Large sets of $K_{2,2}$-decomposition of complete bipartite graphs*

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Abstract: Let H, G be two graphs, where G is a simple subgraph of H. A G-decomposition of H, denoted by G-$GD_\lambda(H)$, is a partition of all the edges of λH into subgraphs (called G-*blocks*), each of which is isomorphic to G. A *large set* of G-$GD_\lambda(H)$, denoted by G-$LGD_\lambda(H)$, is a partition of all subgraphs isomorphic to G of H into G-$GD_\lambda(H)$s. In this paper, we determine the existence spectrums for $K_{2,2}$-$LGD_\lambda(K_{m,n})$.

key words: large set; $K_{2,2}$-decomposition; complete bipartite graph

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges $\{x, y\}$. Let $\lambda K_{n_1,n_2,\cdots,n_h}$ be a *complete multipartite graph* whose vertex set X consists of h disjoint sets $X_1, \cdots, X_h$, where $|X_i| = n_i$ and any two vertices x and y from different sets X_i and X_j are joined by exactly λ edges $\{x, y\}$.

*Research supported by NSFC Grant 11171089, NSFC Grant 10901051, NSFH Grant A2010001481 and Science Foundation Of Hebei Normal University.

Let H, G be two graphs, where G is a simple subgraph of H. An G-$GD_\lambda(H)$ is a partition of all the edges of λH into subgraphs (called *G-blocks*), each of which is isomorphic to G. The G-$GD_\lambda(H)$ is named as *G-decomposition* (or *G-design*) of H. For $H = K_n$ and some simple graphs of G, such as cycle C_k, path P_k, star S_k, k-cube, the graphs with at most five vertices and some graphs with six vertices, the existence of these G-decompositions has been solved(see [2]).

A *large set* of G-$GD_\lambda(H)$, denoted by G-$LGD_\lambda(H)$, is a partition of all subgraphs isomorphic to G of H into G-$GD_\lambda(H)$s. For $\lambda = 1$, the index 1 is often omitted.

A *Steiner triple system* of order n, denoted by $STS(n)$, is a pair $(X, \mathcal{B})$, where X is an n-set and $\mathcal{B}$ is a collection of triples (called *blocks*) on X such that every pair from X appears exactly in one block of $\mathcal{B}$. It is easy to see that an $STS(n)$ is just a C_3-$GD(K_n)$ and a *large set of Steiner triple system* $LSTS(n)$ is just a C_3-$LGD(K_n)$. The existence has been solved by J. Lu and L. Teirlinck (see [6-8]). From then on, the existence problems of *large set* of G-$GD_\lambda(H)$ have been widely researched, see [1,3-5,9-14].

A subgraph H of G is called a *spanning subgraph* of G if $V(H) = V(G)$. A λ-fold *F-factor* of G, is a spanning subgraph of G, which can be partitioned into copies of F (called *F-blocks*), such that each vertex of $V(G)$ appears exactly in λ *F-blocks*. A *λ-fold F-factorization* of G is a set of edge-disjoint *λ-fold F-factors* of G, whose edge sets partition the edges of G. For $\lambda = 1$, it is called an *F-factorization* of G. Particularly, if F is just an edge of G, then the F-factor is called a *one-factor* of G, and the corresponding F-factorization is called a *one-factorization* of G.

A *k-cycle*, denoted by $(x_1, x_2, \cdots, x_k)$, is a subgraph of K_v, which consists of k ($\leq v$) distinct points $x_1, x_2, \cdots, x_k$ and k edges $\{x_1, x_2\}, \cdots, \{x_{k-1}, x_k\}, \{x_k, x_1\}$. When $k = v$, it is called a *Hamilton cycle* of K_v. A *k-cycle system* of order v and index λ, $CS(v, k, \lambda)$, is a collection $\mathcal{C}$ of k-cycles of K_v, such that each edge of K_v appears exactly in λ members of $\mathcal{C}$. In particular, a $CS(v, v, 1)$ is called a *Hamilton cycle decomposition* of K_v.

Lemma 1.1[2] *For $n \geq 1$, there exist a one-factorization of K_{2n} and a*

Hamilton cycle decomposition of K_{2n+1}.

In this paper, we will investigate the existence of $K_{2,2}$-$LGD_\lambda(K_{m,n})$ and obtain its existence spectrum.

2 Main Constructions

A $K_{2,2}$-$GD_\lambda(K_{m,n})$ consists of $\frac{\lambda mn}{4}$ $K_{2,2}$-blocks, each of which consists of four vertices of degree 2. An $K_{2,2}$-$LGD_\lambda(K_{m,n})$ consists of $\frac{(m-1)(n-1)}{\lambda}$ disjoint $K_{2,2}$-$GD_\lambda(K_{m,n})$s. So, we have the following results.

Theorem 2.1 *There exists a* $K_{2,2}$-$LGD_\lambda(K_{m,n})$ *only if* $4|\lambda mn, 2|\lambda m, 2|\lambda n$ *and* $\lambda|(m-1)(n-1)$.

Therefore, in order to determine the existence spectrum for $K_{2,2}$-LGD_λ $(K_{m,n})$, it is enough to construct $K_{2,2}$-$LGD(K_{2m,2n})$, $K_{2,2}$-$LGD_2(K_{2m+1,2n})$, $K_{2,2}$-$LGD_4(K_{4m+1,2n+1})$ and $K_{2,2}$-$LGD_4(K_{4m-1,2n+1})$ for any positive integers m and n.

Theorem 2.2 *There exists a* $K_{2,2}$-$LGD(K_{2m,2n})$ *for any positive integers* m *and* n.

Proof. By Lemma 1.1, there exist one-factorizations

$$\mathcal{P}_i = \{\{a_{i,k}, b_{i,k}\} : 0 \le k \le m-1\}, 1 \le i \le 2m-1,$$
$$\mathcal{Q}_j = \{\{c_{j,l}, d_{j,l}\} : 0 \le l \le n-1\}, 1 \le j \le 2n-1$$

of K_{2m} on Z_{2m} and of K_{2n} on Z_{2n} respectively. Take the point set of $K_{2m,2n}$ as $Z_{2m} \bigcup \overline{Z}_{2n}$. Define the following collections of $K_{2,2}$-blocks of $K_{2m,2n}$, where $i \in Z_{2m}^* = Z_{2m}\backslash\{0\}, j \in Z_{2n}^*$.

$$\mathcal{A}_i^j = \{[a_{i,k}, b_{i,k}; \overline{c}_{j,l}, \overline{d}_{j,l}] : 0 \le k \le m-1, 0 \le l \le n-1\}.$$

Then the following collections form a $K_{2,2}$-$LGD(K_{2m,2n})$:

$$\{\mathcal{A}_i^j : i \in Z_{2m}^*, j \in Z_{2n}^*\}.$$

Firstly, each $\mathcal{A}_i^j$ is just a $K_{2,2}$-$GD(K_{2m,2n})$. And the total number of $\mathcal{A}_i^j$ is $(2m-1)(2n-1)$, as expected. Below we only need to verify that each $K_{2,2}$-block in the form $Q = [a, b; \overline{c}, \overline{d}]$ of $K_{2m,2n}$ on $Z_{2m} \cup \overline{Z}_{2n}$ appears in one $\mathcal{A}_i^j$.

Since $\{\mathcal{P}_i : 1 \le i \le 2m-1\}$ and $\{\mathcal{Q}_j : 1 \le j \le 2n-1\}$ are one-factorization of K_{2m} on Z_{2m} and one-factorization of K_{2n} on Z_{2n} respectively, for the edges $\{a,b\}$ and $\{c,d\}$, there exist $i \in Z_{2m}^*, j \in Z_{2n}^*$, such that $\{a,b\} = \{a_{i,k}, b_{i,k}\} \in \mathcal{P}_i$ and $\{c,d\} = \{c_{j,l}, d_{j,l}\} \in \mathcal{Q}_j$. So, $Q \in \mathcal{A}_i^j$. ∎

Theorem 2.3 *There exists a $K_{2,2}$-$LGD_2(K_{2m+1,2n})$ for any positive integers m and n.*

Proof. By Lemma 1.1, there exists a Hamilton cycle decomposition

$$\mathcal{P}_i = (a_{i,0}, a_{i,1}, \cdots, a_{i,2m}),\ 1 \le i \le m,$$

of K_{2m+1} on Z_{2m+1}. And there exists a one-factorization

$$\mathcal{Q}_j = \{\{c_{j,l}, d_{j,l}\} : 0 \le l \le n-1\}, 1 \le j \le 2n-1$$

of K_{2n} on Z_{2n}. Take the point set of $K_{2m+1,2n}$ as $Z_{2m+1} \bigcup \overline{Z}_{2n}$. Define the following collections of $K_{2,2}$-blocks of $K_{2m+1,2n}$:

$$\mathcal{A}_i^j = \{[a_{i,k}, a_{i,k+1}; \overline{c}_{j,l}, \overline{d}_{j,l}] : 0 \le k \le 2m, 0 \le l \le n-1\},$$

where $i \in Z_{m+1}^*, j \in Z_{2n}^*$ and the index $k+1$ is taken modulo $2m+1$. Then the following collections form a $K_{2,2}$-$LGD_2(K_{2m+1,2n})$:

$$\{\mathcal{A}_i^j : i \in Z_{m+1}^*, j \in Z_{2n}^*\}.$$

Firstly, since each $\mathcal{P}_i$ is a Hamilton cycle, each $\mathcal{A}_i^j$ is just a $K_{2,2}$-GD_2 $(K_{2m+1,2n})$. And the total number of $\mathcal{A}_i^j$ is $m(2n-1)$, as expected. Below we only need to verify that each $K_{2,2}$-block in the form $Q = [a,b;\overline{c},\overline{d}]$ of $K_{2m+1,2n}$ on $Z_{2m+1} \cup \overline{Z}_{2n}$ appears in one $\mathcal{A}_i^j$.

Since $\{\mathcal{P}_i : 1 \le i \le m\}$ is a Hamilton cycle decomposition on Z_{2m+1} and $\{\mathcal{Q}_j : 1 \le j \le 2n-1\}$ is a one-factorization on Z_{2n}, for the edges $\{a,b\}$ and $\{c,d\}$, there exist $i \in Z_{m+1}^*, j \in Z_{2n}^*$, such that $\{a,b\}$ appears in $\mathcal{P}_i$ and $\{c,d\}$ appears in $\mathcal{Q}_j$. So, $Q \in \mathcal{A}_i^j$. ∎

Theorem 2.4 *There exists a $K_{2,2}$-$LGD_4(K_{4m+1,2n+1})$ for any positive integers m and n.*

Proof. By Lemma 1.1, there exist Hamilton cycle decompositions

$$\mathcal{P}_i = (a_{i,0}, a_{i,1}, \cdots, a_{i,4m}),\ 1 \le i \le 2m,$$

$$\mathcal{Q}_j = (b_{i,0}, b_{i,1}, \cdots, b_{i,2n}),\ 1 \le j \le n$$

of K_{4m+1} on Z_{4m+1} and of K_{2n+1} on Z_{2n+1} respectively. Take the point set of $K_{4m+1,2n+1}$ as $Z_{4m+1} \bigcup \overline{Z}_{2n+1}$. Define the following collections of $K_{2,2}$-blocks of $K_{4m+1,2n+1}$:

$$A_i^j = \{[a_{i,k}, a_{i,k+1}; \overline{b}_{j,l}, \overline{b}_{j,l+1}] : 0 \le k \le 4m, 0 \le l \le 2n\},$$

where $i \in Z_{2m+1}^*, j \in Z_{n+1}^*$ and the indices $k+1, l+1$ are taken modulo $4m+1$ and $2n+1$ respectively. Then the following collections form a $K_{2,2}$-$LGD_4(K_{4m+1,2n+1})$:

$$\{\mathcal{A}_i^j : i \in Z_{2m+1}^*, j \in Z_{n+1}^*\}.$$

Firstly, since each $\mathcal{P}_i$ and $\mathcal{Q}_j$ is a Hamilton cycle, each $\mathcal{A}_i^j$ is just a $K_{2,2}$-$GD_4(K_{4m+1,2n+1})$. And the total number of $\mathcal{A}_i^j$ is $2mn$, as expected. Below we only need to verify that each $K_{2,2}$-block in the form $Q = [a, b; \overline{c}, \overline{d}]$ of $K_{4m+1,2n+1}$ on $Z_{4m+1} \cup \overline{Z}_{2n+1}$ appears in one $\mathcal{A}_i^j$.

Since $\{\mathcal{P}_i : 1 \le i \le 2m\}$ and $\{\mathcal{Q}_j : 1 \le i \le n\}$ are Hamilton cycle decompositions of K_{4m+1} and K_{2n+1} respectively, for the edges $\{a, b\}$ and $\{c, d\}$, there exist $i \in Z_{2m+1}^*, j \in Z_{n+1}^*$, such that $\{a, b\}$ appears in $\mathcal{P}_i$ and $\{c, d\}$ appears in $\mathcal{Q}_j$. So, $Q \in \mathcal{A}_i^j$. ∎

Theorem 2.5 *There exists a $K_{2,2}$-$LGD_4(K_{4m-1,2n+1})$ for any positive integers m and n.*

Proof. Similar to Theorem 2.4, we can get the proof. ∎

3 Conclusion

Theorem 3.1 *There exists a $K_{2,2}$-$LGD_\lambda(K_{m,n})$ if and only if $4|\lambda mn, 2|\lambda m, 2|\lambda n, \lambda|(m-1)(n-1)$ and $m, n \ge 2$.*

Proof. By Theorem 2.1, we only need to prove the sufficiency.

If $4|mn$, then we have

Case 1: if $m = 2s+1$, then $n = 4t$ and $2|\lambda|(m-1)(n-1)$.

By Theorem 2.3, there exists a

$K_{2,2}$-$LGD_2(K_{2s+1,4t}) = \{(Z_{2s+1} \cup \overline{Z}_{4t}, \mathcal{A}_i) : 1 \le i \le s(4t-1)\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\frac{\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{A}_i, \ 0 \le k \le \tfrac{2s(4t-1)}{\lambda} - 1,$$

then $\{(Z_{2s+1} \cup \overline{Z}_{4t}, \mathcal{B}_k) : 0 \le k \le \frac{2s(4t-1)}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{2s+1,4t})$.

Case 2: if $m = 4s$, then $\begin{cases} 2|n, & \lambda|(m-1)(n-1); \\ 2 \not| n, & 2|\lambda|(m-1)(n-1). \end{cases}$

By Theorem 2.2, there exists a

$K_{2,2}$-$LGD(K_{4s,2t}) = \{(Z_{4s} \cup \overline{Z}_{2t}, \mathcal{A}_i) : 1 \le i \le (4s-1)(2t-1)\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{A}_i,\ 0 \le k \le \frac{(4s-1)(2t-1)}{\lambda} - 1,$$

then $\{(Z_{4s} \cup \overline{Z}_{2t}, \mathcal{B}_k) :\ 0 \le k \le \frac{2s(4t-1)}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{4s,2t})$.

By Theorem 2.3, there exists a

$K_{2,2}$-$LGD_2(K_{4s,2t+1}) = \{(Z_{4s} \cup \overline{Z}_{2t+1}, \mathcal{A}_i) : 1 \le i \le (4s-1)t\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\frac{\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{A}_i,\ 0 \le k \le \frac{(4s-1)2t}{\lambda} - 1,$$

then $\{(Z_{4s} \cup \overline{Z}_{2t+1}, \mathcal{B}_k) :\ 0 \le k \le \frac{(4s-1)2t}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{4s,2t+1})$.

Case 3: if $m = 4s+2$, then $n = 2t$ and $\lambda|(m-1)(n-1)$.

By Theorem 2.2, there exists a

$K_{2,2}$-$LGD(K_{4s+2,2t}) = \{(Z_{4s+2} \cup \overline{Z}_{2t}, \mathcal{A}_i) : 1 \le i \le (4s+1)(2t-1)\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\lambda+1}^{(k+1)\lambda} \mathcal{A}_i,\ 0 \le k \le \frac{(4s+1)(2t-1)}{\lambda} - 1,$$

then $\{(Z_{4s+2} \cup \overline{Z}_{2t}, \mathcal{B}_k) :\ 0 \le k \le \frac{(4s+1)(2t-1)}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{4s+2,2t})$.

If $4 \not| mn$, then we have

Case 1′: if $2 \not| mn$, then $m = 2s+1, n = 2t+1$ and $4|\lambda|(m-1)(n-1)$.

By Theorem 2.4 and Theorem 2.5, there exists a

$K_{2,2}$-$LGD_4(K_{2s+1,2t+1}) = \{(Z_{2s+1} \cup \overline{Z}_{2t+1}, \mathcal{A}_i) : 1 \le i \le st\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\frac{\lambda}{4}+1}^{(k+1)\frac{\lambda}{4}} \mathcal{A}_i,\ 0 \le k \le \frac{4st}{\lambda} - 1,$$

then $\{(Z_{2s+1} \cup \overline{Z}_{2t+1}, \mathcal{B}_k) :\ 0 \le k \le \frac{4st}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{2s+1,2t+1})$.

Case 2′: if $2|mn$, then $m = 2s+1, n = 2t$ and $2|\lambda|(m-1)(n-1)$.

By Theorem 2.3, there exists a

$K_{2,2}$-$LGD_2(K_{2s+1,2t}) = \{(Z_{2s+1} \cup \overline{Z}_{2t}, \mathcal{A}_i) : 1 \le i \le s(2t-1)\}$.

Define

$$\mathcal{B}_k = \bigcup_{i=k\frac{\lambda}{2}+1}^{(k+1)\frac{\lambda}{2}} \mathcal{A}_i,\ 0 \le k \le \frac{2s(2t-1)}{\lambda} - 1,$$

then $\{(Z_{2s+1} \cup \overline{Z}_{2t}, \mathcal{B}_k) : 0 \leq k \leq \frac{2s(2t-1)}{\lambda} - 1\}$ is just a $K_{2,2}$-$LGD_\lambda(K_{2s+1,2t})$. ■

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