\}} \frac{E_{k+m-j+1}}{k+m-j+1}\left(\binom{k}{j} + (-1)^j\binom{m}{j}\right) \\ &= (-1)^{m+1}\frac{\Gamma(k+1)\Gamma(m+1)}{\Gamma(k+m+2)}. \end{aligned} \tag{28}$$

Therefore, by (22), we obtain the following theorem.

Theorem 1. For $m, k \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{j=0}^{max\{k,m\}} \left(\binom{k}{j}E_{k+m-j}(x) + \binom{m}{j}(-1)^j E_{k+m-j}(x)\right) \\ &= 2x^k(x-1)^m. \end{aligned}$$

By (28) and Theorem 1, we obtain the following corollary.

Corollary 2 . For $m, k \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^j\binom{m}{j}\right)\frac{E_{k+m-j+1}}{k+m-j+1} \\ &= \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} - 2\frac{E_{k+m+1}}{k+m+1}. \end{aligned} \tag{29}$$

Let us take $x = 0$ in (10). Then we see that

$$(30) \qquad \sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right) \frac{B_{k+m-j+1}}{k+m-j+1} = \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}.$$

By (29) and (30), we get

$$(31) \qquad \sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right) \frac{B_{k+m-j+1}}{k+m-j+1} = \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}$$
$$= \sum_{j=1}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^{j} \binom{m}{j} \right) \frac{E_{k+m-j+1}}{k+m-j+1} + 2\frac{E_{k+m+1}}{k+m+1}.$$

Therefore, by (31), we obtain the following theorem.

Theorem 3. For $m, k \in \mathbb{N}$, we have

$$\sum_{j=1}^{max\{k,m\}} \binom{k}{j} \left(\frac{B_{k+m-j+1}}{k+m-j+1} - \frac{E_{k+m-j+1}}{k+m-j+1} \right)$$
$$+ \sum_{j=1}^{max\{k,m\}} (-1)^{j+1} \binom{m}{j} \left(\frac{B_{k+m-j+1}}{k+m-j+1} + \frac{E_{k+m-j+1}}{k+m-j+1} \right)$$
$$= 2\frac{E_{k+m+1}}{k+m+1}.$$

Let $m = k+1$ in Theorem 3. Then we obtain the following corollary.

Corollary 4. For $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{k+1} \binom{k}{j} \left(\frac{B_{2k+2-j}}{2k+2-j} - \frac{E_{2k+2-j}}{2k+2-j} \right)$$
$$+ \sum_{j=1}^{k+1} (-1)^{j+1} \binom{k+1}{j} \left(\frac{B_{2k-j+2}}{2k-j+2} + \frac{E_{2k+2-j}}{2k+2-j} \right)$$
$$= 0.$$

From Theorem 3, we note that

$$(32) \qquad 2 \left(\sum_{j=1}^{[\frac{k}{2}]} \binom{k}{2j-1} \frac{B_{2k-2j+2}}{2k-2j+2} - \sum_{j=1}^{[\frac{k}{2}]} \binom{k}{2j} \frac{E_{2k-2j+1}}{2k-2j+1} \right)$$
$$= 2\frac{E_{2k+1}}{2k+1}.$$

Therefore, by (32), we obtain the following theorem.

Theorem 5. For $k \in \mathbb{N}$, we have

$$\sum_{j=1}^{[\frac{k}{2}]} \binom{k}{2j-1} \frac{B_{2k-2j+2}}{2k-2j+2} - \sum_{j=1}^{[\frac{k}{2}]} \binom{k}{2j} \frac{E_{2k-2j+1}}{2k-2j+1} = \frac{E_{2k+1}}{2k+1}.$$

By (22) and (27), we get

$$\sum_{j=0}^{max\{k,m\}} (k+m-j) \left(\binom{k}{j} + (-1)^j \binom{m}{j} \right) E_{k+m-j-1}(x) = 2x^{k-1}(x-1)^{m-1}((k+m)x-k). \tag{33}$$

Let $m = k$. From (22), we have

$$\sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} E_{2k-2j}(x) = x^k(x-1)^k. \tag{34}$$

Let us take definite integral from 0 to 1 in (34). Then we have

$$-2\sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} \frac{E_{2k-2j+1}}{2k-2j+1} = (-1)^k B(k+1,k+1) = \frac{(-1)^k}{(2k+1)\binom{2k}{k}}. \tag{35}$$

Therefore, by (33) and (34), we obtain the following proposition:

Proposition 6. For $k \in \mathbb{N}$, we have

$$\sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} E_{2k-2j}(x) = x^k(x-1)^k.$$

By (35), we obtain the following corollary.

Corollary 7. For $k \in \mathbb{N}$, we have

$$\sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} \frac{E_{2k-2j+1}}{2k-2j+1} = \frac{(-1)^{k+1}}{(4k+2)\binom{2k}{k}}.$$

From (22), we can derive the following equation:

$$\sum_{j=0}^{k+1} \left(\binom{k}{j} + (-1)^j \binom{k+1}{j} \right) E_{2k+1-j}(x) = 2x^k(x-1)^{k+1}. \tag{36}$$

The left hand side of (36) is given by

$$
\begin{aligned}
(37)\qquad & \sum_{j=0}^{k+1}\left(\binom{k}{j}+(-1)^j\binom{k+1}{j}\right)E_{2k+1-j}(x)\\
&=\sum_{j=0}^{[\frac{k+1}{2}]}\left(\binom{k}{2j+1}-\binom{k+1}{2j+1}\right)E_{2k-2j}(x)\\
&\quad+\sum_{j=0}^{[\frac{k+1}{2}]}\left(\binom{k}{2j}+\binom{k+1}{2j}\right)E_{2k-2j+1}(x)\\
&=-\sum_{j=0}^{[\frac{k+1}{2}]}\binom{k}{2j}E_{2k-2j}(x)+\sum_{j=0}^{[\frac{k+1}{2}]}\left(\binom{k}{2j}+\binom{k+1}{2j}\right)E_{2k-2j+1}(x).
\end{aligned}
$$

By (36) and (37), we get

$$
(38)\qquad \sum_{j=0}^{[\frac{k}{2}]}\left(\binom{k}{2j}+\binom{k+1}{2j}\right)E_{2k-2j+1}(x)=x^k(x-1)^k(2x-1).
$$

Therefore, by (38), we obtain the following theorem.

Theorem 8. For $k\in\mathbb{N}$, we have

$$
\sum_{j=0}^{[\frac{k}{2}]}\left(\binom{k}{2j}+\binom{k+1}{2j}\right)E_{2k-2j+1}(x)=x^k(x-1)^k(2x-1).
$$

Let us take $x=0$ in (22). Then we have

$$
(39)\qquad \sum_{j=0}^{k+1}\left(\binom{k}{j}+(-1)^j\binom{k+1}{j}\right)E_{2k-j+1}=0,
$$

and

$$
\begin{aligned}
(40)\qquad \binom{k}{2j}+\binom{k+1}{2j}&=\frac{1}{k+1}\binom{k+1}{2j}(k-2j+1)+\frac{k+1}{k+1}\binom{k+1}{2j}\\
&=\frac{1}{k+1}\binom{k+1}{2j}(2k-2j+2).
\end{aligned}
$$

By (39) and (40), we get

$$0=\sum_{j=0}^{k+1}\left(\binom{k}{j}+(-1)^j\binom{k+1}{j}\right)E_{2k-j+1} \tag{41}$$

$$=\sum_{j=0}^{[\frac{k+1}{2}]}\left(\binom{k}{2j}+\binom{k+1}{2j}\right)E_{2k-2j+1}$$

$$=\sum_{j=0}^{[\frac{k+1}{2}]}\frac{1}{k+1}\binom{k+1}{2j}(2k-2j+2)E_{2k-2j+1}$$

$$=2E_{2k+1}+\frac{1}{k+1}\sum_{j=1}^{[\frac{k}{2}]}\binom{k+1}{2j}(2k-2j+2)E_{2k-2j+1}.$$

Thus, by (41), we obtain the following corollary.

Corollary 9. For $k\in\mathbb{N}$, we have

$$E_{2k+1}=-\frac{1}{2k+2}\sum_{j=1}^{[\frac{k}{2}]}\binom{k+1}{2j}(2k-2j+2)E_{2k-2j+1}.$$

3. p-adic integral on $\mathbb{Z}_p$ associated with Bernoulli and Euler polynomials

Let $m,k\in\mathbb{N}$. Then, by (4) and (10), we see that

$$I_1=\int_{\mathbb{Z}_p}x^k(x-1)^m d\mu_{-1}(x)+\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \tag{42}$$

$$=\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^{m-\ell}\int_{\mathbb{Z}_p}x^{k+\ell}d\mu_{-1}(x)+\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}$$

$$=\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^{m-\ell}E_{k+\ell}+\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}.$$

From (10), we have

(43)

$$I_1=\sum_{j=1}^{max\{k,m\}}\left(\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right)\frac{1}{k+m+1-j}\int_{\mathbb{Z}_p}B_{k+m-j+1}(x)d\mu_{-1}(x)$$

$$=\sum_{j=1}^{max\{k,m\}}\left(\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right)\frac{1}{k+m+1-j}$$

$$\times\sum_{\ell=0}^{k+m+1-j}\binom{k+m+1-j}{\ell}B_{k+m+1-j-\ell}E_{\ell}.$$

Therefore, by (42) and (43), we obtain the following theorem.

Theorem 10. For $m, k \in \mathbb{N}$, we have

$$\sum_{\ell=0}^{m}\binom{m}{\ell}(-1)^{m-\ell}E_{k+\ell}+\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}$$
$$=\sum_{j=1}^{max\{k,m\}}\left(\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right)\frac{1}{k+m+1-j}$$
$$\times\sum_{\ell=0}^{k+m+1-j}\binom{k+m+1-j}{\ell}B_{k+m+1-j-\ell}E_{\ell}.$$

Let

(44)
$$I_2=\int_{\mathbb{Z}_p}x^k(x-1)^k(x-\frac{1}{2})d\mu_{-1}(x)$$
$$=\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\int_{\mathbb{Z}_p}x^{2k+1-\ell}d\mu_{-1}(x)-\frac{1}{2}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}\int_{\mathbb{Z}_p}x^{2k-\ell}d\mu_{-1}(x)$$
$$=\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}E_{2k+1-l}-\frac{1}{2}\sum_{\ell=0}^{k}\binom{k}{\ell}(-1)^{\ell}E_{2k-\ell}.$$

From (13), we can derive the following equation (45).

$$(45)\quad I_2=\sum_{j=0}^{[\frac{k}{2}]}\left(\binom{k}{2j+1}+\binom{k+1}{2j+1}\right)\frac{1}{2k+1-2j}\int_{\mathbb{Z}_p}B_{2k+1-2j}(x)d\mu_{-1}(x)$$
$$=\sum_{j=0}^{[\frac{k}{2}]}\left(\binom{k}{2j+1}+\binom{k+1}{2j+1}\right)\frac{1}{2k+1-2j}$$
$$\times\sum_{\ell=0}^{2k+1-2j}\binom{2k+1-2j}{\ell}B_{2k+1-2j-\ell}E_{\ell}.$$

Therefore, by (44) and (45), we obtain the following proposition.

Proposition 11. For $k \in \mathbb{N}$, we have

$$\sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} E_{2k-\ell+1} - \frac{1}{2} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{\ell} E_{2k-\ell}$$
$$= \sum_{j=0}^{[\frac{k}{2}]} \left(\binom{k}{2j+1} + \binom{k+1}{2j+1} \right) \frac{1}{2k+1-2j}$$
$$\times \sum_{\ell=0}^{2k+1-2j} \binom{2k+1-2j}{\ell} B_{2k+1-2j-\ell} E_{\ell}.$$

From (4) and (5), we can derive the following equation (46).

$$\int_{\mathbb{Z}_p} x^{2n+1} d\mu_{-1}(x) = -\frac{1}{2n+2} \sum_{j=1}^{[\frac{n}{2}]} \binom{n+1}{2j} (2n-2j+2) \times \int_{\mathbb{Z}_p} x^{2n-2j+1} d\mu_{-1}(x). \tag{46}$$

By (46), we get

$$E_{2n+1} = -\frac{1}{2n+2} \sum_{j=1}^{[\frac{n}{2}]} \binom{n+1}{2j} (2n-2j+2) E_{2n-2j+1}, \quad for \quad n \in \mathbb{N}.$$

Now, we get

$$I_3 = 2 \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu(x) = 2 \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} B_{m+k-\ell}. \tag{47}$$

From (1) and (22), we have

$$\begin{aligned} I_3 &= \sum_{j=0}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^j \binom{m}{j} \right) \int_{\mathbb{Z}_p} E_{k+m-j}(x) d\mu(x) \\ &= \sum_{j=0}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^j \binom{m}{j} \right) \sum_{\ell=0}^{k+m-j} \binom{k+m-j}{\ell} E_{k+m-j-\ell} B_{\ell}. \end{aligned} \tag{48}$$

Therefore, by (47) and (48), we obtain the following proposition.

Proposition 12. For $m, k \in \mathbb{N}$, we have

$$2 \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} B_{m+k-\ell}$$
$$= \sum_{j=0}^{max\{k,m\}} \left(\binom{k}{j} + (-1)^j \binom{m}{j} \right) \sum_{\ell=0}^{k+m-j} \binom{k+m-j}{\ell} E_{k+m-j-\ell} B_{\ell}.$$

By (1), we easily see that

(49)

$$\int_{\mathbb{Z}_p} x^{2k} d\mu(x) = -\frac{1}{(2k+1)(k+1)} \sum_{j=0}^{[\frac{k}{2}]} (2k-2j+1) \binom{k+1}{2j+1} \int_{\mathbb{Z}_p} x^{2k-2j} d\mu(x).$$

Thus, by (49), we get

$$B_{2k} = -\frac{1}{(2k+1)(k+1)} \sum_{j=0}^{[\frac{k}{2}]} (2k-2j+1) \binom{k+1}{2j+1} B_{2k-2j} \text{ for } k \in \mathbb{N}.$$

Let us consider the following p-adic integral on $\mathbb{Z}_p$:

$$I_4 = \int_{\mathbb{Z}_p} x^k (x-1)^k d\mu(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^l B_{2k-l}. \tag{50}$$

From (34), we note that

$$I_4 = \sum_{j=0}^{[\frac{k}{2}]} \binom{k}{2j} \sum_{l=0}^{2k-2j} \binom{2k-2j}{l} E_{2k-2j-l} B_l. \tag{51}$$

Therefore, by (50) and (51), we obtain the following proposition.

Proposition 13. For $k \in \mathbb{N}$, we have

$$\sum_{l=0}^{k} \binom{k}{l} (-1)^l B_{2k-l} = \sum_{j=0}^{[\frac{k}{2}]} \sum_{l=0}^{2k-2j} \binom{k}{2j} \binom{2k-2j}{l} E_{2k-2j-l} B_l.$$

For $m, k \in \mathbb{N}$ with $m, k \geq 2$, we have

(52)

$$\begin{aligned} I_5 &= 2\int_{\mathbb{Z}_p} x^{k-1} (x-1)^{m-1} \left((k+m)x - k\right) d\mu(x) \\ &= 2\sum_{j=0}^{m-1} (k+m) \binom{m-1}{l} (-1)^l B_{k+m-l-1} - 2k \sum_{l=0}^{m-1} \binom{m-1}{l} (-1)^l B_{m+k-l-2}. \end{aligned}$$

By (33), we get

$$\begin{aligned} I_5 = &\sum_{j=0}^{max\{k,m\}} (k+m-j) \left(\binom{k}{j} + (-1)^j \binom{m}{j} \right) \\ &\times \sum_{l=0}^{k+m-j-1} \binom{k+m-j-1}{l} E_{k+m-j-1-l} B_l. \end{aligned} \tag{53}$$

Therefore, by (52) and (53), we can also obtain the following identity.

(53)

$$2(k+m)\sum_{j=0}^{m-1}\binom{m-1}{l}(-1)^l B_{k+m-l-1} - 2k\sum_{l=0}^{m-1}\binom{m-1}{l}(-1)^l B_{m+k-l-2}$$

$$= \sum_{j=0}^{max\{k,m\}} (k+m-j)\left(\binom{k}{j}+(-1)^j\binom{m}{j}\right)$$

$$\times \sum_{l=0}^{k+m-j-1}\binom{k+m-j-1}{l}E_{k+m-j-1-l}B_l,$$

where $m, k \in \mathbb{N}$ with $m, k \geq 2$.

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