Product Cordial Sets of Long Grids

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Abstract

A binary vertex coloring (labeling) $f : V(G) \to \mathbb{Z}_2$ of a graph G is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. This friendly labeling induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(uv) = f(u)f(v)$ for all $uv \in E(G)$. Let $e_f(i) = |\{uv \in E(G) : f^*(uv) = i\}|$ be the number of edges of G that are labeled i. Product-cordial index of the labeling f is the number $pc(f) = |e_f(0) - e_f(1)|$. The product-cordial set of the graph G, denoted by $PC(G)$, is defined by

$$PC(G) = \{pc(f) : \ f \text{ is a friendly labeling of } G\}.$$

In this paper, we will determine the product-cordial sets of long grids $P_m \times P_n$, introduce a class of fully product-cordial trees and suggest new research directions in this topic.

Key Words: friendly coloring, product-cordial index, product-cordial set, grid.
AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [6]. Let G be a graph and $f : V(G) \to \mathbb{Z}_2$ be a binary vertex coloring (labeling) of G. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1.
Any friendly coloring $f : V(G) \to \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(xy) = f(x)f(y)$ $\forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f^{*-1}(i)|$ be the number of edges of G that are labeled i. The number $pc(f) = |e_f(1) - e_f(0)|$ is called the *product-cordial index (or pc-index)* of f. The *product-cordial set (or pc-set)* of the graph G, denoted by $PC(G)$, is defined by

$$PC(G) = \{pc(f) : \ f \text{ is a friendly vertex coloring of } G\ \}.$$

To illustrate the above concepts, consider the graph G of Figure 1, which has 8 vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that four vertices be labeled 0 and the other four 1.

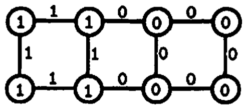
G: 

Figure 1: An example of product-cordial labeling of G.

Figure 1 also shows the associated edge labeling of G, where four edges have label 1 while the other 6 edges have labels 0. Therefore, the product-cordial index (or pc-index) of this labeling is $6 - 4 = 2$. It is easy to see that $PC(G) = \{2, 4, 6, 8, 10\}$. The friendly colorings of G that provide the other four pc-indices are presented in Figure 2.

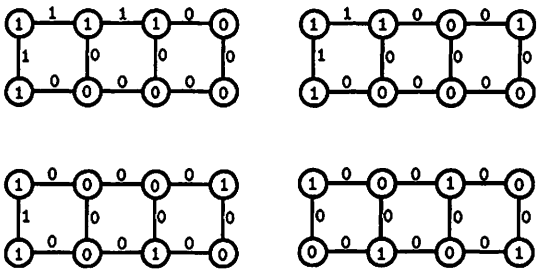


Figure 2: Four friendly labelings of G with pc-indices 4, 6, 8 and 10.

In what follows, whenever there is no ambiguity, we will suppress the index f and denote $e_f(i)$ by simply $e(i)$. For a graph $G = (p, q)$ of size q, and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G, we have

$$pc(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \qquad (1.1)$$

Therefore, to find the pc-index of f it is enough to find $e_f(1)$ (or $e_f(0)$). Moreover, to determine the pc-set of G it is enough to compute $e_f(1)$ for different friendly colorings of G. Another immediate consequence of (1.1) is the following useful fact:

Observation 1.1. *For a graph G of size q, $PC(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.*

Definition 1.2. *A graph G of size q is said to be fully product-cordial (fully pc) if*

$$PC(G) = \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}.$$

For example, the graph G of Figure 1 is not fully pc. However, P_n, the path of order n, is fully pc. In case of P_n, it is easy to observe that $e_f(1) = 0, 1, \cdots, \lfloor \frac{n-1}{2} \rfloor$, which proves that

Theorem 1.3. *For any $n \geq 2$, the graph P_n is fully product-cordial. That is, $PC(P_n) = \{n - 1 - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}$.*

The different friendly labelings of P_7 that provide its pc-set are illustrated in Figure 3.

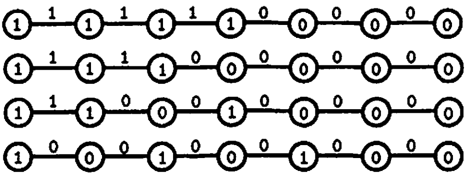


Figure 3: $PC(P_7) = \{0, 2, 4, 6\}$.

In 1987, I. Cahit [2, 3, 4] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labeling. Given a friendly labeling $f : V(G) \to \mathbb{Z}_2$ of a graph G, Cahit introduced an edge labeling $f_+ : E(G) \to \mathbb{Z}_2$ by $f_+(uv) = |f(u) - f(v)|$ and defined the cordial index $c(f)$ of f to be $|f_+^{-1}(0) - f_+^{-1}(1)|$. A graph is called cordial if it admits a friendly labeling with cordial index 0 or 1. Cahit, among other facts, proved that

1. Every tree is cordial;
2. The complete graph K_n is cordial if and only if $n \leq 3$;
3. The complete bipartite graph $K(m, n)$ is cordial $(m, n \in \mathbb{N})$;
4. The wheel W_n is cordial if and only if $n \not\equiv 3 \pmod 4$;
5. In an Eulerian graph $G = (p, q)$ if $p \equiv 0 \pmod 4$, then it is not cordial.

M. Hovay [9], later generalized the concept of cordial graphs and introduced A-cordial labelings, where A is an abelian group. A graph G is said to be A-cordial if it admits a labeling $f : V(G) \to A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 5, 8, 10, 11, 14, 19].
Product cordial labeling of a graph was introduced by Sundaram, Ponraj and Somasundaran [22]. They call a graph G product-cordial if it admits a friendly labeling whose product-cordial index is at most 1. Then Sundaram, Ponraj and Somasundaran [22, 23, 24] investigated whether certain graphs such as trees, cycles, complete

graphs, wheels, etc. are product-cordial. Later E. Salehi [15] introduced the concept of product-cordial set (or pc-set) of a graph and determined the pc-sets of certain classes of graphs such as: complete graphs, complete bipartite graphs, stars and double stars, cycles, and wheels.

2 Trees with Perfect Matching

In general, for a friendly coloring $f : V(G) \to \mathbb{Z}_2$ of a graph G, it is not necessarily true that $e_f(0) \geq e_f(1)$. For example, let $n > 3$ and consider the coronation of the complete graph K_n with K_1, as indicated in Figure 4.

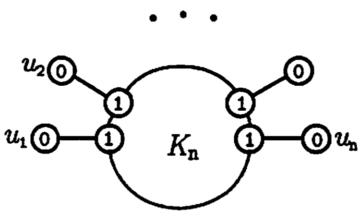


Figure 4: A friendly coloring with $e(1) > e(0)$.

If we color all vertices of K_n by 1 and the end-vertices by 0, then $e(1) = n(n-1)/2$ while $e(0) = n$. However, for certain graphs one can prove that the number of edges labeled 0 is bigger than the number of edges labeled 1. Trees are among such graphs as we will see in the following theorem:

Theorem 2.1. *For any tree T and any friendly coloring of T,* $e(0) \geq e(1)$.

Proof. The statement is true for trees of order $n = 1, 2, 3$. Let T be a tree of order $n \geq 4$ and assume to the contrary that $e(1) > e(0) \geq 2$. Then at least $e(1) + 1$ vertices of T are labeled with 1. Since the coloring is friendly, at least $e(1)$ vertices of T are labeled with 0. This implies that $n \geq 2e(1) + 1$ or $|E| \geq 2e(1)$. Therefore, $2e(1) \leq |E| = e(1) + e(0) < 2e(1)$, a contradiction. □

Definition 2.2. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M.

Note that every graph with perfect matching has even number of vertices. Moreover, if a graph G has a perfect matching M, then every pendent edge of G is in M. Another useful observation about the trees with perfect matching is that they contain at least one P_3, the path of order 3, pendant. That is, there are vertices $u \sim v \sim w$ such that $\deg u = 1$ and $\deg v = 2$. In fact, the two end portions of the longest path of T would have P_3 pendants. Here is another example of a class of fully product-cordial graphs:

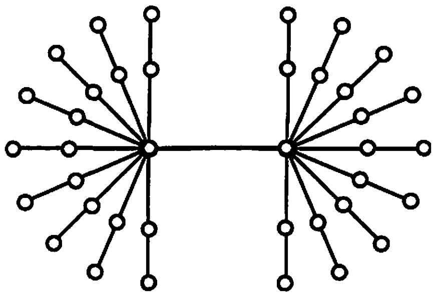


Figure 5: An example of a tree with perfect matching that is fully pc.

Theorem 2.3. *Any tree T of order p with a perfect matching is fully product-cordial. That is,*

$$PC(T) = \{1, 3, 5, \cdots, p-1\}.$$

Proof. Let T be a tree with perfect matching M and $|M| = m$. We proceed by induction on m. Clearly, the theorem is true for $m = 1, 2$. Suppose it is true for any perfect matching tree with $|M| = m$ and let S be a tree with perfect matching M' such that $|M'| = m + 1$. Among the elements of M' there is at least one terminal edge uv of the tree S such that $u \sim v \sim w$, $\deg u = 1$ and $\deg v = 2$. Now if we delete the vertices u and v from S, the result would be a tree T with perfect matching $M' - uv$ and $|M' - uv| = m$. Therefore, by the induction hypothesis, $PC(T) = \{1, 3, \cdots, 2m-1\}$. We need to show that $PC(S) = \{1, 3, \cdots, 2m-1, 2m+1\}$. Consider a friendly coloring $f : V(T) \to \mathbb{Z}_2$ of T and extend it to $g : V(S) \to \mathbb{Z}_2$ by defining $g(v) = 0$, $g(u) = 1$. This becomes a friendly coloring of S with $e_g(1) = e_f(1)$ and $e_g(0) = 2 + e_f(0)$. Therefore, $pc(g) = 2 + pc(f)$. That is, $2 + PC(T) = \{3, 5, \cdots, 2m+1\} \subseteq PC(S)$. To show that $1 \in PC(S)$, we choose a subtree of S with $m+1$ vertices and label all these vertices by 1 and other vertices of S by 0. This is a friendly labeling of S with $e(0) = m+1$ and $e(1) = m$ and has index 1. □

Theorem 2.3 provides a sufficient condition for fully pc trees. However, this condition is not necessary. A simple example would be P_{2n+1} which is fully pc and does not have a perfect matching. We wish to present the following example, illustrated in Figure 6, that can easily be generalized to construct other classes of fully pc trees.

3 Grids and PC-Sets of Ladders

For any $m, n \geq 2$, the Cartesian product $P_m \times P_n$ of two paths is called a *grid*. The grid $P_m \times P_n$ has mn vertices and $2mn - m - n$ edges. Let $v_1 \sim v_2 \sim \cdots \sim v_m$ be the vertices of P_m and $w_1 \sim w_2 \sim \cdots \sim w_n$ be vertices of P_n. In what follows, for convenience, we denote the vertex (v_i, w_j) by u_{ij}, the subgraph $u_{i1} \sim u_{i2} \sim \cdots \sim u_{in}$

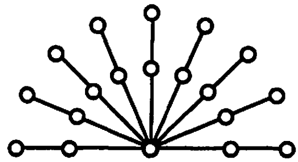


Figure 6: A fully pc tree with pc-set $\{0, 2, 4, \cdots, 16, 18\}$.

of $P_m \times P_n$ by ρ_i (i^{th} Row), and the subgraph $u_{1j} \sim u_{2j} \sim \cdots \sim u_{mj}$ of $P_m \times P_n$ by κ_j (j^{th} Column). Note that two vertices u_{ij} and u_{lk} are adjacent if the difference between $i+j$ and $l+k$ is 1. This leads to our first observation:

Theorem 3.1. *The grid $P_m \times P_n$ has the maximum pc-index $2mn - m - n$.*

Proof. Consider the friendly coloring $f : V(P_m \times P_n) \to \mathbb{Z}_2$ that is 1 on u_{ij} if $i+j$ is even and 0 if $i+j$ is odd. That is, $f(u_{ij}) = \dfrac{1+(-1)^{i+j}}{2}$. Since every two adjacent vertices have opposite colorings, the induced product-cordial edge labeling is identically 0. Therefore, $pc(f) = 2mn - m - n$. □

The coloring that is presented in the proof of Theorem 3.1 will be referred to as *alternating*, by which we mean every two adjacent vertices have different colors.

Theorem 3.2. *For any friendly coloring of $P_m \times P_n$ with $2 \leq m \leq n$, $e(0) > e(1)$.*

Proof. Since $e(0) + e(1) = |E(G)|$ is fixed, it is enough to show that the maximum value of $e(1)$ is less than the minimum value of $e(0)$. Note that the maximum value of $e(1)$ occurs when all the vertices labeled 1 are clustered (adjacent). Likewise, the minimum value of $e(0)$ occurs when all the vertices labeled 0 are clustered. Now, let r, s and t denote the number of edges incident with two vertices that are both labeled 1, have different labeling and are both labeled 0, respectively. We consider two cases:

Case I: $4 \leq 2m \leq n$. Without loss of generality we may assume that all the vertices labeled 1 are vertices of the first $\lfloor n/2 \rfloor$ columns and, if n is odd, the first $\lceil m/2 \rceil$ vertices of the middle column are labeled 1. Thus, $r = t + 1 - (-1)^{mn}$ and $s = m + \frac{1-(-1)^n}{2}$. Therefore,

$$e(0) - e(1) = s + t - r = m + \frac{1-(-1)^n}{2} - 1 + (-1)^{mn} > 0.$$

Case II: $7 \leq m \leq n < 2m$. The conditions $m, n \geq 7$ imply that $(m-2)(n-2) \geq \frac{mn}{2}$. Without loss of generality we may assume that all the vertices labeled 1 are clustered inside the grid and consequently have degree 4. Let H be the subgraph of G induced by all edges that are incident with at least one vertex labeled 1. Then H has $r+s$ edges, $\lceil mn/2 \rceil$ vertices of degree 4 and s end-vertices. Hence

$$\sum_{v \in V(H)} deg(v) = 4\left\lceil \frac{mn}{2} \right\rceil + s = 2r + 2s \text{ or } r = 2\left\lceil \frac{mn}{2} \right\rceil - \frac{s}{2}.$$

Also, we note that the minimum value of s occurs when all the vertices labeled 1 would form a square subgrid. Therefore,

$$s \geq 4\sqrt{\left\lceil \frac{mn}{2} \right\rceil} \text{ and } r \leq 2\left\lceil \frac{mn}{2} \right\rceil - 2\sqrt{\left\lceil \frac{mn}{2} \right\rceil}.$$

Since $r + s + t = |E(G)|$, we have

$$\begin{aligned} e(0) - e(1) &= s + t - r = r + s + t - 2r \\ &\geq 2mn - m - n - 4\left\lceil \frac{mn}{2} \right\rceil + 4\sqrt{\left\lceil \frac{mn}{2} \right\rceil} \\ &\geq 4\sqrt{\frac{mn}{2}} - m - n - 2. \end{aligned}$$

We observe that the function $f(x, y) = 4\sqrt{xy/2} - x - y - 2$ is always positive in the region defined by inequalities $7 \leq x \leq y \leq 2x$ which concludes that $e(0) > e(1)$. Cases I and II do not apply to a finite number of grids, however, the result holds in general and can be verified directly for those cases. □

Corollary 3.3. *For any $m, n \geq 2$, the product-cordial set of $P_m \times P_n$ is* $\{2mn - m - n - 2e_f(1) : f \text{ is a friendly coloring of } P_m \times P_n\}$.

Proof. Note that the number of edges of $P_m \times P_n$ is $2mn - m - n$ and for any friendly coloring f, $pc(f) = |e(0) - e(1)| = e(0) - e(1) = 2mn - m - n - 2e(1)$. □

Before stating the main result concerning grids in the next section, we consider the special case of a ladder, which illustrates the technique and provides us with a tool for the proof of the general case.

Theorem 3.4. $PC(P_2 \times P_n) = \{3n - 2 - 2k : 0 \leq k \leq \lfloor 3n/2 \rfloor - 2\}$.

Proof. For any integer k with $0 \leq k \leq \lfloor 3n/2 \rfloor - 2$, we present a friendly coloring f such that $e_f(1) = k$. By Theorem 3.1, we may assume that $k \geq 1$. We consider the following three cases:

A. $k = 3a + 1$. Since $k \leq \lfloor 3n/2 \rfloor - 2$, then $a \leq \frac{n}{2} - 1$. We label all the vertices of the first $a + 1$ columns by 1 (note this yields k edges labeled 1), label all the vertices of the subsequent a+1 columns by 0 and alternate the coloring of the remaining vertices, as illustrated in Figure 7. That is,

$$f(v_{i,j}) = \begin{cases} 1 & \text{if } 1 \leq j \leq a + 1; \\ 0 & \text{if } a + 2 \leq j \leq 2a + 2; \\ \dfrac{1 - (-1)^{i+j}}{2} & \text{if } 2a + 3 \leq j \leq n. \end{cases}$$

The coloring f is friendly and $e_f(1) = k$.

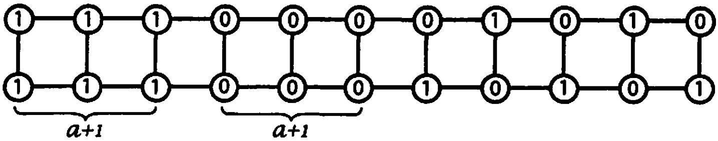


Figure 7: A friendly coloring of $P_2 \times P_{11}$ with index 7.

B. $k = 3a + 2$, where $0 \le a \le \frac{n}{2} - 1$. We modify the coloring of Case **A** on the last two columns of $P_2 \times P_n$ to obtain an extra edge labeled 1. Specifically, let f be defined by

$$f(v_{i,j}) = \begin{cases} 1 & \text{if } 1 \le j \le a+1 \text{ or } j = n; \\ 0 & \text{if } a+2 \le j \le 2a+2 \text{ or } j = n-1; \\ \dfrac{1-(-1)^{i+j}}{2} & \text{if } 2a+3 \le j \le n-2. \end{cases}$$

The coloring f is friendly and $e_f(1) = k$. This friendly coloring is illustrated in Figure 8.

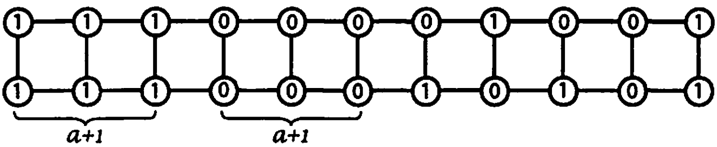


Figure 8: A friendly coloring of $P_2 \times P_{11}$ with index 8.

C. $k = 3a + 3$, where $0 \le a \le \frac{n}{2} - 1$. This time we alter the coloring of Case **A** on the last three columns of $P_2 \times P_n$ to produce two additional edges labeled 1:

$$f(v_{i,j}) = \begin{cases} 1 & \text{if } 1 \le j \le a+1 \text{ or } j = n; \\ 0 & \text{if } a+2 \le j \le 2a+2 \text{ or } j = n-2; \\ \dfrac{1-(-1)^{i+j}}{2} & \text{if } 2a+3 \le j \le n-3 \text{ or } j = n-1. \end{cases}$$

The coloring f is friendly and $e_f(1) = k$. This friendly coloring is illustrated in Figure 9.

This proves that $\{3n - 2 - 2k : 0 \le k \le \lfloor 3n/2 \rfloor - 2\} \subseteq PC(P_2 \times P_n)$. Note that by observation 1.1, $PC(P_2 \times P_n) \subseteq \{3n - 2 - 2k : 0 \le k \le \lfloor 3n/2 \rfloor - 1\}$. To complete the proof, it is enough to show that $k \ne \lfloor 3n/2 \rfloor - 1$, which follows from Theorem 3.2. □

Corollary 3.5. *$P_2 \times P_n$ is not fully product-cordial.*

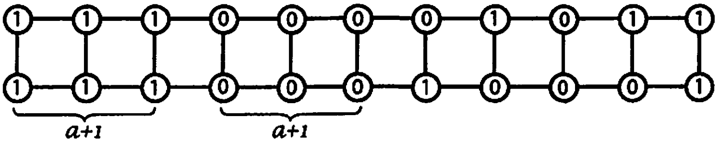


Figure 9: A friendly coloring of $P_2 \times P_{11}$ with index 9.

Proof. It follows from the previous theorem that

$$PC(P_2 \times P_n) = \begin{cases} \{2,4,6,\ldots,3n-2\} & \text{if} \quad \text{n is even;} \\ \{3,5,7,\ldots,3n-2\} & \text{if} \quad \text{n is odd.} \end{cases}$$

4 PC Sets of Long Grids

By a *long grid* we mean the graph $P_n \times P_m$ with $m \geq 2n$. In this section we determine the product-cordial sets of long grids. Before stating the main result, we prove some preliminaries.

Lemma 4.1. *For any grid $P_n \times P_4$ and any integer k with $0 \leq k \leq 2n-2$, there is a friendly coloring such that $e(1) = k$.*

Proof. We consider two cases:
Case I: $0 \leq k \leq n-1$. Label $k+1$ top vertices of κ_4 by 1 (note that this produces k edges labeled 1), $k+1$ top vertices of κ_3 by 0 and alternate coloring of the remaining vertices of $P_n \times P_4$.
Case II: $n \leq k \leq 2n-2$. Label all vertices of κ_4 and $k-n+2$ top vertices of κ_2 1 (note that this produces k edges labeled 1), all vertices of κ_3 and $k-n+2$ top vertices of κ_1 0 and alternate coloring on the remaining vertices of the graph. In each case the coloring is friendly and $e(1) = k$. □

Remark 4.2. Note that the above result is true for any grid $P_n \times P_m$ whenever $m \geq 4$. We simply attach $P_n \times P_{m-4}$ that has alternating coloring to $P_n \times P_4$ by joining the vertices of the last column of $P_n \times P_{m-4}$ to the corresponding vertices of the first column of $P_n \times P_4$, keeping in mind that alternating color of $P_n \times P_{m-4}$ be consistent with the coloring of the first column of $P_n \times P_4$.

Lemma 4.3. *For any long grid $P_n \times P_{2m}$ and any integer k with $2mn-3n-m+1 \leq k \leq 2mn-n-m$, there is a friendly coloring such that $e(1) = k$. Moreover, the maximum value of $e(1)$ is $2mn-m-n$.*

Proof. We consider four cases:
Case I: $k = 2mn-n-m$. The maximum value of $e(1)$, which is $2mn-n-m$, is obtained when all vertices of a subgraph $P_n \times P_m$ are labeled 1 and the remaining vertices of $P_n \times P_{2m}$ are labeled 0. Without loss of generality we may assume that the subgraph $P_n \times P_m$ is induced by the first m columns of $P_n \times P_{2m}$.

Case II: $k = 2mn - n - m - 1$. In the labeling presented in Case I, exchange the colorings of $u_{1,m}$ and $u_{2,m+1}$, which reduces the number of edges labeled 1 by one.
Case III: $k = 2mn - n - m - a$, where $2 \leq a \leq n$. In the labeling presented in Case I, exchange the colorings of $u_{1,m}, \ldots, u_{a,m}$ and $u_{1,m+1}, \ldots, u_{a,m+1}$, which reduces the number of edges labeled 1 by a.
Case IV: $k = 2mn - 3n + m + 1 + a$, where $0 \leq a \leq n - 1$. Label all the vertices of the first $m - 1$ columns of $P_n \times P_{2m}$ by 1 (note that this produces $2mn - 3n - m + 1$ edges labeled 1), all the vertices of $m - 1$ subsequent columns by 0 and let the last two columns of $P_n \times P_{2m}$ have any friendly coloring of $P_n \times P_2$ that has a edges labeled 1, existence of which is ensured by Theorem 3.4. In each case, the coloring is friendly and $e(1) = k$. □

Theorem 3.2 indicates that for any graph G and any friendly coloring f, $pc(f) = |E(G)| - 2e_f(1)$. It follows from the previous lemma that the minimum product-cordial index of $P_n \times P_{2m}$ is n.

Lemma 4.4. $PC(P_n \times P_{2n}) = \{4n^2 - 3n - 2k : 0 \leq k \leq 2n^2 - 2n\}$.

Proof. Note that by Corollary 3.3,

$$PC(P_n \times P_{2n}) = \{4n^2 - 3n - 2e_f(1) : f \text{ is a friendly coloring of } P_n \times P_{2n}\}.$$

To prove the lemma, one has to show that for any k with $0 \leq k \leq 2n^2 - 2n$, there is a friendly coloring f such that $e_f(1) = k$. By Lemmas 4.1 and 4.3 it suffices to consider k with $2n - 1 \leq k \leq 2n^2 - 4n$. Let $k = (n - 1) + (2n - 1)a + r$, where $0 \leq a \leq n - 3$ and $0 \leq r \leq 2n - 2$. Consider the coloring of $P_n \times P_{2n}$ that labels the vertices as follow:

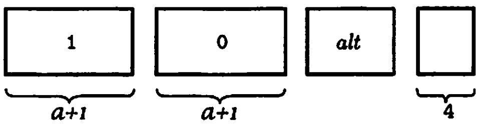


Figure 10: A typical friendly coloring of $P_n \times P_{2n}$.

(1) Label all the vertices of the first $a+1$ columns by 1. The corresponding induced edge labeling will produce $n - 1 + (2n - 1)a$ edges that are labeled 1.

(2) Label all the vertices of the columns $a + 2$ through $2a + 2$ by 0.

(3) Label the vertices of the last four columns according to the Lemma 4.1 to produce r edges with lable 1.

(4) Finally, use alternating labeling for columns $2a + 3$ through $2n - 5$ such that the alternation be consistent with the labels of $(2n - 4)^{\text{th}}$ column.

This coloring is friendly and $e(1) = k$. □

Theorem 4.5. *For any $m \geq n$, the product-cordial set of the long grid $P_n \times P_{2m}$ is* $\{4mn - n - 2m - 2k : 0 \leq k \leq 2mn - n - m\}$.

Proof. We proceed by induction on m. By Lemma 4.4, the statement is true for $m = n$. Suppose it is true for the long grid $P_n \times P_{2m}$ with $m \geq n$. We wish to show that the statement of the theorem is true for $P_n \times P_{2m+2}$.
Let f be any friendly labeling of $P_n \times P_{2m}$. We extend f to a friendly labeling g of $P_n \times P_{2m+2}$ by labeling all the vertices of column $2m+1$ by 0 and all the vertices of column $2m+2$ by 1. Then $e_g(1) = e_f(1) + n - 1$. This implies that for any k with $n-1 \leq k \leq 2mn - m - 1$ there is a friendly labeling of $P_n \times P_{2m+2}$ such that $e(1) = k$. On the other hand, in the view of Lemmas 4.1 and 4.3 it is enough to consider those values of k that satisfy $2n-1 \leq k \leq 2mn + n - m$. This proves the theorem, because $[2n-1, 2mn+n-m] \subseteq [n-1, 2mn+2n-m-1]$. □

Theorem 4.6. *For any $m \geq n$, the product-cordial set of the long grid $P_n \times P_{2m+1}$ is*

$$\{4mn + n - 2m - 1 - 2k \,:\, 0 \leq k \leq 2mn - m - \frac{1+(-1)^n}{2}\}.$$

Proof. By Corollary 3.3, for any friendly coloring f of a grid G, $pc(f) = |E(G)| - 2e_f(1)$. Therefore, the minimum product-cordial index of $P_n \times P_{2m+1}$ is produced by the maximum value of $e(1)$. This maximum value is obtained when all vertices of a subgraph of $P_n \times P_{2m+1}$ induced by the vertices
$\{u_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\} \cup \{u_{ij} : 1 \leq i \leq \lceil n/2 \rceil \text{ and } j = m+1\}$ are labeled 1 and the remaining vertices of $P_n \times P_{2m+1}$ are labeled 0. That is, the maximum value of $e(1)$ is $2mn - m - \frac{1+(-1)^n}{2}$, hence the minimum pc-index is $n + (-1)^n$. To prove the theorem, one has to show that for any k with $0 \leq k \leq 2mn - m - \frac{1+(-1)^n}{2}$ there is a friendly coloring such that $e(1) = k$. By Lemma 4.1 it suffices to consider the values of k with $2n-1 \leq k \leq 2mn - m - \frac{1+(-1)^n}{2}$.
Let f be any friendly labeling of $P_n \times P_{2m}$ such that all vertices of column $2m$ are labeled 1. We extend f to a friendly labeling g of $P_n \times P_{2m+1}$ by labeling all the top $\lceil n/2 \rceil$ vertices of the last column of $P_n \times P_{2m+1}$ by 1 and the remaining vertices of the last column by 0. Then $e_g(1) = e_f(1) + 2\lceil n/2 \rceil - 1 = e_f(1) + n - \frac{1+(-1)^n}{2}$.
This together with Theorem 4.5 imply that for any k with $2n - 1 - \frac{1+(-1)^n}{2} \leq k \leq 2mn - m - \frac{1+(-1)^n}{2}$ there is friendly coloring of $P_m \times P_{2m+1}$ such that $e(1) = k$.
The proof of the theorem is complete, because
$[2n-1, 2mn - m - \frac{1+(-1)^n}{2}] \subseteq [2n - 1 - \frac{1+(-1)^n}{2}, 2mn - m - \frac{1+(-1)^n}{2}]$. □

Corollary 4.7. *The long grid $P_n \times P_m$, $m \geq 2n$, is not fully product-cordial.*

Proof. It follows from Theorems 4.5 and 4.6 that

$$PC(P_n \times P_m) = \begin{cases} \{n, n+2, \ldots, 2mn - n - m\} & \text{if } m \text{ is even;} \\ \{n+1, n+3, \ldots, 2mn - n - m\} & \text{if } m \text{ is odd, } n \text{ is even;} \\ \{n-1, n+1, \ldots, 2mn - n - m\} & \text{if } m \text{ is odd, } n \text{ is odd.} \end{cases}$$

Examples 4.8.

(a) The pc-set of the graph in Figure 5 is $\{35 - 2k : 0 \leq k \leq 14\}$. Because, it is a tree with perfect matching, hence it is fully product-cordial.

(b) $PC(P_2 \times P_7) = \{3, 5, 7, \cdots, 19\}$.

(c) $PC(P_3 \times P_8) = \{37 - 2k : 0 \leq k \leq 17\} = \{3, 5, \cdots, 37\}$.

(d) $PC(P_4 \times P_7) = \{45 - 2k : 0 \leq k \leq 20\} = \{5, 7, \cdots, 45\}$.

(e) $PC(P_5 \times P_7) = \{58 - 2k : 0 \leq k \leq 27\} = \{4, 6, \cdots, 58\}$.

5 Suggestion for Future Research

For the general grid $P_n \times P_m$, depending on the parity of m, its pc-set would *contain* the sets determined in Theorems 4.5 and 4.6. However, we might not have equality. For example, $PC(P_7 \times P_7) = \{4, 6, 8, \ldots, 84\}$, while if we apply Theorem 4.6, we would only obtain $\{6, 8, \ldots, 84\}$ which does not provide the smallest index 4. We wish to find a formula that would apply to all grids.
Also, in this paper, we presented a class of trees, perfect matching trees (Theorem 2.3), that are fully product-cordial. Identification of other fully pc graphs as well as finding necessary and sufficient conditions for fully pc trees would be another research direction.

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